Class 19: Selected Answers

1. The following 8 permutations in $S_4$ are known as the Octic group, $O = \{e, a, a^2, a^3, b, g, d, t\}$, where $e = (1)$, $a = (1234)$, $a^2 = (13)(24)$, $a^3 = (1432)$, $b = (14)(23)$, $g = (12)(34)$, $d = (13)$, and $t = (24)$.

a) Find all the left cosets of $O$ in $S_4$. Solution: The cosets are $O$ itself,

\[(12)O = \{(12), (234), (132), (123), (143), (132), (124)\}\]
\[(14)O = \{(14), (123), (1342), (243), (23), (1342), (143), (142)\}\]

b) What are the right cosets of $0$ in $S_4$? Solution: The cosets are $O$ itself,

\[O(12) = \{(12), (13), (124), (132), (143), (1342), (143), (142)\}\]
\[O(14) = \{(14), (123), (132), (23), (1342), (143), (124)\}\]

c) Is $O$ normal in $S_4$? Solution: No, $(12)O \neq O(12)$

2. Find all the left cosets of $< 4 >$ in $U(15)$. Then find the right cosets. Is $< 4 >$ normal? Solution: The left and right cosets must be the same since $U(15) = \{1, 2, 4, 7, 8, 11, 13, 14\}$ is abelian. So $< 4 >= \{1, 4\}$ is normal. The cosets are: $< 4 >$ itself, $2 < 4 > = \{2, 8\} =$ $< 4 > 2$, $7 < 4 > = \{7, 13\} =$ $< 4 > 7$, and $11 < 4 > = \{11, 14\} =$ $< 4 > 11$.

3. Find all the left and right cosets of $A_3$ in $A_4$. Is $A_3$ normal in $A_4$? Solution: Using the handout, $A_3 = \{(1), (123), (132)\}$ while $A_4 = \{(1), (123), (132), (124), (134), (143), (124), (13)(24), (142), (143), (14)(23)\}$.

So the left cosets are $A_3$ itself,

\[(234)A_3 = \{(234), (13)(24), (142)\}\]
\[(243)A_3 = \{(243), (143), (12)(34)\}\]
\[(124)A_3 = \{(124), (14)(23), (134)\}\]

The right cosets are $A_3$ itself,

\[A_3(234) = \{(234), (12)(34), (134)\}\]
\[A_3(243) = \{(243), (124), (13)(24)\}\]
\[A_3(143) = \{(143), (14)(23), (142)\}\]

$A$ is not normal since $(234)A_3 \neq A_3(234)$.

4. Let $U(R, n) = \{A \in GL(R, n) \mid \det A = \pm 1\}$. Show that $AU(R, n) = BU(R, n) \iff \det a = \det b$.

Solution:

$AU(R, n) = BU(R, n) \iff A^{-1}B \in U(R, n) \iff \det(A^{-1}B) = \pm 1$

$\iff \det A^{-1} \det B = \pm 1$

$\iff \det B \det A = \pm 1$

$\iff \det B = \pm \det A$

$\iff |\det B| = |\det A|$

5. Evaluate the following indices (justify your answers)

a) $|A_n : A_{n-1}| = \frac{\frac{8}{2}}{\frac{1}{n-1}/2} = n$.

b) $|Z_8 : < 2 | = \frac{8}{2} = 2$.

c) $|Z : < n | = n$, if $n \neq 0$, there are $n$ cosets: $0+ < n >, 1+ < n >, \ldots, (n-1)+ < n >$. Otherwise if $n = 0$ the index is infinite.

d) $|D_4 : < v | = \frac{8}{8} = 8$.

e) $|S_4 : O| = \frac{24}{3}$.

f) $|R^* : < -1 | = \infty$ because $|R^*| = \infty$ and $| -1 | = 2$.

g) $|GL(R, n) : U(R, n)| = \infty$ because you proved above that there were an infinite number of cosets, one for each nonnegative real number.
6. Gallian page 143 #14. **Solution:** Given $K < H < G$ with $|K| = 42$ and $|G| = 420$. So by Lagrange’s Theorem, the order of $H$ must be divisible by 42 and a divisor of 420. Possible orders are 84 and 210.

7. Gallian page 143 #20. **Solution:** Given $KG$ and $H < G$, with $|K| = 12$ and $|H| = 35$. Since $K \cap H$ is a subgroup of both $K$ and $H$, $[K \cap H] | 12$ and $[K \cap H] | 35$. Therefore, $[K \cap H] = 1$.

8. Gallian page 143 #24. $|G| = 25$. So by Lagrange, if $a \in G$, then $|a| = 1, 5, \text{ or } 25$. If $|a| = 25$, then $G$ is cyclic. If $G$ is not cyclic, then every element has order 5 (or 1), so $g^5 = e$ for all elements of $G$.

9. a) Suppose that $G$ is a group such that $g^2 = e$ for all $g \in G$. Prove that $G$ is abelian. **Solution:** We must show that $\forall a, b \in G \ ab = ba$. But

$$ab = (ab)e = (ab)(ba)^2 = ab((ba)ba) = a(bb)a(ba) = aca(ba) = (aa)(ba) = e(ba) = ba.$$ 

b) Suppose that $G$ is a non-abelian group of order 10. Prove that $G$ has an element $a$ of order 5. **Solution:** If all elements $a \in G$ have the property that $a^2 = e$, then we just showed that $G$ would be abelian, a contradiction. So there exists some $a \in G$ with $a \neq e$ and $|x| \neq 2$. By Lagrange $|a| = 5$ or 10. If $|a| = 10$, then $G$ is cyclic, hence abelian. Contradiction. So there is some $a \in G$ so that $|a| = 5$.

c) (Continuation of part b): Let $g \in G$ such that $g \notin < a >$. Prove that there are only two left cosets of $< a >$ in $G$, namely $< a >$ and $g < a >$. **Solution:** Since $| < a > | = |a| = 5$, then $[G : < a >] = \frac{|G|}{|a|} = 2$. So let $g$ be any element of $G$ not in $< a >$. Then $g < a >$ is the other coset.

d) (Continuation of part b and c): Prove that $g^2 < a > = < a >$. **Solution:** Since there are only two cosets, $g^2 < a >$ is either $< a >$ or $g < a >$. But if $g < a > = g^2 < a >$, then by the Coset Property Theorem, $g^{-1}g^2 = g \in < a >$ which contradicts that $< a >$ and $g < a >$ are distinct cosets.

e) (Continuation of part b and d): Prove that $g^2 = e$. **Solution:** Use a proof by contradiction. If $g^2 \neq e$, then $|g| = 5$ or 10. But the latter is impossible since $G$ is not cyclic. But if $|g| = 5$, then $g^5 = gg^5 = ge = g$. On the other hand, $g^2 < a > = < a > \Rightarrow g^2 = a^k$. So $g^6 = (g^2)^3 = (a^k)^3$ and so $g \in < a >$. But $< a >$ and $g < a >$ are distinct cosets.

f) Ok, look at what you’ve now shown: If $G$ is a non-abelian group of order 10, then $G$ has an element $a$ of order 5. Further, any element $g \notin < a >$ has order 2. But there are 5 such elements. Since none of the elements of $< a >$ have order 2 (because elements of the cyclic group $< a >$ must have order 5 or 1), then $G$ has exactly 5 elements of order 2. (For example, $D_5$ has 5 flips.) Can you conjecture how many elements a non-abelian group of order $2p$ has? Conjecture $p$.

10. a) Let $G$ be a group of order $n > 1$. Suppose that the only subgroups of $G$ are $\{e\}$ and $G$ itself. Prove that $G$ is cyclic. **Solution:** Take any element $a \neq e$ in $G$. Since $a \neq e$, then $< a > \neq \{e\}$ is a subgroup of $G$. By assumption, $< a >$ must be $G$, itself. That is, $G$ is cyclic.

b) Extra Credit: Show that the number $n$ would have to be prime for the hypothesis in the part above to be true. **Solution:** Let $|G| = n$. If $n$ is not prime, then $n = km$, where $2 \leq k, m \leq n - 1$. Since we have shown that $G$ is cyclic, by the Fundamental Theorem of Cyclic Groups, $G$ has cyclic subgroups of orders both $k$ and $m$. This contradicts the fact that the only subgroups of $G$ are $\{e\}$ and $G$ itself.

11. Extra Credit: Gallian page 143 #26. This is not hard; there are just several cases to check.