A. The aim of this problem is to show that a solution of the Schrödinger equation with a vector potential \( A \) is related to the solution of the Schrödinger equation with a “gauge equivalent” vector potential \( A + \nabla \chi \).

a. If \( g(x, y, z) \) is a differentiable and invertible function (i.e., one that doesn’t have zeroes), and \( f(x, y, z) \) is any differentiable function, show the following relation:

\[
g^{-1}(-i\hbar \nabla - eA)gf = (-i\hbar \nabla - \varrho - eA)f.
\]

where \( \varrho \) is the vector function \( \varrho = i\hbar g^{-1}\nabla g \).

b. Let \( g = e^{-ix/\hbar} \), with \( \chi = \chi(x, y, z) \) a real-valued function. Show that

\[
i\hbar g^{-1}\nabla g = e\nabla \chi.
\]

(Note that \( e \) is used both for the base of the natural logarithms, \( 2.7182818 \ldots \), as well as the charge on the electron, \( -1.6 \times 10^{-19} \text{C} \). You need to use context to distinguish them!)

c. For any \( n > 1 \), show that

\[
g^{-1}(-i\hbar \frac{\partial}{\partial x^i} - eA_i)^n gf = (-i\hbar \frac{\partial}{\partial x^i} - i\hbar g^{-1}\frac{\partial g}{\partial x^i} - eA_i)^n f.
\]

Here \( x^i \) stands for \( x^1 = x, x^2 = y, x^3 = z \), and \( A_1 = A_x, A_2 = A_y, \text{ etc.} \) Hint: it is useful to write out the product and insert \( 1 = gg^{-1} \) in strategic places.

d. Show that if \( \psi \) is a solution of the Schrödinger equation with energy \( E \),

\[
\frac{1}{2m} (\hat{p} - eA)^2 \psi = E\psi,
\]

then

\[
\tilde{\psi} = g^{-1}\psi
\]

is a solution of the Schrödinger equation

\[
\frac{1}{2m} (\hat{p} - e(A + \nabla \chi))^2 \tilde{\psi} = E\tilde{\psi},
\]

with the vector potential \( A + \nabla \chi \). Hint: It is useful to use the notation \( (\hat{p} - eA)^2 = \sum_{i=1}^{3}(\hat{p}_i - eA_i)^2 \).