Let the FTF A G to determine which product of cyclic groups is isomorphic to.

Consider the following group table with 8 elements

\[
\begin{array}{|c|cccccccc|}
\hline
    & e & a & b & c & d & x & f & g \\
\hline
  e & e & a & b & c & d & x & f & g \\
  a & a & b & c & e & x & f & g & d \\
  b & b & c & e & a & f & g & d & x \\
  c & c & e & a & b & g & d & x & f \\
  d & d & g & f & x & e & c & b & a \\
  x & x & d & g & f & a & e & c & b \\
  f & f & x & d & g & b & a & e & c \\
  g & g & f & x & d & c & b & a & e \\
\hline
\end{array}
\]

a) Use the table to find \( C(G) \) (the center). Solution: \( C(G) = \{e, b\} \).

b) Show that \( H = \langle b \rangle \) is normal in \( G \). Solution: \( \langle b \rangle = C(G) \).

Let \( x \in G \). Prove: If for each \( g \in G \) there is some integer \( n_g \) so that \( g x g^{-1} = x^{n_g} \), then \( x < x \triangleleft G \).

Solution: Let \( x^k \in \langle x \rangle \). Then using the given condition

\[
g x^k g^{-1} = g x g^{-1} \cdot g x g^{-1} \cdots g x g^{-1} = (g x g^{-1})^k = (x^{n_g})^k = x^{kn_g} \in \langle x \rangle.
\]

This means that \( g x g^{-1} \in \langle x \rangle \). By condition (iii) for normality, this means that \( x < x \triangleleft G \).

Use the FTFAG to determine which product of cyclic groups is isomorphic to

a) \( U(150) \cong \mathbb{Z}_{20} \oplus \mathbb{Z}_2 \cong \mathbb{Z}_5 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \).

b) \( U(320) \cong \mathbb{Z}_{16} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \)

c) a group \( G \) of order 48 with 1 element of order 1, 7 elements of order 2, 2 elements of order 3, 8 elements of order 4, 14 14 elements of order 6, and 16 elements of order 12. Solution: \( \mathbb{Z}_{12} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)

d) Here's a useful fact: Suppose that

\[
\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k} \oplus \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_j},
\]

where all the \( n \)'s are even and all the \( m \)'s are odd. Prove that \( G \) has \( 2^k - 1 \) elements of order 2.

Solution: None of the \( Z_m \) have elements of order 2, so in manufacturing an element of order 2 for \( G \) we will have to use the identity element from each of these. Next, each \( Z_{n_i} \) has 1 element of order 2. Then, we have two choices in the first \( k \) slots (either the identity or the element of order 2) and 1 choice in the last \( j \) slots for a total of \( 2^k \) choices. But this counts the identity element which has order 1. Tossing that out, we get \( 2^k - 1 \) elements of order 2.
7. a) Find an element of order 4 in \( \mathbb{Z}_{10} \oplus \mathbb{Z}_{14} \) or explain why none exists. **Solution:** None exists. \( \mathbb{Z}_{10} \) has elements of order 1, 2, 5, and 10 while \( \mathbb{Z}_{14} \) has elements of 1, 2, 7, and 14. There is no combination of these orders that has an lcm of 4.

b) Find a subgroup of order 4 in \( \mathbb{Z}_{10} \oplus \mathbb{Z}_{14} \) or explain why none exists. **Solution:** Use \( \langle 5 \rangle \oplus \langle 7 \rangle \).

8. The Cayley Table for the dihedral group \( D_3 \) is given below.

<table>
<thead>
<tr>
<th></th>
<th>( r_0 )</th>
<th>( r_{120} )</th>
<th>( r_{240} )</th>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_0 )</td>
<td>( r_0 )</td>
<td>( r_{120} )</td>
<td>( r_{240} )</td>
<td>( a )</td>
<td>( b )</td>
<td>( c )</td>
</tr>
<tr>
<td>( r_{120} )</td>
<td>( r_{120} )</td>
<td>( r_{240} )</td>
<td>( r_0 )</td>
<td>( c )</td>
<td>( a )</td>
<td>( b )</td>
</tr>
<tr>
<td>( r_{240} )</td>
<td>( r_{240} )</td>
<td>( r_0 )</td>
<td>( r_{120} )</td>
<td>( b )</td>
<td>( c )</td>
<td>( a )</td>
</tr>
<tr>
<td>( a )</td>
<td>( a )</td>
<td>( b )</td>
<td>( c )</td>
<td>( r_0 )</td>
<td>( r_{120} )</td>
<td>( r_{240} )</td>
</tr>
<tr>
<td>( b )</td>
<td>( b )</td>
<td>( c )</td>
<td>( a )</td>
<td>( r_{240} )</td>
<td>( r_0 )</td>
<td>( r_{120} )</td>
</tr>
<tr>
<td>( c )</td>
<td>( c )</td>
<td>( a )</td>
<td>( b )</td>
<td>( r_{120} )</td>
<td>( r_{240} )</td>
<td>( r_0 )</td>
</tr>
</tbody>
</table>

a) Write out the left and right cosets of \( H = \langle a \rangle \). **Solution:** The left cosets are: \( H = \{ r_0, a \} = aH = r_0H, r_{120}H = \{ r_{120}, c \} = cH, \) and \( r_{240}H = \{ r_{240}, b \} = bH \). The right cosets are: \( H = \{ r_0, a \} = Ha = Hr_0, Hr_{120} = \{ r_{120}, b \} = Hb, \) and \( Hr_{240} = \{ r_{240}, c \} = Hc \).

b) Determine whether \( H \) is normal in \( G \). **Solution:** It is not since \( bH \neq Hb \), for example.

c) Find elements \( x \) and \( y, z \) and \( w \) in \( D_3 \) so that all of the following conditions hold simultaneously (or explain why this is impossible):

i. \( Hx = Hy \);

ii. \( Hz = Hw \);

iii. \( Hxz \neq Hyw \).

**Solution:** \( Hr_{120} = Hb \) and \( Hr_{240} = Hc \). But \( Hr_{120}r_{240} = H \) while \( Hbc = Hr_{120} \).

9. Give an example of a group \( G \) and a subgroup \( H \) of \( G \) such that \([G : H] = 3\) and \( H \triangleleft G \), or explain why no such example exists. **Solution:** There are lots. Use \([\mathbb{Z} : 3\mathbb{Z}] = 3\) or \([\mathbb{Z}_3n : \langle n \rangle] = 3\) for any positive integer \( n \).

10. On the first exam I asked you to show that: If \( H \) and \( K \) are subgroups of \( G \), then \( H \cap K \) is subgroup of \( G \). Now prove that if \( H \) and \( K \) are both normal subgroups of \( G \), then \( H \cap K \) is normal in \( G \). **Solution:** Let \( x \in H \cap K \). Then \( x \in H \) which is normal in \( G \), so for any \( g \in G \), \( gxg^{-1} \in H \). Similarly, \( gxg^{-1} \in K \). So \( gxg^{-1} \in H \cap K \). By condition (iii), \( H \cap K \) is normal in \( G \).

11. As usual, let \( SL(R, n) \) denote the \( n \times n \) matrices whose determinants are 1. Determine whether \( SL(R, n) \) is normal in \( GL(R, n) \). **Solution:** See class notes from Wednesday, 3 November 1999.