1. Let $\beta$ be the product of the following elements of $S_6$:

$$
\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 6 & 4 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 4 & 6 \end{pmatrix}.
$$

a) What is $|\beta|$?

b) What is $\beta^{-1}$?

c) Is $\beta \in A_6$? Explain, please.
2. a) Give an example of non-abelian cyclic group and its generator, or explain why no such example exists.

b) Assume $G = \langle a \rangle$, where $|a| = 18$. What are all the other generators of $G$?

c) Find $[200]$ in $\mathbb{Z}_{216}$.
3. a) Let $\phi : G_1 \to G_2$ be a group homomorphism. The **kernel of** $\phi$ is the set

$$K_1 = \{ a_1 \in G_1 \mid \phi(a_1) = e_2 \}. $$

(In words, $K_1$ is the stuff in $G_1$ that gets mapped to the identity in $G_2$.) Carefully prove that $K_1$ is a subgroup of $G_1$.

b) Let $\phi : G_1 \to G_2$ be a group homomorphism. Let $H_2$ be a subgroup of $G_2$. Define

$$H_1 = \{ a_1 \in G_1 \mid \phi(a_1) = a_2 \in H_2 \}. $$

(In words, $H_1$ is the stuff in $G_1$ that gets mapped to the subgroup $H_2$ in $G_2$.) Carefully prove that $H_1$ is a subgroup of $G_1$.

c) Can you use (b) to give a one sentence proof of (a)?
4. Briefly justify your answers to any 3 of the following.
   a) Let \( a \in S_n \). Is \( a^2 \in A_n \)?

   b) Are \( \mathbb{Z} \) and \( \mathbb{R}^* \) isomorphic groups? Make sure to explain your answer.

   c) Give an example of finite groups \( G \) and \( H \) such that \( |G| = |H| \) but \( G \) is not isomorphic to \( H \) or explain why no example exists.

   d) Find an element \( \beta \in S_5 \) such that \( \beta(23)(45) = (34)\beta \), or explain why no such element exists.
5. a) Suppose that $\phi : D_4 \to \mathbb{Z}_8$ is a group homomorphism. What is $\phi(r_0)$? Explain how you know this.

b) Suppose, again, that $\phi : D_4 \to \mathbb{Z}_8$ is a group homomorphism. Let $v$ be the vertical flip. If $\phi(v) = k$, what are the possible choices for $k$ in $\mathbb{Z}_8$? Again, explain your answer.

c) Can a homomorphism $\phi : D_4 \to \mathbb{Z}_8$ be an isomorphism? Explain.

d) Can a homomorphism $\phi : D_4 \to \mathbb{Z}_8$ be onto? Explain.
6. Is the mapping \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) by \( \phi(a) = \sqrt{a} \) injective? Surjective? A group homomorphism? An isomorphism? Show all details. (Caution: The operation is multiplication.)

7. Let \( \phi : \mathbb{Z}_6 \to \mathbb{Z}_3 \) be the homomorphism determined by \( \phi(5_6) = 2_3 \). Make a list of the elements of \( \mathbb{Z}_6 \) and determine what each is mapped to under \( \phi \). Is \( \phi \) injective? Surjective? An isomorphism?

8. (Extra Credit) Prove Colin’s Theorem: If \( H < S_n \) and \( |H| \) is odd, then \( H \) contains only even elements. Hint: Think about the homework problem with odd and even elements in subgroups of \( S_n \).
1. Let $\beta$ be the product of the following elements of $S_6$:

$$\beta = \left( \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 5 & 6 & 4 & 3 & 2
\end{array} \right) \left( \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
2 & 3 & 1 & 5 & 4 & 6
\end{array} \right).$$

a) What is $|\beta|$?

$\beta = (1543)(26)$, so $|\beta| = \text{lcm}(4, 2) = 4$.

b) What is $\beta^{-1}$?

$\beta^{-1} = (62)(3451)$.

c) Is $\beta \in A_6$? Explain, please.

$\beta = (13)(14)(15)(26)$ is the product of an even number of transpositions, so $\beta \in A_6$.

2. a) Give an example of non-abelian cyclic group and its generator, or explain why no such example exists.

Impossible. Cyclic groups are always abelian.

b) Assume $G = \langle a \rangle$, where $|a| = 18$. What are all the other generators of $G$?

By Sam’s Theorem, we need elements whose powers are relatively prime to 18; so the generators are: $a^5, a^7, a^{11}, a^{13}, a^{17}$.

c) Find $[200]$ in $\mathbb{Z}_{216}$.

Find the gcd: $216 = 2 \cdot 100 + 16 \Rightarrow 200 = 12 \cdot 16 + 8 \Rightarrow 16 = 2 \cdot 8 + 0$. So $\gcd(216, 200) = 8$.

Therefore $[200] = \frac{216}{\gcd(216, 200)} = 216 = 27$.

3. a) Let $\phi : G_1 \to G_2$ be a group homomorphism. The kernel of $\phi$ is the set $K_1 = \{a_1 \in G_1 \mid \phi(a_1) = e_2\}$. Carefully prove that $K_1$ a subgroup of $G_1$.

Use the two-step method. Closure. Let $a, b \in K_1$. Show that $ab \in K_1$. But $a, b \in K_1$ means that $\phi(a) = \phi(b) = e_2$. But then $\phi(ab) = \phi(a)\phi(b) = e_2e_2 = e_2$. Therefore, $ab \in K_1$. Inverses: Let $a \in K$. Show $a^{-1} \in K_1$. But $a \in K_1$ so $\phi(a) = e_1$. Therefore, $\phi(a^{-1}) = [\phi(a)]^{-1} = e_2^{-1} = e_2$. Yes, it is a subgroup.

b) Let $\phi : G_1 \to G_2$ be a group homomorphism. Let $H_2$ be a subgroup of $G_2$. Define $H_1 = \{a_1 \in G_1 \mid \phi(a_1) = a_2 \in H_2\}$. Carefully prove that $H_1$ a subgroup of $G_1$.

Use the two-step method. Closure. Let $a_1, b_1 \in H_1$. Show that $a_1b_1 \in H_1$. But $a_1, b_1 \in H_1$ means that $\phi(a_1) = a_2 \in H_2$ and $\phi(b_1) = b_2 \in H_2$. But then $\phi(a_1b_1) = \phi(a_1)\phi(b_1) = a_2b_2 \in H_2$, since $H_2$ is closed. Therefore, $a_1b_1 \in H_1$. Inverses: Let $a_1 \in H_1$. Show $a_1^{-1} \in H_1$. But $a_1 \in H_1$ so $\phi(a_1) = a_2 \in H_2$. Therefore, $\phi(a_1^{-1}) = [\phi(a_1)]^{-1} = a_2^{-1} \in H_2$ because $H_2$ is a subgroup. So $a_1^{-1} \in H_1$. Yes, it is a subgroup.

c) Can you use (b) to give a one sentence proof of (a)?

Yes, $\{e_2\} < G_2$, so $K_1 = \{a_1 \in G_1 \mid \phi(a_1) = e_2\}$ is a subgroup of $G_1$ by (b).
4. Briefly justify your answers to any 3 of the following.
   a) Let \( a \in S_3 \). Is \( a^2 \in A_4 \)?
      Yes. If \( a \) is the product of \( k \) two-cycles, then \( a^2 = a \) is the product of \( 2k \) two-cycles.
   b) Are \( \mathbb{Z} \) and \( \mathbb{R}^+ \) isomorphic groups? Make sure to explain your answer.
      No. \( \mathbb{R}^+ \) has an element of order 2 (viz., \(-1\)) but the only element of finite order in \( \mathbb{Z} \) is the identity which has order 1.
   c) Give an example of finite groups \( G \) and \( H \) such that \( |G| = |H| \) but \( G \) is not isomorphic to \( H \) or explain why no example exists.
      There are many such. See the next problem to which shows that \( \mathbb{Z}_8 \) and \( D_4 \) are not isomorphic. One is abelian and cyclic. The other is neither.
   d) Find an element \( \beta \in S_5 \) such that \( \beta(23)(45) = (34)\beta \), or explain why no such element exists.
      Impossible. If \( \beta \) is even, then the left side is even and the right side is odd. Similarly, if \( \beta \) is odd, then the left side is odd and the right side is even.

5. a) Suppose that \( \phi : D_4 \to \mathbb{Z}_8 \) is a group homomorphism. What is \( \phi(r_0) \)? Explain how you know this.
      \( \phi(r_0) = 0 \) since the identity is mapped to the identity under any homomorphism.
   b) Suppose, again, that \( \phi : D_4 \to \mathbb{Z}_8 \) is a group homomorphism. Let \( v \) be the vertical flip. If \( \phi(v) = k \), what are the possible choices for \( k \) in \( \mathbb{Z}_8 \)? Again, explain your answer.
      From the basic properties of a homomorphism, we must have \( |\phi(v)||v| \). But \( |v| = 2 \), so the only elements of \( \mathbb{Z}_8 \) that have orders which divide 2 are 0 and 4 which have orders 1 and 2, respectively.
   c) Can a homomorphism \( \phi : D_4 \to \mathbb{Z}_8 \) be an isomorphism? Explain.
      No. \( D_4 \) is not abelian or cyclic, but \( \mathbb{Z}_8 \) is.
   d) Can a homomorphism \( \phi : D_4 \to \mathbb{Z}_8 \) be onto? Explain.
      No. Since both groups have 8 elements, if \( \phi \) were onto, it would also be injective. Since we are told that \( \phi \) is a homomorphism, this would make an isomorphism. But that's impossible.

6. Is the mapping \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) by \( \phi(a) = \sqrt{a} \) injective? Surjective? A group homomorphism? Show all details. (Caution: The operation is multiplication.)
   Injective. Notice that \( \phi(a) = \phi(b) \iff \sqrt{a} = \sqrt{b} \iff a = b \). Surjective: Let \( c \in \mathbb{R}^+ \) (codomain). Find \( a \in \mathbb{R}^+ \) (domain) so that \( \phi(a) = c \). But \( \phi(a) = c \iff \sqrt{\sqrt{a}} = c \iff a = c^2 \). Notice that \( c^2 \in \mathbb{R}^+ \), so \( \phi \) is surjective. Homomorphism: Let \( a, b \in \mathbb{R}^+ \). Show \( \phi(ab) = \phi(a)\phi(b) \). But \( \phi(ab) = \sqrt{ab} = \sqrt{a}\sqrt{b} = \phi(a)\phi(b) \). Isomorphism: Yes, because we have shown that \( \phi \) is injective, surjective, and a homomorphism.

7. Let \( \phi : \mathbb{Z}_6 \to \mathbb{Z}_3 \) be the homomorphism determined by \( \phi(5_6) = 2_3 \). Make a list of the elements of \( \mathbb{Z}_6 \) and determine what each is mapped to under \( \phi \). Is \( \phi \) injective? Surjective? An isomorphism?
   Use the fact that \( \phi \) is a homomorphism and that \( 1_6 \) is the inverse of \( 5_6 \) so \( \phi(1_6) = \phi(-5_6) = -\phi(5_6) = -2_3 = 1_3 \). Then \( \phi(2_6) = \phi(2 \cdot 1_6) \to 2 \cdot 1_3 = 2_3, \phi(3_6) = \phi(3 \cdot 1_6) \to 3 \cdot 1_3 = 0_3, \phi(4_6) = \phi(4 \cdot 1_6) \to 4 \cdot 1_3 = 1_3 \), and of course \( \phi(0_6) \to 0_3 \).
8. (Extra Credit) Prove Colin’s Theorem: If $H < S_n$ and $|H|$ is odd, then $H$ contains only even elements. Hint: Think about the homework problem with odd and even elements in subgroups of $S_n$.

You proved for homework that: If $H$ is any subgroup of $S_n$, then either every element of $H$ is even or that exactly half the members of $H$ are even. But in this case the second option is not possible since $H$ has an odd number of elements (half can’t be odd and half even). So $H$ must contain only even permutations.