Physics 150 Laboratory Manual

Hobart & William Smith Colleges

HWS Physics Department

Spring Semester, 2015

General Instructions For Physics Lab

Come to lab prepared – Bring quadrille-ruled, bound lab notebook, lab instructions, calculator, pen, and fine-lead pencil.

Recording data

Each student must record all data and observations in ink in his or her own lab notebook. Mistaken data entries should be crossed out with a single line, accompanied by a brief explanation of the problem, not thrown away. Do not record data on anything but your lab notebooks.

Put the date, your name, and the name of your lab partners at the top of your first data page. Your data and observations should be neat and well organized. You will rely on these to make your report, and may need to refer to them later, too.

Lab reports

To make any kind of scientific or technical work accessible, it will have to be accompanied by appropriate documentation. Your physics lab report will cause you to review and reflect on what you do in lab, give you practice in rendering scientific results clearly, as well as give your instructors a way to evaluate your mastery of the concepts for that lab.

Broadly speaking, the report should describe what you learn during a given lab. What were your goals? What observations did you make, and what were the dominant sources of uncertainty? How did you analyze your data, and what were the results? Write for colleagues who may not have encountered the topic you are investigating so they can understand what you conclude—it will have to be both complete and concise. You will generally have four main tasks in the report: introducing and motivating your work, describing your procedure, stating the results, and interpreting the results. You of course will also have to provide a title for your report and give your name, the date, and the names of your partners.

Repetitive calculations should not be written out in your report; however, you should include representative examples of any calculations that are important for your analysis of the experiment.

A detailed description of the requirements for the lab report is given in the "Lab Report Guide," which is included in your course materials.

Graphing

Graphs are an excellent way to represent data. You may draw them by hand, for example in your lab notebook for reference and preliminary analysis. For presenation, however they should be plotted by computer graphing software such as Matlab, Mathematica, or Excel. Hand drawn graphs should be drawn with a fine-lead pencil. Any complete graph will have clearly labeled axes, a title, and will include units on the axes.

When you are asked to plot T^2 versus M, it is implied that T^2 will be plotted on the vertical axis and M on the horizontal axis.

Do not choose awkward scales for your graph; e.g., 3 graph divisions to represent 10 or 100. Always choose 1, 2, or 5 divisions to represent a decimal value. This will make your graphs easier to read and plot. When finding the slope of a best-fit straight line drawn through experimental points:

1. Use a best-fit line to your data. For a hand-drawn graph you can get very good fits just by judging the best line and using a ruler to sketch it in.
2. When measuring the slope of a best fit line, do not use differences in data point values for the rise and the run. You should use points that are on your best-fit line, not actual data points, which have larger errors than your best-fit line.

   Also, use as large a triangle as possible that has the straight line as its hypotenuse. Then use the lengths of the sides of the triangle as the rise and the run. This will yield the most significant figures for the value of your slope.
3. Most computer graphing software can compute the slope and intercept of the best fit line. You will also have to output the uncertainty of the fitting parameters, which is computed from the total variance of the data points from the line.

More information on graphing can be found in the appendix to this manual, the "Notes on Graphing."

Reporting of Experimental Quantities

Use the rules for significant figures to state your results to the correct number of significant figures. Do not quote your calculated results to eight or more digits! Not only is this bad form, it is actually dishonest as it implies that your errors are very small! Generally you measure quantities to about 3 significant figures, so you can usually carry out your calculations to 4 significant figures and then round off to 3 figures at the final result. After you calculate the uncertainties in your results, you should round off your calculated result values so that no digits are given beyond the digit(s) occupied by the uncertainty value. The uncertainty value should usually be only one significant figure, perhaps two in the event that its most significant figure is 1 or 2. For example: $T = 1.54 \pm .06\,\mathrm{s}$, or $T = 23.8 \pm 1.5\,\mathrm{s}$. If you report a quantity without an explicit error, it is assumed that the error is in the last decimal place. Thus, $T = 2.17\,\mathrm{s}$ implies that your error is $\pm 0.005\,\mathrm{s}$. Because the number of significant figures is tied implicitly to the error, it is very important to take care in the use of significant figures, unlike in pure mathematics.

More information on computing and stating uncertainty can be found in the appendices "Notes on Experimental Errors" and "Notes on Propagation of Errors."

Collaborative and Individual Work

You may collaborate with your partner(s) and other students in collecting and analyzing the data, and discussing your results, but for the report each student must write his or her own summary and discussion **in his or her own words.** Images or ideas from the lab manual and the textbook must be cited as such.

General Admonition

Don't take your data and then leave the lab early. If you have completed data collection early, stay in the lab and begin your analysis of the data—you may even be able to complete the analysis before leaving. This is good practice because the analysis is easier while the work is still fresh in your mind, and perhaps more importantly, you may discover that some of the data is faulty and you will be able to repeat the measurements immediately.

Experiment 1

Random Error & Experimental Precision

Goals & General Approach

We will measure the period of a pendulum repeatedly, observe the random errors, treat them statistically, and look to see if the measurements distribute themselves as expected. Then we will attempt to determine experimentally whether or not the period of a simple pendulum depends on the amplitude of its motion.

Apparatus

A simple pendulum (a metal bob suspended from a string), a stop-watch, and a photogate timer, with USB interface.

Procedure I: Timing a Single Period

You and your partner should *each* (independently) make twenty manual measurements (40 measurements altogether if there are two lab partners) of the time taken by a single swing of the pendulum.

1. For each measurement, draw the pendulum to one side about 20 degrees and release it in order to start it swinging.
2. Try to draw it approximately the same distance to the side in every case, so that you will be measuring the period for the same amplitude in all cases.
3. Also try to release it smoothly and consistently each time.
4. Allow the pendulum to complete at least one swing before beginning your period measurement, so as to allow it to settle down a bit.

5. Then, using the stop-watch, measure the time taken by a *single complete* swing. Give some thought as to how best to accomplish this. You might start and stop the watch when the ball reaches the top of its swing, or you may prefer marking time as it passes over a fixed point on the table. In the latter case be sure that the ball is be moving in the same direction when you stop your watch as it was when you started your watch so that the measurement is for a complete swing. You may want to practice several ways before deciding which one to employ.

6. **Do not measure the time for multiple swings.** Of course, using multiple swings would result in a better measure of the period, but the point of this exercise is to emphasize the variability of the measurements.

7. Now read the "Notes on Experimental Errors" to be found at the end of this manual before proceeding to the analysis section.

Analysis I

1. Create a histogram representing your and your partner's data sets. Do this by plotting the position of each of your time measurements on a time axis as shown in Figure 1. In order to distinguish the different sets, use a different plotting mark for each (say x for you and o for your partner). This makes for easier comparison.

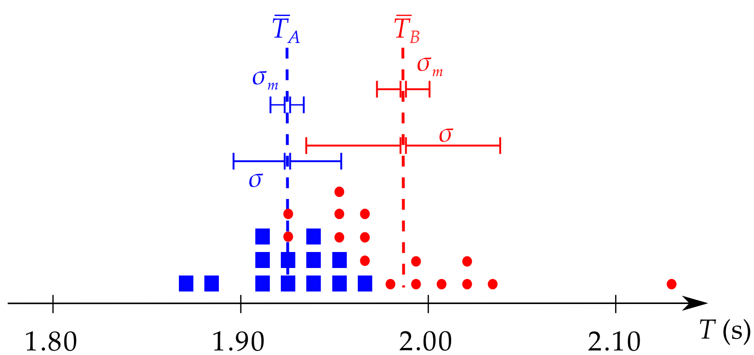

Figure 1

2. Calculate the mean (average) and standard deviation for each of your data sets. *Do not use a programmed function on a calculator to perform these calculations.* For each calculation, set up a table with columns for the measured values, the difference between each measured value and the average, and the squares of these differences. At the bottom of the T_i and $(T_i - \bar{T})^2$ columns enter the average value of the numbers in that column. The mean is the average of the T_i column and the standard deviation is the square root of the average of the last column. It is not necessary that each of you do all the calculations. You may divide up the work between or among you.

Measurement # (i)	T_i	$T_i - \bar{T}$	$(T_i - \bar{T})^2$
1			
2			
3			
⋮			
Average		0	

3. Create a table giving the average ($\bar{T}$), the standard deviation (σ), and the uncertainty in the average (σ_m) for each of your data sets. It is a common mistake to think that the standard deviation *is* the uncertainty in the average. *It is not.* If you thought so, go back and read the notes on error again more carefully.

4. On your time graphs mark with vertical dotted lines the average value for each data set ($\bar{T}$), and then draw horizontal bars centered on each average value, representing $\pm\sigma$, the standard deviation, and $\pm\sigma_m$, the uncertainty in the average (see Fig. 1).

Notes on Precision & Accuracy

The precision of a measurement refers to its reproducibility. How close together do repeated measurements of a quantity lie? This is measured by the standard deviation. Precise measurements have small random uncertainties reflected in a small standard deviation. For example, a person's weight fluctuates during the day, so a measurement the average weight over a week might vary depending on the specific times of measurement.

Precision is to be distinguished from accuracy. An accurate measurement not only has small random errors, but is also free from significant systematic errors. Thus a measurement may be very precise, in that repeated measurements give values lying very close together, but it may be very inaccurate if those values in fact misrepresent the value of the quantity being measured. This would be the case, for example, if the person's weight were measured every day after lunch to get an average for the week. The readings that go into the average may not differ from one another very much, but they would overestimate the average.

Questions I

1. Are your and your partner's values for the period in agreement? Justify your answer. Here it is crucial that you take the uncertainty in the average (σ_m) into account.

2. Do you think you can detect any difference in the precision with which you and your partner measured the period of the pendulum?

3. For each of your data sets, how many of the individual measurements lie within ± one standard deviation of the mean? Compare that with what the theory predicts, namely two thirds of the measurements.

4. What is the difference between accuracy and precision, and what

5. Considering that the standard deviation is a measure of the average deviation of the data from the mean, is it possible that all the data points lie within ± one standard deviation?

Procedure II: Dependence of Period on Amplitude

Galileo conjectured that the period of a pendulum is independent of the amplitude of the swing, but he could not come up with a theoretical proof. (He *was* able to prove that if instead of an arc of a circle, an object falls along a chord, the time to fall is independent of the length of the chord.) Not having a theoretical proof, we will put Galileo's conjecture to experimental test.

First, you will use single-swing stop-watch measurements of the sort employed in Procedure I, for two different amplitudes of the motion of the pendulum. Then you will use a photogate and electronic timer to measure the period for the same amplitudes.

1. Choose two rather different amplitudes (I suggest 5° and 20°) and make multiple measurements for each one. (You already have some data from Procedure I that need not be ignored here, as long as you were careful to control the release angle.)

2. Repeat the amplitude experiment using the photogate and computer interface. Your laboratory instructor will show you how it works. Note that the results from the photogate timer may be highly reproducible. If you find many measurements of exactly the same result, the error is not zero even though the standard deviation is. In this case, the correct error to use is the so-called "least count" error. For example, if the least significant digit is 0.001 second, the correct error in any given measurement is half that, or 0.0005 second. Use this value for σ, and not the standard deviation.

Analysis II

You now have four data sets, two sets acquired using the stop-watch and two sets using the photogate. Using the statistical function on your calculator or a computer, find the mean, standard deviation and standard error of the mean for each data set. For each timing method, first stop-watch and then the photogate, determine whether the difference in period between the two amplitudes is statistically significant.

Report

You have just put Galileo's conjecture to the test using two timing methods. Report on whether the conjecture is plausible based on your results from the stop-watch method. Also, report on it's plausibility given your photogate data. Your answers rely on your analysis of the precision and accuracy of your measurements, so in your discussion please address Questions 1–4 of Procedure I. Also show any analysis you did in Procedure I required to answer those questions.

Experiment 2

Instantaneous Velocity

Goals & General Approach

This experiment will help you to understand the concept of instantaneous velocity. You may want to read the relevant sections in your text before embarking on the experiment. Intuitively we think of instantaneous velocity as the velocity that an object has at a particular instant in time and at a particular point in space. Now in physics, the definition of a concept must contain within it, either implicitly or explicitly, directions for measuring it. But how does one go about measuring a velocity at a point in time?

We know that average velocities are measured by dividing a distance traversed by the time taken. An instant in time presumably has no duration, and a point in space presumably has no extension. Thus velocity at an instant seems to defy our powers of measurement, and perhaps also our powers of imagination. (The ancient philosophers struggled mightily with this problem, and you will be rewarded by reading about some of their efforts[1].) So we will have to enlarge our intuition to include the notion of instantaneous velocity.

We will define instantaneous velocity in terms of the experiment that you are going to do today so as to be as clear and concrete as we can. Fig. 1 will help you to follow the explanation. You will drop a steel ball from the ceiling and attempt to measure its instantaneous velocity at point P along its path. We propose to do that in the following way.

We establish a higher point A and measure the distance between A and P, as well as the time it takes for the ball to fall from A to P. Dividing the distance by the time will give us the average velocity between these two points. We suspect that this average velocity will be less than the velocity at P, since the velocity of the ball is presumably increasing as it falls. (Note: When we use the word velocity without a modifier, we are referring to *instantaneous* velocity.)

[1]See, for example, Zeno's arrow paradox (Aristotle *Physics* VI:9, 239b5), among others.

Now choose a new point, call it A', which is not as high as the first, but still above point P. Measure the average velocity between A' and P using the same technique. Once again one expects this average velocity to be less than the velocity at P, but greater than the average velocity between A and P, because it samples velocities closer to P. Repeating this procedure for additional points, each successively closer to point P, results in a series of generally increasing average velocities, all of which we suspect to be less than the velocity at P. Now we are in a position to offer a practical definition of the instantaneous velocity at P.

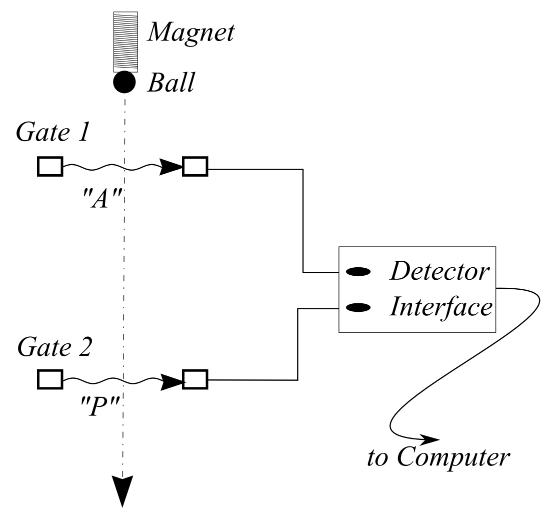

Figure 1 Schematic diagram of the apparatus.

Definition Plot the average velocity values that you measured, as a function of the corresponding time intervals. Draw a smooth curve through the data points and extrapolate it to zero time interval. The velocity value given by this curve for zero time interval (the y-intercept) is defined to be the instantaneous velocity at P (see Fig. 2). More formally and mathematically one says this as follows (see your text book): the instantaneous velocity at a point is the limit that the average velocity between that point and a neighboring point approaches, as the neighboring point is brought successively closer to it. Or,

$$v = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} . \qquad (1)$$

You no doubt recognize this as the definition of the derivative of the position y with respect to the time t. So the instantaneous velocity is the derivative of the position with respect to time, evaluated at the point of interest.

We will also use a second method for determining the instantaneous velocity at point P. This second method relies upon the assumption that the ball is undergoing constant acceleration. We first choose a point A just a small distance above point P. We will measure the time the ball

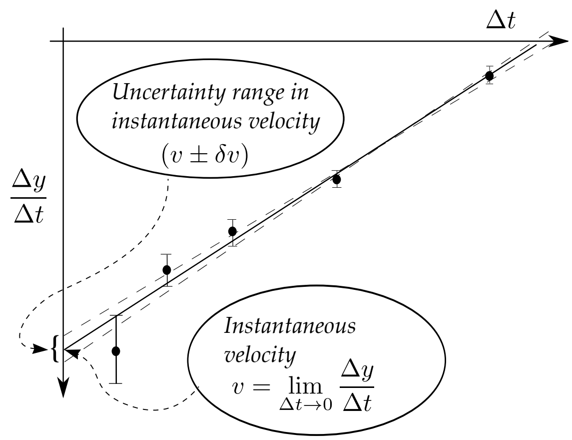

Figure 2 Graph of the average velocity $\Delta y/\Delta t$ versus time interval Δt. The intercept of the "best fit" straight line through the data is the instantaneous velocity. The dashed straight lines are reasonable alternative "best fits" that are used to find the uncertainty in the instantaneous velocity.

takes in dropping between points A and P and then, keeping the point A fixed, we will find another point, call it B, such that the time the ball takes falling between A and B is exactly twice the time taken between points A and P. If the ball is accelerating uniformly, the average velocity over the interval from A to B is equal to the instantaneous velocity of the ball at a time halfway between the times at which the ball is at the points A and B, or, the time at which the ball is at point P. This situation is depicted in Fig. 3.

Apparatus

The steel ball is initially held near the ceiling by an electromagnet, and can be released by activating a switch which interrupts the current to the magnet. Two photogates are mounted on a track down which the ball falls (See Fig. 1). Each gate consists of a light source, focusing lenses, and a light detector. (The light is infrared so you will not be able to see it.) As the ball falls through each light beam, the light is momentarily interrupted and the photo-detector sends out a signal to the computer, which records the time of each event. The data logging software computes and displays the time interval between events.

Procedure I: Instantaneous Velocity

1. Measure the distance from the floor to the bottom of the magnet (use a 1-meter and 2-meter stick together with the sliding pointer mounted on the top stick). Measure the diameter of the steel ball with a pair of calipers. These two numbers will enable you to find the distance from the bottom of the ball to the floor when the ball is hanging from

the magnet. **REMINDER**: **Don't forget that all measurements need to have an uncertainty value attached to them before they are complete.** Take 2 or 3 independent measurements of this position in order to estimate your uncertainty.

2. Set Photogate 2 on the track not more than a meter above the floor, make sure it is oriented perpendicular to the track, and then tighten it very securely to the track. This determines the position P and the gate will remain at P throughout the experiment.

3. Place Photogate 1 at a position above P.

4. Use a 2-meter stick with sliding pointer to measure the height above the floor of the light beams of the two gates. To do this, first start the software collecting data in "Preview" mode. Then rest the end of the meter stick on the floor and slide the slider down (the way the ball will be moving) until the flat edge interrupts the light beam, at which point the state of Gate 1 will go from 0 to 1. At that point, read the position of the slider on the meter stick. Record this in your lab notebook. Again, estimate your uncertainty by taking 2 or 3 independent measurements.

5. Repeat the procedure to find when the slider interrupts Gate 2. Record the height of Gate 2 in your lab notebook along with your uncertainty.

6. Stop previewing data, without keeping any timing information. Then click "Preview" again to restart data collection.

7. Press the button to momentarily interrupt the electromagnet and drop the ball. The time the ball spent between gates should be displayed on screen. Click "Keep" to record the interval.

8. Repeat your timing measurement at least three times in order to check for reproducibility. If slightly different times are measured, an average can be taken and an uncertainty assigned sufficient to span the range of the individual readings. If the times are significantly different (more than a few tenths of a millisecond), something is wrong. If the times are identical, take the uncertainty to be one in the least significant figure.

9. Stop recording when you have several measurements at this height.

10. Begin working on the analysis section. Doing the analysis as you go will allow you to see if there are gaps in your data set you want to fill and if there are questionable points you want to revisit.

11. Now loosen the bolt and slide Photogate 1 to a new position. Measure and record the new height of the gate and your uncertainty.

12. Then start a new run and measure the elapsed time during the fall, taking enough data to estimate the uncertainty in both.

13. Take a set of measurements for not fewer than five positions of Gate 1. Try to get a range of positions, including one as close to Gate 2 as you

can get (when their brackets touch—but be careful not to disturb the position of Gate 2). Don't use equal distances between the Gate 1 positions, as that will make the elapsed times change by rather unequal amounts. Rather, change the position less when Gate 1 is high, and change it more as it gets closer to Gate 2.

Analysis I

1. For each run, the mean and standard deviation of the measured time intervals are displayed. Enter the mean into the column for "Mean Time Between Gates."

2. Also use the displayed standard deviation to estimate your uncertainty in the mean, and enter it into the column "Timing Uncertainty."

3. From your measurements of the heights of the photogates, find the mean and your uncertainty for the height of the ball at the beginning and end of the interval you recorded.

4. Subtract the initial height from the final to find the change in height of the ball during the recorded interval, Δy. Adopt the convention that upward is positive. Go to Page 2 in the experiment, and enter your value in the table for "Change in Height."

5. From your uncertainty in each height, estimate the uncertainty in Δy. If you do not know how to combine the uncertainties of each height to get the uncertainty in the difference, consult the appendix *Notes on Propagation of Errors*. Enter the result in the table for "Δy Uncertainty."

6. Then go to Page 3 to calculate the average velocity of the ball over the time interval spent between the gates. Click on the calculator on the left, and enter the formula for the velocity. The polychrome triangle button allows you to use your measured quantities in the calculation (e.g. displacement and time interval). The velocity should appear in the appropriate column.

   You will be able to export your data for further analysis at home. However, also record your data in a table like the one below, so as to be sure to have access to it.

	Δy (m)	Δt (s)	$\bar{v}$ (m/s)
1	1.752 ± 0.001	0.7759 ± 0.00005	2.258 ± 0.001
2			
3			
4			
5			
6			
7			

7. While you are taking and recording the data for the various positions of Gate 1, plot your average velocity values against the corresponding times (Δt). You may do the plot using the Capstone software on Page 4 or with the computer plotting software of your choice. You may also want to copy the plot into your lab notebook. Don't wait to do this after all of the velocities have been measured. Be sure to choose appropriate axes and scale so the data take up the full allotted space.

8. If you do not already know, **read the notes on propagation of error** to learn how to calculate the uncertainty (error) for each calculated velocity value, given the uncertainties in displacement and interval.

9. In order to compute the uncertainty in velocity, you will first have to compute the *relative* uncertainty in your measured quantities (the fractional error). Go back to Page 2 of the experiment and enter a formula for the relative uncertainty in Δy.

10. Do the same for the relative uncertainty of the time interval.

11. Enter a formula for the relative uncertainty of velocity, using the relative uncertainties in displacement and interval you just calculated. The result will appear on Page 3.

12. Finally, compute the (absolute) uncertainty in velocity from the relative uncertainty in velocity. Make sure to include the absolute uncertainties for each measurement in your written table, as shown.

13. Using the uncertainties calculated above, place vertical error bars on each of your plotted velocity values. If you are using the Capstone graph, the blue gear above the plot gives you access to the graph properties. Choose "Data Appearance" to turn on error bars. Under "Error Bar Type," select "Measurement" to use your calculated uncertainty as the length of the bars.

14. The average velocity in this case is expected to be a linear function of time interval. (Can you see how to show this theoretically?) Therefore the data would lie on a single line, were it not for random error. The line that best fits the data is our estimate of the line in the absence of random error. You can find the best fit line using the toolbar of the Capstone graph (the button is a set of points with a line). Choose a weighted linear fit. Microsoft Excel can also compute a fit ("trendline") but you will have to use the LINEST function to find the uncertainty in the fit parameters.

    Caution: The LINEST function and the linear fit function in Capstone as well as most other fitting software, assumes that each point is equally likely to stray from the line. You know that this is not true because the uncertainty is different for each point. Unless you use a weighted fit, your fit parameters are only rough estimates. The course website has the add-in "`leastsquaresaddin.xla`," which provides the function LSFit to do weighted fits in Excel. There is also information about how to use both the LINEST and LSFIT functions.

15. After making the fit, tell Capstone to use your measured uncertainties as the errors when weighting the fit. Go to the rainbow triangle for the "Data Summary," and click the pencil to access your user entered data. Then for the velocity data, click the blue gear to access the properties. Find the "Errors" section, and choose "Measurement" for the errors. Finally select the velocity uncertainty.

16. Locate intersection of the fit line with the velocity axis. This intercept value is the limit of the average velocities as the time interval approaches 0 and is by definition the instantaneous velocity at the position of Gate 2.

17. The intercept with the velocity axis shows the instantaneous velocity at the point of the second photogate. In your lab report, please also discuss the slope of your fit line. In quoting its value, of course, give both the units and uncertainty. What is the significance of the slope?

Before starting Procedure II, complete your plot from Procedure I and make sure that the results are reasonable. Once you begin Procedure II, it will be difficult to restore your apparatus to the Procedure I configuration.

Procedure II: Average Velocity

In subsequent experiments, we will want to measure a whole set of instantaneous velocities. That will be very time consuming if we have to use the procedure employed in this experiment. A simpler procedure would be to measure the average velocity over a very small interval centered on the point of interest. But would this average value be sufficiently close to the instantaneous value that we want? Let's check it out against the data we have just taken.

1. Your two photogates should still be in their last positions (as close to one another as possible). Re-measure the elapsed time for this position to make sure that it hasn't changed, and make a note of it in your notebook.

2. Leaving Gate 1 fixed, drop Gate 2 slightly. (This should be the first time that Gate 2 has been moved.)

3. Check the new time interval for the ball to traverse the gates. Your goal is for this new interval to be twice its previous value (this requires some trial and error). If it is not readjust the position of Gate 2 until it is. This ensures that the time the ball spends falling from point A (Gate 1) to point P (the original position of Gate 2) equals the time the ball spends falling from point P to point B (the new position of Gate 2.) With gates 1 and 2 in these positions, we are sampling velocities above and below the velocity at P for equal times, and the average should then be the instantaneous velocity at P. Figure 3 shows a graph of the velocity versus time.

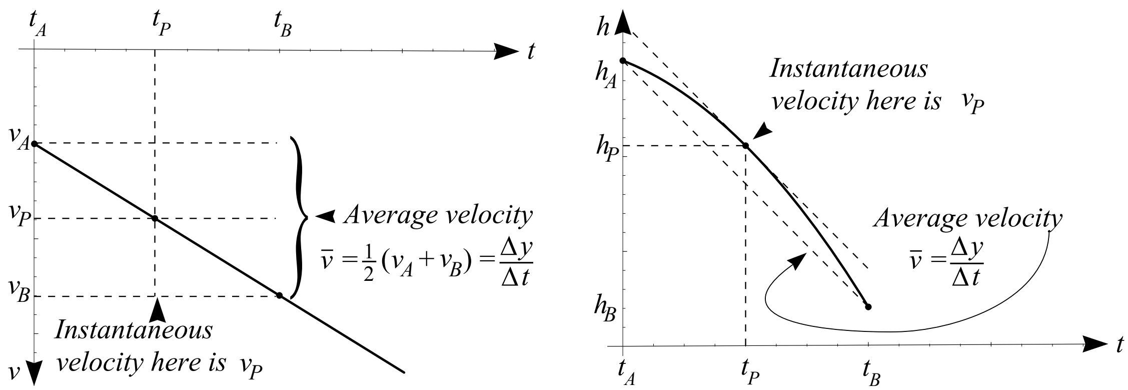

Figure 3 For constant acceleration, the average velocity equals the instantaneous velocity at a time halfway between the endpoints. Thus the *average* velocity between points A and B is equal to the *instantaneous* velocity at the point P. You cannot simply use $\bar{v} = (v_A + v_B)/2$, since you do not have the instantaneous velocities at A or B, but you *can* use $\Delta y/\Delta t$.

4. Measure carefully the height of the two light beams above the floor.

5. Calculate the average velocity of the ball between the two gates. Also calculate the estimated uncertainty (error) in this value.

Analysis II

1. Compare the values you have obtained for the instantaneous velocity of the ball at point P from the first procedure and the second procedure. Are they in agreement?

2. You have taken sufficient data to allow you to calculate the distance the bottom of the ball has fallen from rest (at the magnet) to point P (Photogate 2). From the theory of freely falling bodies, and the accepted value for the acceleration of gravity, calculate the velocity that the ball is predicted to have after falling that distance, and use the uncertainty in the distance fallen to calculate an uncertainty in your calculated velocity value. (*Caution: think carefully about how to get the uncertainty in velocity from that in height.*) Does this calculated value agree with your two experimental values?

Report

What is instantaneous velocity, and how can you measure it? Discuss whether the results of the two methods of measurement agree and whether either or both agree with your expectation from theory. Give detail about how you determine "agreement" and in case you find disagreement, which method you trust more and why.

Experiment 3

Force Table

Goals and General Approach

In this experiment we will attempt to prove for ourselves that forces are vectors. By this we mean that if we represent a set of forces acting on an object by vectors, and then add the vectors together, the resultant vector accurately represents a force which is equivalent to the original set of forces. By equivalent we mean that this single force would have the same physical effect on the object as the original set of forces. In addition we will get some practice at adding vectors, both graphically and analytically.

Apparatus

The apparatus for today's experiment is called a force table, and consists of a horizontal circular table with a removable peg at its center. An angular scale is provided around its edge, where pulleys may be clamped. A string passes over each pulley, one end of which is connected to a ring at the center of the table and the other to a weight hanger.

Procedure

The ring is acted on by three forces. The magnitude of each force depends on how much weight is placed on each weight hanger, and the direction depends on the directions assumed by the strings. If the three forces do not add up to zero, the ring will tend to move in the direction of the net force, and it will move until it reaches a new position where the new forces (new because the directions have changed) do add up to zero. We call this the equilibrium position, and say that the ring is in equilibrium because it has no tendency to move away from this point.

1. You will be assigned a set of three angles. Set your pulleys at those angles.

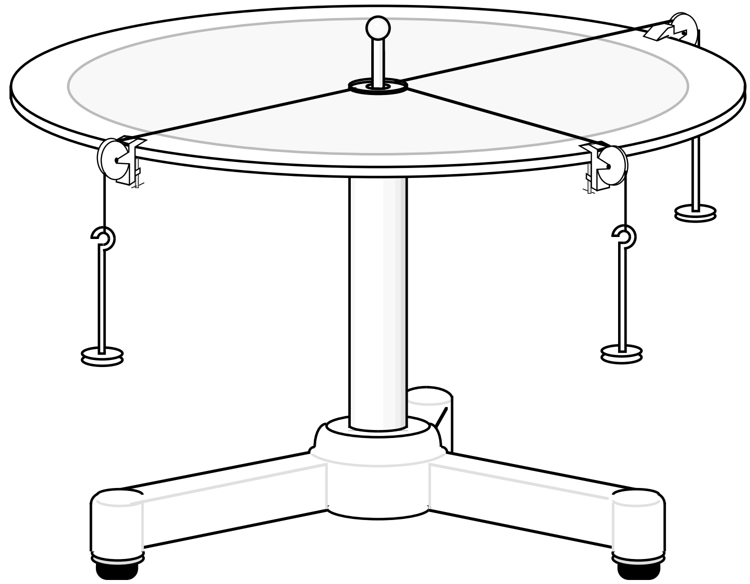

Figure 1 Force table apparatus used in this experiment.

2. With the pin in place in the center of the table, place some weights on the weight hangers. (The pin keeps the ring from flying off the table while you are getting the weights in place.) **When choosing weights in this experiment avoid choosing very light ones, but do not exceed about 500 grams.** Adjust the amount of weight on each hanger until you succeed in stabilizing the ring exactly at the center of the table (the pin will be at the center of the ring). Jiggle the strings after each weight change to aid the system in coming to equilibrium.

3. After achieving ring equilibrium at the center of the table, pick a string and shift the ring away from equilibrium in the direction of that string. When you let go, you should find that the forces are not balanced, and the ring returns to equilibrium. However, it will not return exactly to the former position (why?).

4. Adjust the weight on the string you picked until it exactly returns to equilibrium. The amount you adjusted the weight is an estimate of your uncertainty in the weight required on that string.

5. Repeat the shifting and readjustment process for the other two strings.

6. Record the final best weights, weight uncertainties, and angles. (Don't forget to include the weight of the hanger in your weight value.) This information provides you with the magnitude and direction of each force.

Analysis

By the definition of the equilibrium position, the ring remains at rest there. If you could replace all the forces acting on the ring by a single force that has the same effect, what force would you choose? (Don't overthink this: what force is required on an object at rest to get it to stay at rest?)

Now to show that forces act like vectors, you must show that the vector sum of the actual forces you measured on the ring is indeed identical to the equivalent force. Of course you will not find exact agreement, because of experimental uncertainty. The best you can do is show that the difference between the vector sum of forces and your expected equivalent force is attributable to your measurement uncertainty.

Represent each of the three forces by a vector whose length is determined by the weight applied to the string, and whose direction is along the direction of the string. Then find the sum of the three vectors. The summation should be performed in two different ways—graphically and analytically.

Graphical Addition of Forces

1. Draw two *rough* sketches in your notebook, one representing the forces as they act on the ring and the other representing the same forces laid tail-head, tail-head for addition. Choose the positive x-axis to correspond to zero degrees of angle. Make the length of each vector proportional to the magnitude of the corresponding force and make the angles representative of the actual physical angles. Your sketches will look something like Figure 2.

2. Calculate the angles that your vectors make with the x-axis and label the angles on your sketches accordingly. Also label each vector with the weight in grams.

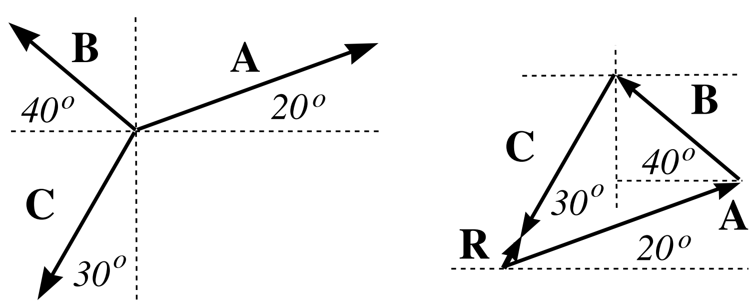

Figure 2 The leftmost figure shows the forces $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$, and $\vec{\mathbf{C}}$ acting on a point. The rightmost figure shows the graphical method of adding the vectors to produce their resultant $\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}} + \vec{\mathbf{C}}$.

3. Now plot a very careful and accurate version of your second sketch that you can include in your report. Let it take up a full page. You may choose to generate it with a fine lead pencil on good graph paper. In that case, use a protractor to draw the vector with the correct angle and ruler to set its length. You will have to make a high quality scan of your plot to include it in your lab report. (No phone photos!

Force	x-component (g)	y-component (g)
A		
B		
C		
Resultant R		

The copiers in the library can scan and email your documents free of charge).

Alternatively, you may computer generate the plot. In that case you will draw three vectors, scale and rotate them appropriately and set them tail-head like in your second sketch.

4. Whatever method you choose, the positive x-axis should correspond to zero degrees. You will need to choose a scale for your plot, on which a distance will represent a force. Let each centimeter represent some number of grams such that the scale is easy to use (for instance, multiples of 10, 20, or 25 g) and such that diagram is as large as your paper will allow. (Of course grams are not units of force; they must be divided by 1000 and multiplied by 9.8 m/s^2 to obtain the corresponding weight in Newtons. However, since the mass is proportional to the weight, *we can enjoy the convenience of using gram units without affecting our analysis and conclusions.*)

5. The *resultant* $\vec{\mathbf{R}}$ is the vector sum of the three vectors. Draw $\vec{\mathbf{R}}$ diagram and measure its magnitude and direction (an angle with respect to some convenient axis). Remember that the resultant is the vector drawn from the tail of the first of your three vectors to the head of the third.

Analytical Addition of Forces

1. Make a table in your notebook with two columns, the first for x-components and the second for y-components. Label the rows A, B, and C to represent your three force vectors.

2. Use trigonometry to calculate the x and y components of each vector (your first rough sketch will assist you at this). Be sure to include the correct algebraic sign in each case, which you can determine by inspection of the figure. Do each calculation neatly in your notebook. Enter these component values (in grams) into the table you have created.

3. Add a fourth row to your table labeled "Resultant" and find the x and y components of the vector sum $\vec{\mathbf{R}}$ by adding the components in each column and recording the result in the "Total" row.

4. Use the Pythagorean Theorem and trigonometry to find the magnitude and direction of the resultant from its x and y components (again all force values expressed in grams).

Report

In the results section of your report show your work in adding the forces graphically (your plot) and analytically (your table). Then discuss the following points.

1. Construct another small table giving the magnitude and direction of the vector sum by each method. For both addition methods, state whether your resultant force is in agreement with the expected equivalent force.

Resultant $\vec{\mathbf{R}}$	Graphical Sum	Analytical sum	Expected value
Magnitude (g)			
Direction (°)			—

   Remember *agreement* is determined by the uncertainties. Square the uncertainty you estimate for each force, add the squares, and take the square root. This represents the uncertainty in the resultant, the distance you expect the tip of the resultant to be affected by random error. (Uncertainties in the angles will also affect the magnitude of the sum, but it is rather difficult to calculate their effects, and so we will not be able to treat them.)

   If the distance between your resultant vector and your expected equivalent vector is less than your uncertainty, you can claim that forces behave as vectors to within the precision of this experiment.

2. You can also obtain very good values for the x and y components of each individual by reading them directly from the careful scale graph you created in doing the graphical summation. Compare the values obtained this way with the values you obtained by the analytical method. Do they seem to be equal to within the limitations of the scale used in the graph?

Experiment 4

Momentum and Kinetic Energy in Collisions

Finding Momenta and Energy of Colliding Gliders: Goals and General Approach

Making use of gliders on a nearly frictionless air track, we will study the total momentum and kinetic energy of colliding masses. We generally distinguish between two types of collisions: elastic and inelastic. In an elastic collision, the total kinetic energy is the same before and after the collision. In an inelastic collision, the total kinetic energy is less after the collision. In practice, no macroscopic collision is ever perfectly elastic, since some energy losses will always be incurred, but our system can come pretty close.

Total momentum should be the same before and after elastic and inelastic collisions, so long as there are no external forces acting on the system in the direction of motion. Of course the air track cannot be perfectly frictionless, and if the track is not perfectly horizontal, a component of the gravitational force will be non zero along the track. Also, if the collision is too violent, the gliders may be forced to scrape on the air track, introducing another external force with a component along the track. These effects will introduce some change in the total momentum during the course of the collision, and so you should try to keep them to a minimum.

Apparatus

Air track, small, medium and large gliders, photogates, computer and interface, and triple beam balance.

Procedure

In this lab we will use photogate timers to find the velocity of the gliders before and after they collide, in various situations. Then after measuring the mass of the gliders, we'll be able to find the momentum and energy of each. We'll compare the total momentum and energy before and after the collision. The linear air track enables us to study motion that is almost frictionless, so we will ignore friction in our analysis, and therefore we expect that the forces external to the system of gliders will be negligible. Of course we will also need to get an estimate of how reliable that assumption is. We'll do this by looking at the change in speed of a single glider, which is an indication of the force acting on the glider.

Preliminary set up

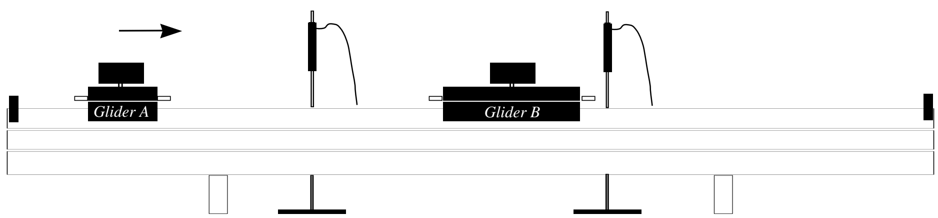

Figure 1 Gliders on the airtrack. Glider "B" is between the two light gates.

1. Before beginning, make sure that the track is level. With the air flowing, place a glider at different points on the track and look for any tendency for it to drift one way or the other. If necessary, adjust the leveling screw at one end of the track until such drifts are negligibly small.

2. Measure the flag length. Most (but not all) of your photogates have indicator lights to show when they are blocked. Slide the glider until the flag first blocks one of your photogates. Look at the ruler built into the track, and note the position of the glider. Then keep sliding the glider until the gate is unblocked, and note that position. The length of the flag is the difference in these positions.

   You may take the flags all to be of the same length for all carts and both shutters. How much uncertainty in the flag length does this introduce? (That is, how close in length are all your flags?) Write down the length of the flag and your estimate of the uncertainty.

3. If not already open, run the Capstone software and open the file "pandKE".

4. Go to "Timer Setup" on the left panel and enter your measured value under "Flag Length." This allows the computer to calculate the speed of the glider from the time the photogate is blocked. (How?)

Estimating the effect of external forces

1. We expect the total momentum and energy of the glider (or cart) system to be conserved only under the condition that there is no external force or work. As a test of this condition, look at how the velocity of a single glider changes as it slides down the track:

2. Take the smallest glider and fit it with two similar bumpers so it is balanced.

3. Press record and start the glider from one end of the track so it intercepts both photogates. When the glider interrupts the light beams as they slide by, the amount of time the gate is blocked is sent to the computer, which then calcuates the velocity of the glider (v = shutter length/transit time).

4. Once you get data from both photogates, stop recording.

5. To within the precision of the photogate does the velocity change? If so, there is an external force.

6. Record the two velocities in your lab notebook, and calculate the change in velocity. Also note the sign of any velocity change. Is it an increase or decrease in magnitude?

7. Start the cart from the other end of the track and perform the same test. Again, make note of the velocity and its change.

8. Lift the cart off the track, flip it, and set it back. Perform the test one more time.

9. For each of the three runs, also compute the percent change in velocity: (final - initial)/initial.

Colliding the gliders

1. Select two gliders to collide, and set bumpers on the gliders. You can have a nearly elastic collision (little lost KE) if you use the metal band bumper. You can have a totally inelastic collision (maximum lost KE) if you put a bumper with a pin on one glider and the wax-filled cup on the other glider. Whatever bumper you choose for one end of the glider, make sure the other end is loaded with a similar weight. This keeps the gliders balanced—if they are unbalanced, more air will escape from under one end than the other resulting in a net horizontal force, in which case the glider system momentum is not conserved.

2. Practice a few collisions before trying to take data just to get the hang of it. Keep the velocities fairly high so that the effects of track friction, imperfect leveling, etc., will not loom large compared with the velocities that you are measuring. On the other hand, if the velocities are too great, the gliders may scrape the track briefly during the collision and destroy the condition necessary for momentum conservation to hold true.

3. Now you will take measurements. The following table summarizes eight situations that you are expected to measure. Feel free to invent others that interest you as well. In several of these situations, Glider B will be initially at rest. You can insure that this is so (and therefore insure that its initial velocity is zero) by holding on to it lightly until just before the other glider collides with it.

	GLIDER A		GLIDER B		
Case	Mass	Initial Vel.	Mass	Initial Vel.	Bumper
1)	2	moving right	3	zero	elastic
2)	1	moving right	2	zero	elastic
3)	1	moving right	3	zero	elastic
4)	3	moving right	1	zero	elastic
5)	1	moving right	2	moving right	elastic
6)	2	moving right	3	zero	inelastic
7)	1	moving right	3	zero	inelastic
8)	1	moving right	2	moving left	inelastic

Table 1 Investigate at least these eight collisions. You will be working with three different size gliders, with relative masses of approximately 3:2:1. The number under mass in the table indicates which glider to use. In case 5, the lighter glider should catch up with the heavier glider and collide with it somewhere between the photogates. You may need some practice to accomplish this.

4. For each collision you measure, choose the appropriate pair of gliders, attach the appropriate bumpers, and then weigh each glider. Be sure that you have checked the zero on your balance. **Remember that when you change bumpers on a glider, it must be weighed again.**

5. Select "Record" to measure the photogate times for the gliders to pass the photogates before and after the undergoing the collision you have chosen. Stop recording after both final velocities have been measured.

6. You will need to make sure the collision happens between the photogates, so that the initial and final velocities of both gliders can be measured. In some circumstances, one of the gliders will bounce off an end and a second collision will occur before the other glider intercepts a photogate. In that case, you'll have to catch the faster glider before it hits the slower one.

7. The computer outputs the time the flag spent blocking the gate. It also shows the speed of the cart given the flag length you provided. It has no way of knowing the direction of the cart, so you will have to make a note of it and change the sign of the velocity accordingly. If you are satisfied you have all four velocities (initial and final of both

gliders), record the data in your lab notebook, and copy and paste them to an Excel notebook. If not, try that trial again.

8. **After each measurement, do the momentum calculations and check for conservation of momentum before going on the next case.** If there are any questionable results, you may want to confirm your data by repeating the trial.

Analysis

1. For each collision create a small pictorial diagram in your lab notebook that summarizes the nature of the collision by showing the relative sizes of the gliders involved, and the directions and relative magnitudes of their velocities before and after the collision.

2. From the mass of each glider, and their initial and final velocities, compute the following quantities (you may want to use an excel spreadsheet to do the calculations, so that you won't have to input the formulas each time):

    - Momentum and kinetic energy for each glider before and after the collision. Don't forget the algebraic sign for the momenta.
    - Total momentum and total kinetic energy for the two glider system before and after the collision.
    - The change in total momentum and total kinetic energy in the course of the collision, expressed both as their actual values and as a percentage of the total initial value.

3. Place your results in your lab notebook in a table. A form such as the following would be convenient.

	Momentum		Kinetic Energy	
	before	after	before	after
A				
B				
Total				

Change in total momentum: _____ Change in total KE: _____
% Change _________ % Change _________

4. When you have completed your measurements, create a table that summarizes all of your results. It should look like this, where p and KE refer to total momentum and total kinetic energy for the two glider system:

Identifier	p_i	p_f	$\Delta p(\%)$	KE_i	KE_f	$\Delta KE(\%)$
Case 1						
Case 2						
⋮						
Case 8						

The Identifier should refer to the enumeration employed in the table on the second page of this experiment.

Questions

1. Do the total momenta and total kinetic energies behave the way you expect them to for each collision? Explain. If your results do not seem to make sense, you may want to go back and repeat the relevant measurements.
2. If a glider is to collide elastically with a stationary glider, what size glider would you choose to maximize the KE the moving glider gives up to the stationary one? Use your data to justify your answer.
3. If a glider is to inelastically collide with a stationary glider, what size glider would you choose to maximize the KE lost by the moving glider? Use your data to justify your answer.

Note: The answers to questions 2 and 3 do not lie in the values for ΔKE in your table, which represents the loss in KE for the whole two-glider system. **You must look at the change in KE of the initially moving glider only** to answer these questions.

Report

Clearly present the drawings and tables summarizing your results and answer the questions that have been posed.

Experiment 5

Simple Harmonic Motion

When an object in stable equilibrium is displaced, a force acts to return it to its equilibrium position. If the restoring force is proportional to the displacement (a "linear" force), disturbing the object will result in a sinusoidal motion, also called "simple harmonic motion (SHM)." Simple harmonic oscillations are important because even when restoring forces are not linear with respect to displacement, they approach linearity as displacements become smaller and smaller. Therefore, small amplitude oscillations are usually harmonic even when oscillations of larger amplitude are not. Today's SHM experiment will employ a mass suspended from a spring, a common example of simple harmonic motion.

Goals and General Approach

1. Determine the extent to which the frequency of the mass/spring system oscillations is dependent on the amplitude of the oscillations. (Recall your exploration of this question for the simple pendulum earlier in the term. The frequency is completely independent of amplitude for an ideal harmonic oscillator.)
2. Determine the spring constant two different ways, firstly by directly observing the effect of stretching the spring, and secondly by inferring it from the motion of the oscillator.

Apparatus

A weight hanger suspended from a vertically hanging spring, an assortment of weights, a meter stick with sliding pointer, a triple beam balance, a stop-watch, and a motion sensor with wire guard and computer interface.

Procedure I: Effect of Amplitude on Period

For a given spring and mass, how does the period of oscillation depend on the amplitude?

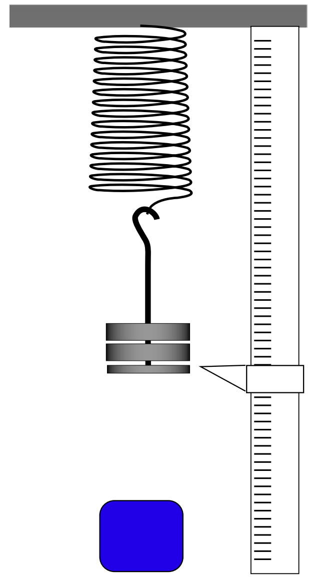

Figure 1 The apparatus used in this experiment consists of a spring, a hanger with masses, a meter stick with a sliding pointer, and an ultrasonic motion detector.

1. Make sure the spring is hanging with its small end at the top. It should remain this way for the duration of the lab. Hang some mass between 200 and 350 g on the mass hanger of the spring. Record the amount of mass you chose.

2. Use a stop-watch to measure the period of the oscillations for two very different amplitudes but the same mass. The large amplitude should not be more than about 12 cm, and the small amplitude can be as small as 1 or 2 cm. To increase the precision of your measurements you will want to measure the total time needed for many oscillations and then divide that number by the number of oscillations. (This is in constrast to your pendulum measurements where we insisted on your measuring the period of individual swings).

3. While you are timing the period with the stop-watch, start recording the position of the hanging mass as a function of time using the motion sensor.

4. Check the reproducibility of your measurements by making several trials for each of the two amplitudes. The uncertainty values that you assign to your measurements here will be very important.

Analysis I

1. For both the large and small values of amplitude, calculate the period of oscillator from your stop-watch data, along with the uncertainty in your measurement.

   (As you know, the average of all your values is the best estimate of the period. What statistic estimates the uncertainty in the average?)

2. Also use the position data you acquired using the motion sensor to measure the period. You can get the time and position of a given point with the cross-hairs button on the toolbar. You can then drag the cross-hairs to find other points. Use the table to calculate the period from the total time and the number of oscillations.

3. Compare the two measurement methods. For a given amplitude are the two methods in agreement?

4. Compare the periods for the different amplitudes along with their uncertainties. Do the periods differ by more than can be accounted for by experimental error?

Procedure II: Static Determination of Spring Constant

An ideal spring has a force that is directly proportional to the displacement from its relaxed position x_0. The constant of proportionality is the "spring constant" k. This can be expressed by

$$F_{\text{spring}} = -k(x - x_0) \tag{1}$$

Here you will determine the spring constant by measuring the force the spring exerts as a function of how much it is stretched.

Hang various masses on the spring and measure the corresponding positions of the bottom of the weight pan with a vertically held meter stick and sliding pointer. Be sure that the small end of the spring is at the top. Take enough data to make a good plot of applied force, F, versus position. **Do not hang more than 350 grams on the spring.**

Analysis II

1. For each weight you put on, the hanger came to a new equilibrium position, which you measured. What was the net force on the mass at that new equilibrium position? Knowing the net force, find the force the spring exerted given the weight you hung.

2. Plot the spring's force against the position of the bottom of the weight pan. (This means the independent variable is position.)

   You can make your plot on the second page of the experiment file, or by any other method. Whatever method you choose, you will have to report the uncertainty in your fitting parameters.

3. What function do you expect for your data? Fit your curve to the function you expect.

   You may find that the extreme points do not match your expectation since the spring is nearly ideal only over a limited range. In that case, limit your fit to include only points within that range.

Ideally you would use a weighted curve fit, with the weights coming from the uncertainty in the average of several measurements. As a shortcut we will assume the uncertainty of each point is about the same and use the scatter in the data about the fit line to estimate the uncertainty. This is how the uncertainty is calculated in Excel's LINEST function, and in the Capstone linear fit.

4. You should see a correspondence between your fit parameters and the constants k and x_0 in Equation 1. Given the numerical values of your fit, determine the spring constant—that is, find the amount of force per unit displacement. Include proper units and uncertainty.

5. From your fit, determine the relaxed position as well—that is, predict the position where the force of the spring would to zero. Include units and uncertainty. As a reminder, if your x_0 is a product or quotient of two parameters, a and b, its uncertainty is

$$\delta x_0 = x_0 \sqrt{\left(\frac{\delta a}{a}\right)^2 + \left(\frac{\delta b}{b}\right)^2}$$

Procedure III: Dynamical Determination of Spring Constant

The period of oscillation depends on the spring constant and the mass, so by measuring the period for several different masses, we can determine the spring constant.

1. Before you start *predict* how the period will change with mass. Does increasing the mass increase or decrease the period?

   (a) How does the *net* force change by using a larger mass? Compare the net force at equilibrium for a small mass to the net force at the new equilibrium for the larger mass. Compare the net force if the small mass is shifted a centimeter from equilibrium to the net force if the large mass is shifted a centimeter from the new equilibrium.

   (b) Knowing the change in net force, how does the acceleration change as mass increases? In that case how does the period change?

2. In Procedure I, you found the period of oscillation for a certain mass. You can include the period for small amplitude oscillation in this data set.

3. For that same (small) amplitude, measure the period T for at least four additional hanging masses. Use whatever timing method you prefer. **Do not hang more than 350 grams on the spring.**

4. Measure the mass of the spring, m_s.

Analysis III

1. Make a plot of the period T against the hanging mass M. Be sure to include the mass of the hanger.

2. Check your prediction. How does increasing the mass change the period?

3. Can you fit a line to this data set? If you try you should find that the data points are not randomly scattered about the line, but curve around it. This is because the expected functional form is not linear. To make our fitting easy we will try to "linearize" our dataset:

   It can be shown that the period of a harmonic oscillator is given by

   $$T = 2\pi\sqrt{\frac{m_{\text{total}}}{k}}\ .$$

   However, some of the total mass that is oscillating comes from the spring. We cannot add all of the mass of the spring because not all of the spring is moving the same amount. The very top of the spring does not move at all, while the very bottom of the spring moves just as much as the hanging mass. The spring's contribution will be some fraction of m_s, the entire mass of the spring.

   The total mass then is the hanging mass M plus the contribution from the spring m:

   $$T = 2\pi\sqrt{\frac{M+m}{k}}\ .$$

   Clearly T is not a linear function of M. But if we square both sides we do get a linear function of M:

   $$T^2 = 4\pi^2\,\frac{M+m}{k} \qquad \text{or}$$
   $$T^2 = \frac{4\pi^2}{k}\,M + \frac{4\pi^2 m}{k}\ . \qquad (2)$$

   According to this equation, the quantity T^2 is a linear function of M; it has the form $y = ax + b$. What group of constants in equation (2) correspond to the slope a? What group corresponds to the intercept b?

4. Now change the y axis of your plot to show T^2 as a function M. This is a linear function, so fit a line to the data. Set the slope a (including units!) equal to the slope of the equation 2 and do likewise for the y-intercept b. Then solve for the values of k and m.

5. Once again, you may let the plotting software estimate the uncertainty in slope from the scatter in the data about the fit line. As a check, make a mental estimate of the uncertainty you expect if you were to measure each point multiple times. Most of your data points should lie within that distance from the fit line. From the uncertainty in the slope calculate the uncertainty in your value of k. If you don't remember how you might want to refer to your notes on error propagation.

6. Compare this value of k with that obtained in Procedure II, the force versus position curve. In order to do this, of course, you will need an uncertainty value for each of your k values. Are they in agreement?

7. From your values for m_s and m calculate the *fraction* of the spring's total mass that must be added to the hanging mass in this experiment in order to obtain the correct value of the period from equation (2.

Questions

1. Explain in your own words (don't use equations), using the laws of physics, why the period of an oscillator should increase as the oscillating mass increases.

2. Doesn't it seem counterintuitive that the period *does not* depend on the amplitude of the oscillations? After all, the mass has farther to travel. Can you give a convincing explanation of this phenomenon?

Report

In your report say whether you can discern a difference in the period as a function of amplitude, and display the evidence for your claim. Show your plot and analysis for the static determination of the spring constant. Then do the same for your dynamic determination; you need not show your plot of T versus M, but you should show T^2 vs M and how you determined the constants k and m. Compare your two spring constants (using the uncertainty!). As part of your discussion, raise the questions posed and write out clear, readable answers.

Experiment 6

Standing Waves

Goals and General Approach

Standing waves occur when waves of equal velocity (v) and wavelength (λ) pass over one another while moving in opposite directions. This situation commonly occurs when a wave moving in one direction is reflected back on itself by a barrier. There is not time here to give a general treatment of this phenomenon, so you should consult your text on these matters if you need to.

In this experiment you will set up standing waves on a string and on a column of air, observe the relationship between their frequencies, and check the values of the observed wave velocities against theory and handbook values.

Strings

Strings are most usually held firmly at each end, as for example in stringed musical instruments, so that the ends of the string are nodes of the standing wave pattern (where no motion occurs). The points midway between the nodes, where maximum motion occurs, are called antinodes. Typical standing wave patterns for a string fixed at both ends are shown Fig. 1.

From the diagrams it should be clear that each complete "loop" is $\lambda/2$ long, and some integer number of such loops fit exactly on the string with length L. From this observation we can write the condition for a standing wave as:

$$n(\lambda/2) = L \ , \tag{1}$$

where n is any positive integer. This means that in principle there is an infinite number of possible standing waves, but in practice only a finite number are achievable. You will be able to observe about ten.

Now for any wave, the wavelength, λ, is related to the frequency, f, and velocity, v, in the following way

$$f\lambda = v \ .$$

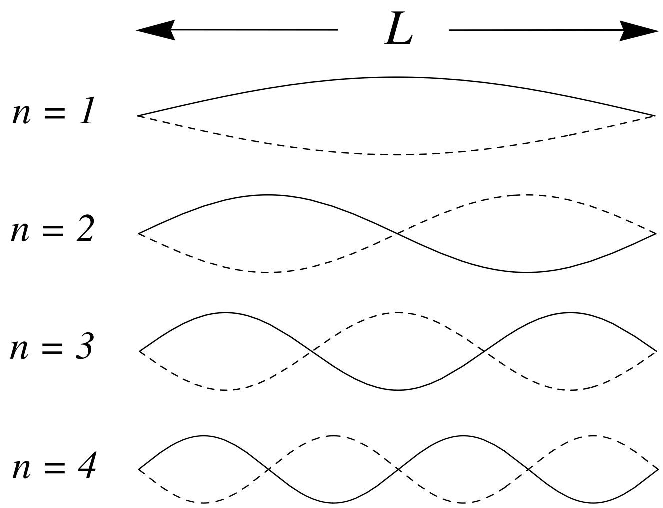

Figure 1 The first four modes of a standing wave that has nodes at the ends.

Solving this equation for λ and substituting the resulting expression into equation (1) we obtain

$$n(v/2f) = L \ , \quad \text{or} \quad f = n(v/2L) \ . \tag{2}$$

Equation (2) tells us that the frequencies of the various standing waves on a string of length L with wave velocity v, are integer multiples of a fundamental frequency whose value is $v/2L$. This fundamental frequency is also the lowest standing wave frequency for the string. Another common name for these various standing waves is "normal modes."

Each such normal mode (standing wave) motion is a natural motion for the string. If we try to drive the string into motion, it will respond very strongly if our driving force varies with one of these frequencies. We say that each of the normal mode frequencies is a "resonant" frequency for the string.

Notice that by measuring the resonant frequencies of the string and its length L, equation (2) allows us to calculate the velocity of waves on that string. It must have the same value no matter which normal mode is excited. Theory predicts that the velocity of a wave on a string depends on its mass density and the tension applied to the string in the following way

$$v = \sqrt{T/\mu} \ , \tag{3}$$

where T is the tension in Newtons and μ is the linear density of the string in kg/m.

Air Columns

An air column can be formed by a tube of any cross section, inside of which the air molecules vibrate in a direction parallel to the long axis of the tube when a sound wave passes through it. Thus we have longitudinal waves in the case of a column of air, in contrast to the string which supports transverse waves (string motion is perpendicular to the direction of wave motion).

When a sound wave traveling through a tube reaches the end of the tube, most of the wave is reflected back down the tube if the end of the tube is closed off. Surprisingly enough, even if the end of the tube is open, some of the wave is reflected back down the pipe. So whether the ends of the tube are open or closed, a wave is reflected back and forth between the ends, generating a standing sound wave (or normal mode). In practical applications at least one end of the tube must remain open to let the sound out, and so the most common situations are both ends open, or just one end open. Brass and reed instruments are examples of the former, while flutes are an example of the latter. Organ pipes can be of either the open (both ends open) or closed (one end closed) variety.

Open Pipes

Consider first a tube that is open at both ends. It can be shown that the molecules at the ends must vibrate the maximum amount, and therefore the antinodes of the standing wave occur at the ends of the tube. Thus the allowed normal modes for an air column open at both ends can be represented by the following diagrams. Bear in mind that the standing waves in this case are longitudinal, even though the diagrams which represent them are patterned after those for transverse waves. There is no other clearer way to represent a longitudinal standing wave.

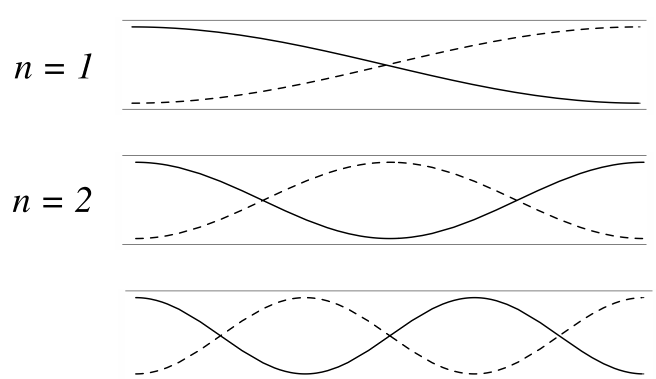

Figure 2 The first three modes of a standing wave in an air column that is open at both ends.

From the diagrams it should be clear that each complete "loop" is $\lambda/2$ long, and some integer number of such loops fit exactly on the air column with length L (in this case we have a half loop at each end with complete loops in between). Therefore the derivation of the formula for the frequencies of the normal modes is identical to that for the string, resulting in

$$f = n(v/2L) \ , \tag{4}$$

where n is any integer.

Equation (4) tells us that the frequencies of the various standing waves in an open air column of length L with wave velocity v, are integer multiples of a fundamental frequency whose value is $v/2L$. This fundamental frequency is also the lowest standing wave frequency for the air column. In this case v is the velocity of sound in air.

Now L is not exactly the same as the length of the tube that enclosed the air column. It can be shown that the antinode that occurs at the open end of a tube occurs at a point slightly beyond the end. (It takes a little while for the wave to realize that the tube has come to an end.) How far beyond the tube the antinode extends depends on the diameter of the tube. For a tube of circular cross section, the antinode occurs approximately $0.6\,r$ beyond the end, where r is the radius of the cross section. So for a tube open at both ends,

$$L = L_{\text{tube}} + 2 \times 0.6r \ . \tag{5}$$

Closed Pipes

If the pipe is closed at one end and open at the other, there must be a node at the closed end and an antinode at the open end. In this case the allowed normal modes of vibration can be represented by the following diagrams.

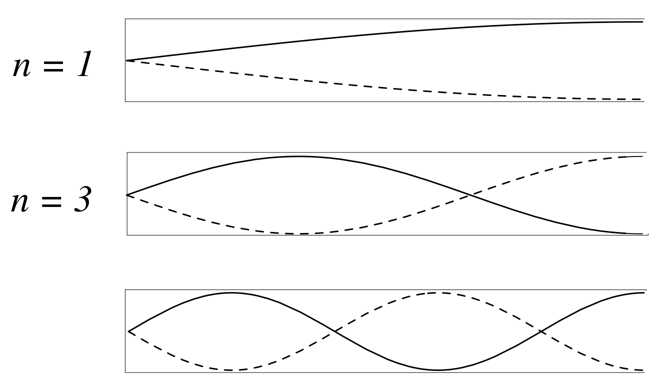

Figure 3 The first three modes of a standing wave in an air column that is closed at one end.

From the diagrams it should be clear that in this case, an odd number of half loops must fit exactly on the air column of length L, each half loop being $\lambda/4$ long. From this observation we can write the condition for a standing wave as

$$n'(\lambda/4) = L \; , \tag{6}$$

where n' is any odd integer.

Again, solving $f\lambda = v$ for λ and substituting the resulting expression for λ into this equation, one obtains the result

$$f = n'(v/4L) \; , \tag{7}$$

where n' is any odd integer.

Equation (7) tells us that the frequencies of the various standing waves in a closed air column of length L with wave velocity v, are odd integer multiples of a fundamental frequency whose value is $v/4L$. This fundamental frequency is also the lowest standing wave frequency for the air column. Here again v is the velocity of sound in air, but L is the length of the tube + $0.6\,r$,

$$L = L_{\text{tube}} + 0.6r \; . \tag{8}$$

Apparatus

A function generator, wave driver, loudspeaker, meter stick, glass tube, cord, pulley, and weights.

Procedure I: Strings

1. Use banana plug leads to connect the function generator to the wave driver. The cord should be anchored to the post clamp at the end of the mechanical wave driver, pushed down into the slot of the drive rod, and passed over the pulley at the other end of the table. Use the weight hanger to hang a 500 gram weight on the end of the cord.

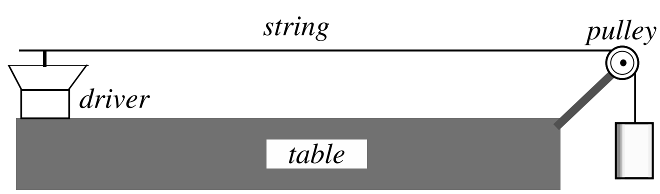

Figure 4 A string fixed at both ends is driven by a small speaker to produce standing waves. A mass at the end provides tension.

2. Turn on the function generator, make sure the wave shape is set to sine wave, and the amplitude (voltage) is set to zero (counterclockwise). Turn down the frequency to 5 Hz. You can use switch below the frequency knob to change the digit the knob adjusts. Turn up

the amplitude control to achieve a reasonable level of vibration of the string (1 to 2 V should suffice). Avoid overdriving the mechanical driver and damaging it. When the sound becomes harsh, raspy, or clicky, you should turn the amplitude down.

3. Adjust the frequency control knob until you achieve a clear standing wave pattern. You are looking for a "resonance," which means that the amplitude of the vibrations will reach a maximum when the frequency matches the resonance frequency of the mode. Carefully tune the frequency until the maximum amplitude of vibration is achieved. This will be the frequency of this mode.
4. Adjust the amplitude to the smallest possible value necessary to observe the resonance.
5. Before looking for another normal mode (standing wave pattern), detune the function generator and re-tune it for a maximum again. You will want to do this more than once. In this way you will be able to estimate the uncertainty in your measured frequency values.
6. Adjust the frequency controls to observe yet other normal modes and their frequencies. Always adjust the amplitude to the smallest possible value necessary to observe the resonance.
7. Record the frequencies you observed along with the integer number for each mode. This integer will be one less than the total number of nodes (including the nodes at the ends); it is also equal to the number of loops on the string. Also include a simple diagram of the string for each mode.
8. Find at least six modes. It is possible to find ten or more.
9. Carefully measure the length of the string between the two end nodes.
10. Measure the total mass and total length of your string (along with uncertainties). Make sure the total length is that when it is stretched.

Analysis I

1. Plot the frequency of each mode against the mode integer. Include error bars on your points.
2. According to equation (2) the slope of this line is equal to $v/2L$. Use Eq. (3) to calculate a value for v (and its uncertainty). Use your measured slope $v/2L$ along with its uncertainty, to calculate a value for the speed v and its uncertainty.

Questions I

1. Does your graph confirm the dependence of frequency on mode number predicted by equation (2)? Explain.
2. Are your two values for the velocity of the wave in agreement within experimental uncertainties?

Procedure II: Air Columns

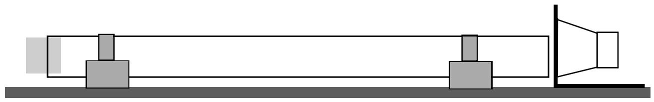

Figure 5 The modes of a tube are excited with a small speaker. The tube can be corked or uncorked, depending on whether a closed or open tube is desired.

1. Connect the output of the function generator to the loudspeaker.
2. Hold the loud speaker close to the end of your glass tube and turn up the amplitude control until you hear a low level tone. Now adjust the frequency, and with your ear close to the end, listen for a noticeable swelling of the volume at the resonant frequencies. If the far end of the tube is corked off, the first resonance should occur somewhere between 50 and 100 Hz. If both ends of the tube are open, the first resonance should occur somewhere between 100 and 200 Hz. The resonance swelling will be more noticeable if you keep the sound level of the loudspeaker quite low.

   If you are unsure that you have a resonance, you can check for it by moving the speaker back and forth in front of the tube in a direction perpendicular to the axis of the tube. The amplitude of sound should noticeably decrease as the loud speaker moves away, and increase again as it is brought back to its position in front of the mouth of the tube.
3. Find as many normal mode frequencies as you can, both with one end closed off with the cork, and with both ends open. Don't forget to measure the uncertainties in those frequencies as well. When you locate each frequency by ear using a low level tone, then turn the amplitude of the driving sound up high and observe the action of the cork dust lying along the bottom of the tube. It will make the standing wave behavior of the gas molecules visible by bouncing about where the gas is vibrating at large amplitude, and lying still where the gas is relatively quiescent (nodes).
4. Record the frequencies, the appropriate integer, and the amplitude diagram for each normal mode. The integer and diagram are to be obtained from the cork-dust pattern. Remember, only odd integer modes exist for this case
5. Try to find some even integer modes, and convince yourself that they don't exist.
6. Measure the length of the glass tube (from end to end for the open ends case, but from end to cork barrier for the closed end case). Also measure the radius of the cross section of the tube.
7. Finally, measure the temperature in the lab. Temperature affects the speed of sound and will have an effect on your results. There is a thermometer on the barometer, located near the sink.

Analysis II

1. Plot frequency versus mode integer for each of the two cases (open and closed). These two curves should be plotted on the same graph.
2. From the slopes of the plotted lines and the measured value of L (don't forget the end correction) calculate the velocity of sound in air.

Questions II

1. Do your graphs confirm the dependence of frequency on mode number predicted by equations (4) and (7)? Explain.
2. Compare your values for the velocity of sound in air with each other and with the professionally measured value, which is 344 m/s at 20°C. The velocity of sound rises 0.6 m/s for each degree centigrade temperature rise, so you can correct this value for the temperature of the laboratory.

Report

Give a clear presentation of the data and the analysis called for in the Procedure and Analysis sections. Write out clear, readable answers to the four questions.

Appendix A

Notes on Experimental Errors

Experimental Error and Uncertainty

We generally speak of two broad classes of errors—random and systematic. The term random error refers to the random and unpredictable variations that occur when a measurement is repeated several times with non-identical results, although the results do cluster around the true value. The stopwatch or photogate time measurements that you have performed are very good examples of this.

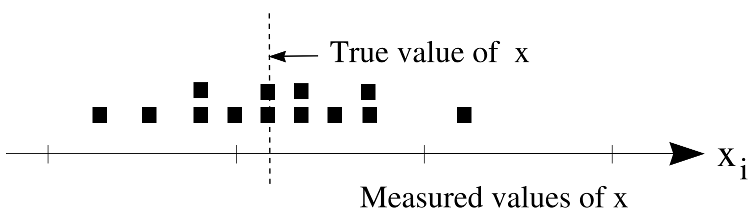

Figure 1 A set of measurements that has a fair amount of random error, but little, if any, systematic error.

Systematic errors are those errors in a measurement that are regular and consistent in the sense that the measurements are consistently too large or too small, and usually by about the same amount for each measurement. The manual measurement of a time period would be an example of systematic error if the person is consistently late in stopping the clock.

Other examples of sources of systematic error would be a stop watch that ran consistently fast or slow, and a meter stick that was mistakenly manufactured too long or too short.

Any actual measurement will ordinarily be liable to both random and systematic errors. The random errors reveal their presence much more readily, and can usually be observed by repeating the measurement several times and observing the differences in the values obtained. The measured

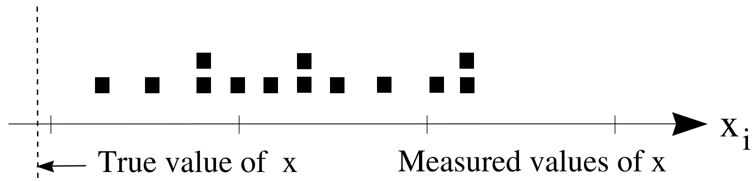

Figure 2 A set of measurements that has a fair amount of random error, as well as significant systematic error.

values will be observed to scatter themselves in some random way about an average value.

Systematic errors are usually much more difficult to detect and measure. We are often unaware of their existence. If one is measuring something whose value is already thought to be known, and consistently obtains a value that is significantly different, one is certainly led to suspect the presence of one or more systematic errors. (The alternative is that the experiments that led to the accepted value contained the systematic error.) However, if one is measuring something that has never been measured before, any systematic error present will not be obvious, and great care must be taken to investigate and eliminate every source of systematic error that one can think of. Experience and ingenuity are the valuable assets in such an investigation. Obviously sometimes things are overlooked and values published that later turn out to be wrong.

The remainder of these notes will confine themselves to a discussion of random errors. There are several reasons for doing this.

- A discussion of systematic error depends critically on the experiment being performed, and cannot easily be generalized.
- Random errors are present and significant in most experiments.
- Random errors are susceptible to a general mathematical treatment by means of which their effects can be reduced.

Before going on, it would be well to comment briefly on the use of the word "error." Random error might better be called random uncertainty. It is the amount by which our measured values uncontrollably fluctuate, and thus provides a measure of our lack of certainty about the value of the quantity we are attempting to measure. The word error connotes mistake, and random error should not be thought of in those terms. The word error is more appropriate in the term, systematic error, where something more like a mistake is actually being made. Nevertheless, we will continue to use the word "error" to denote random uncertainties, because it is so firmly entrenched in our customary usage.

We will frequently use the term experimental error. *Experimental error is simply total uncertainty in any measured quantity, or any quantity calculated from measured quantities. It is not the amount by which your value differs from the generally accepted value.* Experimental error will generally have both systematic and random components, and one of the most important tasks of the experimenter is to measure, estimate, and assess the experimental *"uncertainty"* of all experimental and experimentally derived quantities.

How does one go about doing that? Consider first, directly measured quantities like length of time. One may repeat the experiment several times, taking care to make each measurement independent of the others. The variation in these values will enable us to measure the random "error" in the measurement, using a method to be discussed later in these notes. Many measurements will involve reading a scale and interpolating between the marks on the scale, as well as zeroing the scale at one end or the other. Such interpolations cannot be perfect and one is often content to record as the experimental error simply the uncertainty in one's mind when making the interpolation. For example, consider measuring the length of a solid body. If a meter stick is used to measure this length and every measurement yields the same value, we would say that the "error" in the measurement is the "least count" of the meter stick, namely 1 mm. So, we might quote the error as ± 1 mm. Sometimes the uncertainty will be determined by consulting the specifications for the equipment being used, where the manufacturer will state the limits of accuracy of the equipment. All of the possibilities cannot be covered here, and you will have to use your common sense and ingenuity as particular situations arise.

In any event, *no experimental measurement is complete until a value of the uncertainty, or experimental error, is determined.* Thus every entry in the data pages of your lab book must be accompanied by a value of its uncertainty. This is usually expressed in the following form:

$$x = 35.7 \pm 0.3 \text{ cm}$$

If the experimental uncertainty is the same for a whole series of data entries, it would be sufficient to record the uncertainty for one entry and indicate that it is the same for all of the others.

One never needs more than two significant figures to specify an experimental error, and usually a single significant figure is sufficient. The value of the measured quantity should be carried out as many decimal places as necessary so as to include the decimal places used in the experimental error—no more and no less. The following examples illustrate this point.

Use only one sig. fig. for uncertainty.

Correct	*Incorrect*
3.4423 ± 0.0002 s	3.54073 ± 0.12 m
3.54 ± 0.12 m	3.5 ± 0.12 kg
3.5 ± 0.1 kg	3 ± 0.1 m/s

As a final note, NEVER use the term "human error" in discussing your experimental error. It is meaningless; which is why so many students are tempted to use it. All measurements are made by human beings, by the use of various instruments, so all uncertainty is

Never cite "human error" as a source of uncertainty!

human. **If you do not know the cause of your uncertainty, it is better to simply say so.**

Mean & Standard Deviation

Very often an experimentalist will measure a quantity repeatedly, and then use an average of the measured values as the final result. Why is that done? what is the justification for it? and what is the uncertainty or error associated with that average? This is the classic example of random error and its treatment. If we plot each individual measurement on an axis with appropriate scale and units, it might look something like this.

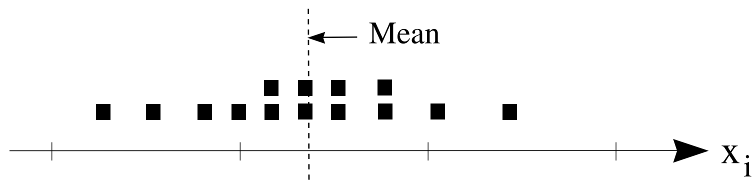

Figure 3 A set of measurements and its mean.

Such a plot provides a clear and graphic description of the uncertainty, or experimental error, associated with this measurement. If the errors in these measurements are truly random, then one would expect that measured values would be larger than the "true value" just as often as they would be smaller, and by similar amounts. This in turn implies that the "true value" lies somewhere in the center of the distribution of measured values. You might choose a best value by simply using your eye to find the center of the distribution. A mathematical procedure that everyone would agree upon would of course be better. There is such a procedure, and it is called calculating the mean, or average. You already know how to do that. In what sense does that find the center of the distribution?

Let us call the value of the center of the distribution $\bar{x}$, and define it so that if we add up all the differences between it and the measured data points, we will get zero. (A data point to the left of $\bar{x}$ will produce a negative difference while a data point to the right of $\bar{x}$ will produce a positive difference.) This can be expressed mathematically in the following way.

Definition of the Mean

The mean, $\bar{x}$, is defined by the relation

$$\sum_{i=1}^{N}(x_i - \bar{x}) = 0 \ , \tag{1}$$

where the x_i are individual measurements.

Since the order in which we do the additions and subtractions does not affect the outcome, this can be rewritten as

$$\sum_{i=1}^{N} x_i - \sum_{i=1}^{N} \bar{x} = 0, \quad \text{but} \quad \sum_{i=1}^{N} \bar{x} = \bar{x} \sum_{i=1}^{N} 1 = N\bar{x}$$

$$\sum_{i=1}^{N} x_i = N\bar{x} \ , \quad \text{which implies} \quad \bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i \ .$$

This last equation is just the ordinary prescription for calculating an average. Thus we have shown that the familiar average value of a set of measurements coincides with a reasonable definition of the center of the distribution of measured values.

Now the question arises, what experimental error should one associate with each of the measured values in our example? A crude way of doing this would be to choose the uncertainty value to give a range of values just as wide as that covered by the actual measurements. But this would probably overestimate the uncertainty, since a single widely divergent measurement would make the spread quite large, and would not be truly representative of most of the values.

Someone might suggest finding, on average, how much each individual value differs from the mean. But we have already seen that the sum of all the deviations from the mean is zero, and therefore such an average would be zero — not a very helpful experimental error value. This method could be saved by finding the average of the absolute values of the deviations from the mean, thus eliminating all of the negative signs from the negative deviations. Although this method could be used, another method has been practically universally accepted; that of the standard deviation.

Definition of the Standard Deviation

The standard deviation of the distribution is calculated in the following way.

a) *Square* each deviation. (This insures that we have only positive quantities to deal with.)

b) Find the *mean* of these squared deviations.

c) Take the *square root* of this mean. (We want a measure of the deviations, not their squares.)

d) The result of these calculations is the standard deviation, σ, also called the root mean square deviation (r.m.s. deviation) for obvious reasons.

Mathematically we can represent this as follows:

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2} \tag{2}$$

We will use the standard deviation as our best estimate of the random experimental error associated with each of the individual measurements. *That is, each of the individual measurements is uncertain by an amount* $\pm\sigma$*, which we express by writing* $x_i \pm \sigma$*.*

What justification do we have for using the standard deviation in this way? It has been found that the random errors associated with many types of measurements follow what is know as a normal probability distribution, which is illustrated in Fig. 1. The height of the curve above any point on the x-axis is proportional to the probability of obtaining an experimental result very close to that value of x. This curve meets our crude expectations that the likelihood of obtaining any measured value will grow less the more the value differs from the average value that lies at the center of the probability curve. The left-right symmetry of the curve insures that measurements are equally likely to be too small or too large.

More precisely, the area under the probability curve between any two x values, x_1 and x_2, gives the probability that a measurement will result in a value lying between x_1 and x_2. The total area under the curve out to $\pm\infty$ must equal 1, since the probability that any measured value will lie between $-\infty$ and $+\infty$ must be 1. But what does this have to do with the standard deviation?

It can be shown that the area under the normal distribution between $\bar{x} - \sigma$ and $\bar{x} + \sigma$ equals 0.67. This means that any individual measurement has a probability of about 67% of lying within one standard deviation from the average of many measurements. Similarly it can be shown that any individual measurement (which obeys the normal distribution) has a probability of 95% of lying within two standard deviations from the average. Thus by using the standard deviation to represent the uncertainty in our measured values, we are providing some very specific information about how much this value is likely to depart from the true average value.

Now the *average of all our measured values* is certainly less uncertain than the individual measurements. What error shall we associate with the average? It can easily be shown (see notes on propagation of error) that the error to be associated with the average is approximately given by the standard deviation of the measurements divided by the square root of the number of measurements included in the average calculation. We will call this uncertainty for the average σ_m. We can summarize our conventions as follows:

If we have N independent measurements, x_i, of a quantity x, then the important quantities and their uncertainties are:

- $x_i \pm \sigma$, where x_i is any individual measurement,

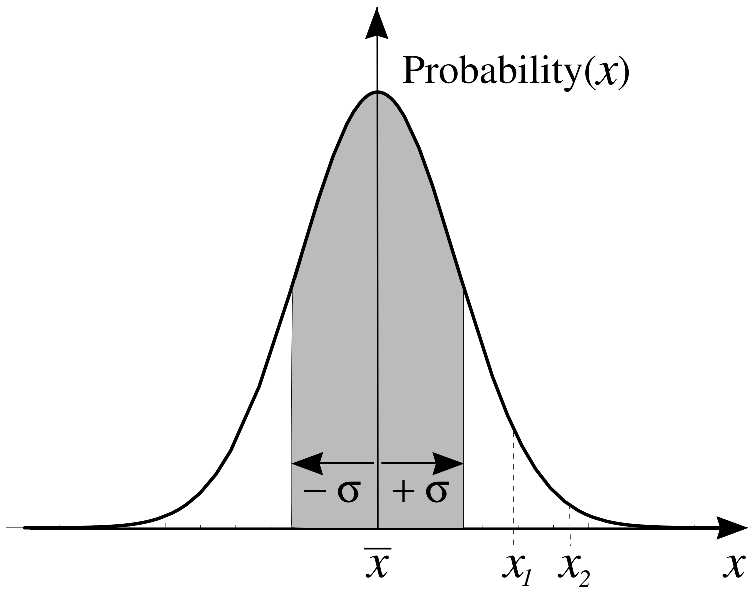

Figure 4 A normal probability distribution curve. The total area under the curve = 1. Two-thirds of the area lies between $\bar{x} - \sigma$ and $\bar{x} + \sigma$, which implies that 2/3 of the measurements should fall within $\pm\sigma$ of the mean. The probability of obtaining a value between x_1 and x_2 equals the area under the curve between x_1 and x_2.

- and $\bar{x} \pm \sigma_m$, where $\sigma_m = \sigma/\sqrt{N}$.

Two words of caution are in order. First, it is a common mistake to think that the standard deviation σ represents the uncertainty in the average. *It does not.* Rather, it represent the uncertainties associated with each of the individual measurements. Second, these statistical methods work well only when N is large, say 10 or more, and the larger N is, the better they work. Occasionally we use them for smaller values of N because we have no other choice. In such cases we must take our conclusions with a grain of salt.

Agreement of Experimental Results

Why do we lay such stress on measuring and treating our experimental uncertainties? This becomes quite clear when we go to compare two experimentally derived numbers. The two numbers might represent two different measurements of the same quantity, taken from the same experiment; or they might represent the values arrived at by two different pairs of students in the laboratory; or they might represent your experimentally determined value and a value measured and published in the literature by a professional physicist. The two numbers will almost invariably be different. What does that mean? They don't agree? Does "agreement" mean identical? You will realize immediately that this cannot be the case. Then what does it mean for values to be in agreement?

For our purposes, the following crude criteria will be sufficient to draw conclusions about the agreement (or lack thereof) of two measured values.

1. **Agreement:** Two values are in agreement if either value lies within the error range of the other.
2. **Disagreement:** Two values disagree if their respective error ranges do not overlap at all.
3. **Inconclusive:** No clear conclusion can be drawn if the ranges overlap, but neither value lies within the range of the other.

(Technically, what one should check is that the difference between two values is consistent with zero. This requires that one use propagation of error to find the error for the difference of two values. If the error is greater than the difference of the values, then the results are said to be in agreement.)

The following examples and diagrams will help to make this idea clearer.

Values in agreement:

$$
\begin{aligned}
x_A &= 5.7 \pm 0.4 \\
x_B &= 6.0 \pm 0.2
\end{aligned}
$$

Here x_B lies within the uncertainty range of x_A, though x_A does not lie within the uncertainty range of x_B.

Values in disagreement:

$$
\begin{aligned}
x_A &= 3.5 \pm 0.5 \\
x_B &= 4.3 \pm 0.2
\end{aligned}
$$

The two uncertainty ranges do not overlap.

Inconclusive:

$$
\begin{aligned}
x_A &= 8.3 \pm 0.3 \\
x_B &= 8.8 \pm 0.3
\end{aligned}
$$

Neither value lies within the uncertainty range of the other, although the two uncertainty ranges do overlap.

So we see that no experimental measurement can be complete until its experimental uncertainty is determined. Otherwise, the measurement is of little or no use to us.

Appendix B

Notes on Propagation of Errors

We have learned something about measuring and estimating the uncertainties in experimentally measured values. We know that this is important because it allows us to compare two different measurements of the same quantity. It is also important because it allows us to calculate the uncertainties in other quantities which are calculated from the directly measured quantities. In what follows we will determine how to calculate the errors in a function $G(x, y)$ of the experimentally measured values x, and y.

Sums of Measurements

For example, suppose that $G(x, y) = x + y$, where x and y are experimentally measured quantities with uncertainties dx and dy respectively. What is the uncertainty, dG, in the value of G that is calculated from the experimentally measured values for x and y? The largest and smallest values for G would be:

$$\begin{aligned} G_{\text{max}} &= (x + dx) + (y + dy) = x + y + (dx + dy) \quad \text{and} \\ G_{\text{min}} &= (x - dx) + (y - dy) = x + y - (dx + dy) \ . \end{aligned}$$

In this case we could write: $dG = (\pm)(dx + dy)$. Graphically the uncertainty in G could be expressed as follows:

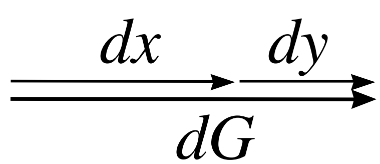

Figure 1

However, in order for the value of G to be this far off, both x and y would have to be in error by their maximum amounts and in the same sense; i.e., dx and dy both positive or both negative. If the errors in x and y had opposite signs something like this would happen:

$$G = (x + dx) + (y - dy) = x + y + (dx - dy) \ .$$

Here the error in G is smaller than in the previous case, and could be zero if $dx = dy$. This case can be represented graphically as follows:

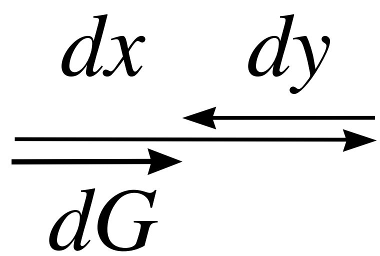

Figure 2

Statistically speaking, it is unlikely that the error in G will be either as large as the first assessment or as small as the second. The truth must lie somewhere in between. Our rule for quantities calculated from sums (and also differences by the way) will be:

$$dG^2 = dx^2 + dy^2 \ .$$

Graphically this can be represented as "error vectors" being added at right angles to each other rather than parallel or anti-parallel: **If $\mathbf{G(x, y) = ax \pm by}$, then the error in G is given by:**

$$\boxed{dG^2 = (a\,dx)^2 + (b\,dy)^2 \ .} \tag{1}$$

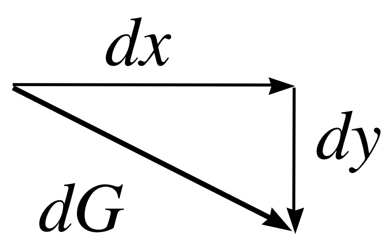

Figure 3

Products and Powers of Measurements

Suppose that $G(x, y) = Kx^p y^q$, where p and q can be any positive or negative numbers. For example, if we were considering the relation

$$v = R\sqrt{\frac{g}{2h}},$$

which could be rewritten as $v = (\sqrt{g/2})R^1 h^{-1/2}$, where R and h are the experimentally measured quantities (corresponding to x and y), then $p = 1$, $q = -1/2$, and $K = \sqrt{g/2}$.

The rule in this case will be quite similar to that for sums of measurements, but in this case it is the *fractional* uncertainties in G, x, and y (dG/G, dx/x, and dy/y) that will be important.
When $\mathbf{G(x, y) = Kx^p y^q}$, then the fractional uncertainty in G is given by

$$\boxed{\left(\frac{dG}{G}\right)^2 = \left(p\,\frac{dx}{x}\right)^2 + \left(q\,\frac{dy}{y}\right)^2 \ .} \tag{2}$$

An Example Computation

As an example of the application of these rules, we consider finding the uncertainty in the average velocity over an interval when we know the uncertainties in the length and the time.

Suppose that the length of the interval is $x = 1.752 \pm 0.001$ m and the time is $t = 0.7759 \pm 0.00005$ s. The average velocity is $\bar{v} = x/t = 2.258$ m/s. The uncertainty is calculated from Eq. (2) above:

$$\begin{aligned} \left(\frac{d\bar{v}}{\bar{v}}\right) &= \sqrt{\left(\frac{dx}{x}\right)^2 + \left(\frac{dt}{t}\right)^2} \\ \left(\frac{d\bar{v}}{2.258\ m/s}\right) &= \sqrt{\left(\frac{0.001\ m}{1.752\ m}\right)^2 + \left(\frac{0.00005\ s}{0.7759\ s}\right)^2} = 0.00057. \end{aligned}$$

Thus, the uncertainty in the velocity is $dv = 0.00057 \times 2.258\ m/s = 0.0013\ m/s$.

An Application: Uncertainty in the Mean

Both rules, for sums and products, are easily generalized to cases with more than two independent variables, i.e., x, y, z, ... etc.

The rule for dG when G is a sum or difference has a nice application to the uncertainty in an average value of several measurements. Suppose that $G_{\text{ave}} = (g_1 + g_2 + g_3 + \ldots + g_N)/N$. Then by equation (1) the uncertainty in G_{ave} is given by:

$$(dG_{\text{ave}})^2 = \frac{1}{N^2}\left(dg_1^2 + dg_2^2 + dg_3^2 + \ldots + dg_N^2\right) \ . \tag{3}$$

But $(1/N)(dg_1^2 + dg_2^2 + dg_3^2 + \ldots + dg_N^2)$ is the mean square deviation (σ^2) of the g measurements, or the square of the standard deviation of the g

measurements. Using this identification, we can transform equation (3) to: $(dG_{\mathrm{ave}})^2 = \sigma^2/N$, and therefore the uncertainty in the average, dG_{ave}, which we sometimes call σ_m, is

$$dG_{\mathrm{ave}} = \frac{\sigma}{\sqrt{N}} \ . \tag{4}$$

This confirms our feeling that the uncertainty in an average value should be less than the uncertainties in the individual measurements (otherwise, why take averages on repeated measurement?), and confirms the formula for the uncertainty in a mean which was given without proof in the Notes on Experimental Errors.

We summarize these notes with a table showing the propagation of errors for a few simple cases.

Table 1 Propagation of Error for a few simple algebraic expressions.

Expression	Error in Expression
$ax \pm by$	$\sqrt{(a\,dx)^2 + (b\,dy)^2}$
xy	$\sqrt{(x\,dy)^2 + (y\,dx)^2}$
$\frac{x}{y}$	$\frac{x}{y}\sqrt{\left(\frac{dx}{x}\right)^2 + \left(\frac{dy}{y}\right)^2}$
$\sqrt{ax}$	$\frac{1}{2}\sqrt{ax}\,\left\lvert\frac{dx}{x}\right\rvert$
x^2	$2\,\lvert x\,dx\rvert$

Appendix C

Notes on Graphing

Graphical Representation and Analysis

Graphs are used scientifically for two purposes. The first is simply to represent data in a form that can be understood visually. That is, a graph shows schematically how one variable depends on another. This is the usage that is most familiar to most people. Perhaps the most common graph of this kind is that of the Dow Jones Industrial Average versus time. One might look at such a graph to see whether stock prices were falling throughout the day or to see how much volatility there was in the Dow. Beyond these general trends, most people do not look further, except perhaps to see if the low or high for the day reached or exceeded some particular level. The size of such a graph does not particularly matter for such purposes so newspaper editors generally print the graph small to save space.

The second and less common use of a graph is to analyze data. This is a more sophisticated use of graphing and more stringent criteria apply to making and using such graphs. In a graph of the Dow Jones Industrial Average versus time, it is not so important to be able to read the value of the Dow to great precision. This is not the case with a scientific graph with which one will analyze data. Just as most students are reluctant to leave out any of the ten digits in a calculated result (even though they must according to the rules of significant figures!), a good scientist ought to be reluctant to lose any precision in analyzing data graphically.

Linearizing the Data

In analyzing experimental data, we most often will be interested in finding a representation of the data that results in a **linear** relationship between the quantities. This is because our visual system is acutely sensitive to lines and fairly bad at distinguishing other curves. To the eye, the crest or trough of a sine curve looks identical to a parabola until they are laid on top of one another. In addition, it is much more difficult to find the curvature of a parabola, for instance, than it is to find the slope of a line!

The two values that are of interest in a linear graph are the slope and the y-intercept. For instance, the period of a pendulum, T, and its length L are related theoretically by

$$T = 2\pi\sqrt{L/g}$$

when the amplitude of the motion is very small. If we were to test this relationship experimentally by graphing data, we would square the equation to get

$$T^2 = (4\pi^2/g)L \; . \tag{1}$$

Eq. (1) indicates that if we were to graph T^2 versus L, we should find a line with slope $4\pi^2/g$. In fact, this would be a reasonable way of measuring the acceleration of gravity, g. (Recall that when we say a graph of y versus x, we mean a graph with y along the vertical axis and x along the horizontal axis.) According to Eq. (1) we should also find an intercept of zero: when L is zero, T^2 should also be zero. However, we might find that there is a non-zero intercept, which would indicate that perhaps we had not measured the length of the pendulum correctly: we may have a non-zero offset.

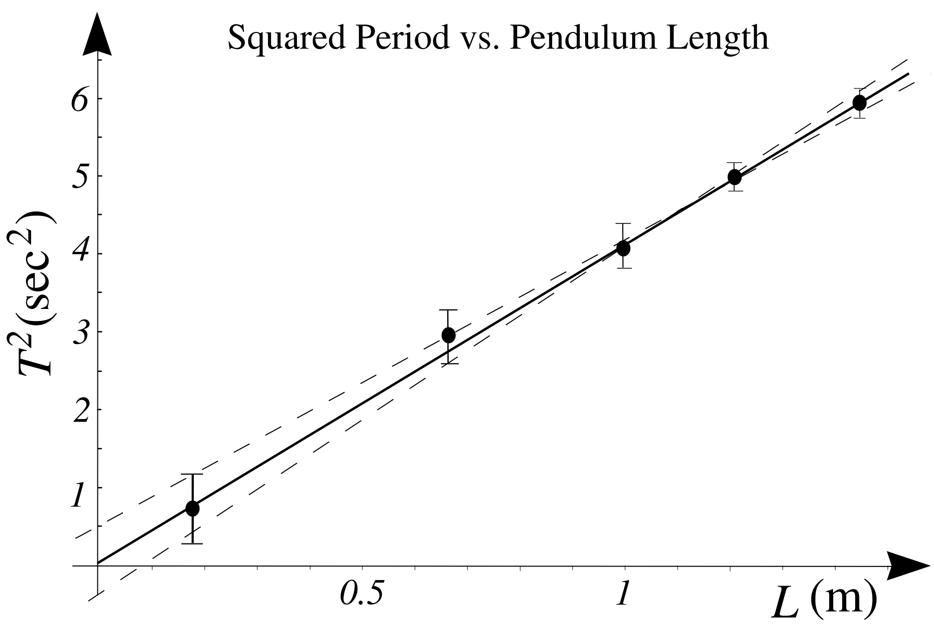

Figure 1 A well-drawn graph of data for the squared period of a pendulum versus length of the pendulum. The solid line is the best-fit line. The dashed lines are lines of good fit. Note that the line and the data points fill the whole range of the graph.

Making Good Graphs

Make Graphs Large

The of scientific graphing is to make the graph easy to read accurately and precisely. This means that **graphs must be made as large as**

practicable. In practice you should make the graph as large or nearly as large as your sheet of graph paper will allow, i.e., the the whole or nearly the whole sheet. Included in this is the requirement to make the range of the axes no larger than is necessary; you want your line to fill up the whole range. Just as it is more difficult to measure very small objects, it is difficult to read very small graphs accurately. In addition, graphs made very small lack the precision of larger graphs simply due to the size of the mark made by even a sharp pencil. In addition to being as large as practicable, a graph must be labeled in such a way as to be easy to read quickly and accurately. This means that **the axes must be labeled in a simple and regular fashion. It is best to use 1, 2, or 5 divisions to represent a decimal value.** That way the graph is easiest to read and to make; one need not guess where a value of 1.50 lies between marks at 0.85 and 1.70!

Number Axes Simply

In addition to being of good size and having well labeled axes, to be complete graphs must have a descriptive title and the quantities being graphed must be labeled on the axes with their proper units.

Title Graphs, Label Axes, Use Units

Figure 1 shows a good graph with all the features necessary to a good graph. These good features include

Attributes of a Well Drawn Graph

- Large size
- Range of the axes just covers that of data
- Well numbered axes
- Short and Clear Title
- Axes labeled with the appropriate quantity
- Units on the axes
- Data points with uncertainties (error bars)
- A best-fit line
- One or two alternate good-fit lines

Measuring the Slope of a Line

In most of the graphs that we will be using in the introductory laboratory the most important piece of information will be the slope of the "best-fit line." There are many ways of obtaining this slope. One way is to use the formulas at the end of these notes into which one may substitute the coordinates (x_i, y_i) and errors σ_i of the data points. If the errors σ_i are all of the same size, one might use Microsoft Excel to calculate the slope and intercept. (Excel is not recommended because it is a difficult tool to master and is not appropriate if there are large error bars on some data points and not on others.) Last, one may use a hand-drawn graph that is carefully made. On the hand-drawn graph there should be a best-fit line to the data. It is this line whose slope we wish to calculate. Note that the best-fit line does not necessarily go through any of the data points so it is not correct to use the data points to calculate its slope.

Why use widely separated points?

Choose Points Far Apart to Minimize Error in Measuring the Slope

The slope of the line is given by the familiar rule "rise over run." Now, mathematically, it does not matter which two points you choose on a line to calculate slope; any two points will give the same answer. However, **we cannot read the coordinates of the points to arbitrary precision, hence there is inherent error in calculating the slope from a real graph.** The rise is difference in the heights of the two points,

$$\Delta y = y_2 - y_1 \ ,$$

so the error in the rise is found from the uncertainty in the y-coordinates themselves,

$$\sigma_{\rm rise} = \sigma_{\Delta y} = \sqrt{\sigma_{y_2}^2 + \sigma_{y_1}^2} \ ,$$

and is independent of Δy. **The uncertainty in the rise depends only on the uncertainty in reading the points on the line, which is minimized by making the graph as large as practicable.** Similar results hold for the run, Δx. The slope, being the ratio of rise to run, has an error

$$\frac{\sigma_{\text{best fit slope}}}{\text{best fit slope}} = \sqrt{\left(\frac{\sigma_{\rm rise}}{\rm rise}\right)^2 + \left(\frac{\sigma_{\rm run}}{\rm run}\right)^2} \ . \qquad (2)$$

It is useful to pause here to think about Eq. (2). To minimize the error in the slope, we should minimize the errors in the rise and run, but we should also **maximize the rise and the run themselves**. The rise and run are maximized when the points on the line taken to compute the slope are **as far apart as possible** while still being on the graph. In practice, one will not choose the points to be way at the ends of the line, but rather one will choose points quite far apart but lying near the crossing of two lines on the graph paper so that their coordinates may be read accurately. A graphical illustration of the Eq. (2) is shown in Fig. 2.

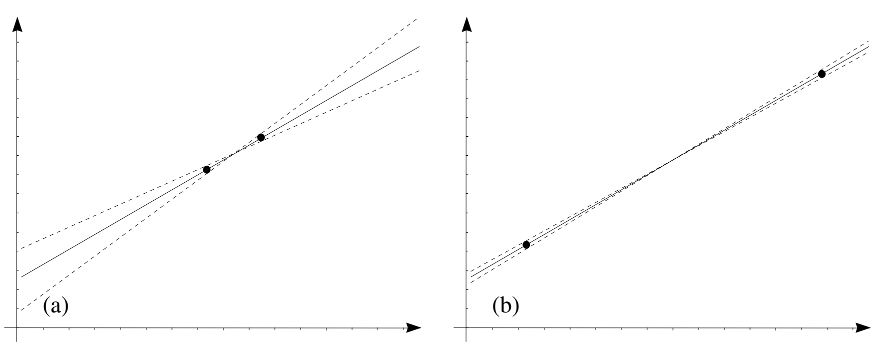

Figure 2 The separation of the points chosen on a line for the computation of the slope has a strong effect on the error in the computed slope. In figure (a) the points, whose widths are exaggerated for clarity, are close together, resulting in a large error in the slope. In figure (b) the points are far apart, resulting in a more precise determination of the slope.

Finding the range of slopes for lines of good fit

Error in Reporting the Slope of the Best Fit Line

It is important to point out that the uncertainty given in Eq. (2) is the uncertainty in measuring from your graph the actual slope of the best-fit line. This is logically different from quoting a range of values for lines of good fit! In other words, even if all the points were to lie perfectly along a line, there is a limitation in finding the slope of that line due to the fact that graphing is a physical process and reading the graph has an uncertainty associated with it. In practice, however, the points generally will not lie along a line and there may be many lines that have a reasonable fit to the data. We'd like to find the range of the slopes for these lines that fit the data reasonably well. The procedure in this case is to draw two more lines in addition to the best-fit line. One line will have the largest possible slope and still fit the data reasonably well and the other will have the smallest possible slope and still fit the data reasonably well. Example lines are drawn Figure 1. To find the quoted error in the slope of the best-fit line, take half the difference in the slopes of the two outlying dashed lines. For example, if the slope of the best-fit line in the figure is 4.00 sec^2/m, the slope of the steeper dashed line is 4.16 sec^2/m, and the slope of the shallower dashed line is 3.90 sec^2/m, you would quote the slope as 4.00 ± 0.13 sec^2/m, because $(4.16 - 3.90)/2 = 0.13$. In a pinch, you could draw just one other good-fit line and then the error in the slope is just the difference between the best-fit line slope and the slope of the good-fit line.

Least Squares Method

If graphing seems to imprecise to you or you just like algebra better, there is a method that will give you the slope and the intercept of the best-fit line directly without graphing. Of course, you will also forgo the benefits of seeing the relationship between the data points and the visual estimate of the goodness of fit, but you avoid the artistic perils of graphing.

The idea is that the best-fit line ought to minimize the sum of the squared vertical deviations of the data from the line. That is, you want a line $y = mx + b$ that has the least value for

$$\Delta(m, b) \equiv \sum_{i=1}^{N} (y_i - (mx_i + b))^2 \ .$$

If the data points have vertical error bars that are $\pm\sigma_i$ for data point (x_i, y_i), then the correct quantity to minimize is chi-squared, or

$$\chi^2 \equiv \sum_{i=1}^{N} \left(\frac{y_i - (mx_i + b)}{\sigma_i} \right)^2 \ . \qquad (3)$$

The algebra is not difficult to do and the result of minimizing with respect to both m and b yields values for the slope and the y-intercept is

$$m \quad = \quad \frac{S_1 S_{xy} - S_x S_y}{S_1 S_{x^2} - S_x^2} \ , \qquad (4)$$

$$b = \frac{S_y S_{x^2} - S_x S_{xy}}{S_1 S_{x^2} - S_x^2} \ , \tag{5}$$

where S_1, S_x, S_y, S_{xy}, S_{x^2}, and S_{y^2} are defined by

$$\begin{aligned} S_1 &= \sum_{i=1}^{N} \sigma_i^{-2} \ , & S_x &= \sum_{i=1}^{N} \sigma_i^{-2} x_i \ , \\ S_y &= \sum_{i=1}^{N} \sigma_i^{-2} y_i \ , & S_{xy} &= \sum_{i=1}^{N} \sigma_i^{-2} x_i y_i \ , \\ S_{x^2} &= \sum_{i=1}^{N} \sigma_i^{-2} x_i^2 \ , & S_{y^2} &= \sum_{i=1}^{N} \sigma_i^{-2} y_i^2 \ . \end{aligned} \tag{6}$$

The errors in the slope and intercept are given by

$$dm = \frac{\sqrt{S_1(S_1 S_{x^2} + S_x^2)}}{S_1 S_{x^2} - S_x^2} \ , \tag{7}$$

$$db = \frac{\sqrt{S_{x^2}(S_1 S_{x^2} + S_x^2)}}{S_1 S_{x^2} - S_x^2} \ . \tag{8}$$

The goodness of fit can be found by evaluating the correlation coefficient

$$R = \frac{S_1 S_{xy} - S_x S_y}{\sqrt{(S_1 S_{x^2} - S_x^2)(S_1 S_{y^2} - S_y^2)}} \ , \tag{9}$$

in the notation of Eq. (6). When the $R = 0$, there is essentially no linear relationship present, while a value of $R = 1$ occurs for perfect correlation, or all points lying on a line.