1. In this problem we will investigate the wave equation:

\[ \frac{\partial^2}{\partial t^2} \psi(x, t) - c^2 \frac{\partial^2}{\partial x^2} \psi(x, t) = 0. \]  \hspace{1cm} (1)

Here \( c \) denotes the speed of light. Show that a sum of left-moving and right-moving waves, namely

\[ \psi(x, t) = g(x + ct) + f(x - ct) \]  \hspace{1cm} (2)

with \( f \) and \( g \) being arbitrary functions, is a solution of the wave equation, (1). (Hint: you can deal with \( f \) and \( g \) separately, showing each to be a solution of the wave equation. The sum is then also a solution. To show that \( f(x - ct) \) is a solution, let \( u = x - ct \) and then use the chain rule to find \( \partial f/\partial x = (df/du)(\partial u/\partial x) \), etc.)

2. Next, consider the complex exponential

\[ \psi(x, t) = \exp(i(kx - \omega t)), \]  \hspace{1cm} (3)

where \( k \) is the wave number and \( \omega \) is the angular frequency. How are \( k \) and \( \omega \) related to \( c \)?

3. If we add a mass term to the wave equation, we would find a wave equation for a massive particle, known as the Klein-Gordon equation

\[ \frac{\partial^2}{\partial t^2} \psi(x, t) - c^2 \frac{\partial^2}{\partial x^2} \psi(x, t) + \frac{m^2c^4}{\hbar^2} \psi = 0. \]  \hspace{1cm} (4)

a. Is \( \psi(x, t) = f(x - ct) \) a solution to Eq. (4)?

b. Can you find a solution in the form \( \psi(x, t) = f(x - vt) \) for any \( v \)? (Hint: by using the chain rule, reduce Eq. (4) to a single differential equation \( f'' = \beta f \) and try functions of the form \( f(u) = \exp(\alpha u) \) for solutions.)

c. What is the character of the solution for \( v < c \)? (That is, is it exponential or sinusoidal?)

d. What is the character of the solution for \( v > c \)?
4. In class, we looked at waves on a string fixed at both ends. We showed that a solution of the wave equation on a finite string that lies between $x = 0$ and $x = L$ and is fixed at boundary $x = 0$ is

$$y(x, t) = A \sin(kx - \omega t) + A \sin(kx + \omega t). \quad (5)$$

By using trigonometric identities, we reduced this to

$$y(x, t) = 2A \sin(kx) \cos(\omega t). \quad (6)$$

Then the boundary condition that $y(L, t) = 0$ fixed the value of $k$ so that $kL = n\pi$.

If we change the boundary conditions to having one end fixed at $x = 0$, $y(x = 0, t) = 0$, and one end free at $x = L$, which implies that $\frac{\partial y}{\partial x}|_{x=L} = 0$, what are the allowed values of $k$ and from those, what are the allowed wavelengths? Does this correspond to what you learned in Physics 150?