Problem Set 7
due in class Friday April 18th

Reading: Last week you read through section 9.5 in Bennett. Pick up there and read 9.6-9.8. For problems 7 and beyond, which concern reflections at interfaces, go back to Smith and read Chapter 10 through 10.7. To my mind the electrical transmission line (coax cable) is the system to focus on since the concept of impedance is already familiar in the context of voltage and current. Also the various kinds of boundaries come up more naturally there than they do for strings (where one has to invent rings sliding on poles and so forth). Our approach to EM waves (discussed in 10.6) will be to assume that the expressions for transmission and reflection carry over from the coax cable, so we won’t go back to Maxwell’s equations and derive these from scratch. Note that section 10.7 contains a handy chart gathering together the corresponding formulas for waves on a string (rope), EM waves, and waves on a transmission line (as well as sound waves that we have not discussed). (in the chart he uses the symbol $H$ for $B/\mu$; we are not going to talk about any cases where $\mu \neq \mu_0$.

1. Warm up on Jones Vectors.
   a. Show that the Jones vectors for LHC and RHC (9.20) are orthonormal (i.e. that each is normalized and the dot product of the two is zero).
   b. Show that the normalized Jones vector for light linearly polarized in the vertical direction (9.18) can be represented as a combination of RHC and LHC light (9.20).
   c. If the phase of the RHC component were to be magically shifted by 180 degrees (i.e. the RHC component was multiplied by -1), what would be the polarization state of the resulting light?

   Hint: your answers for b and c should be consistent with the claims in the text 9.21 & 9.22.

2. Circular Polarizer. In one of his problems (9.14) Bennet talks about a “left-circular polarizer” with a Jones matrix:

   $$M = \frac{1}{2} \begin{pmatrix} +1 & -i \\ +i & +1 \end{pmatrix}$$

   a. What output does this device give when the input is vertically polarized light? (give the polarization state and the relative intensity of the output).

   b. What output does this device give for any linearly polarized input? (give the polarization state and relative intensity of the output).

   c. Find the eigenvalues and eigenvectors of the matrix $M$.

   d. Show that such a device could be constructed by combining wave plates and polarizers (i.e. show that some combination of wave plates and polarizers has the right Jones matrix. Try all possible combinations does not seem fruitful, instead use what you have established about this device and what you learned in earlier problems and in lab to make in informed guess and then show that it works. I claim that a couple of quarter wave plates and a polarizer will do the trick).
3. **Electro-optic modulators.** Some materials which are not inherently birefringent will become so upon application of an electric field. By switching the electric field on and off one can create a variable wave plate which switches on and off. Suppose that the material is set so that the induced axes are in the $\hat{x}$ and $\hat{y}$ directions. By switching the electric field off and on the index difference $n_x - n_y = \Delta n$ can be varied between 0 and some maximum value $\Delta n_{on}$ and thus the phase retardation $\phi$ can be switched between 0 and $\phi_{on} = k_0 \Delta n_{on} L$ where $L$ is the length of the material and $k_0$ is the vacuum wavenumber of the light.

Section 9.6 of Bennett discusses modulators of this type. He gives some more details about the construction and how the retardation (his $\delta$, our $\phi_{on}$) depends on the applied voltage. None of that is necessary to answer this question, but you may still find it useful to pause and read this section to get an orientation to the problem.

The variable wave plate is placed between crossed polarizers. The first polarizer is set to have its pass axis in the $\hat{x} + \hat{y}$ direction. The intensity of the light after this first polarizer is $I_0$. The polarizer is followed by the variable wave plate (axes along $\hat{x}$ and $\hat{y}$) and then by a second linear polarizer ("the analyzer") with pass axis in the $\hat{x} - \hat{y}$ direction. Two limiting cases are easy to analyze without any calculations: 1) When the electric field is off ($\Delta n = 0$), the modulator has no effect on the polarization of the light and thus no light is transmitted by the analyzer. 2) If the applied electric field is such as to produce a phase shift of $\phi_{on} = \pi$, then the variable waveplate is a $\lambda/2$ plate, it rotates the plane of polarization by 90 degrees, and the transmitted intensity, after the analyzer, is $I_0$.

a. Show that if the electric field produces a phase retardation of $\phi_{on}$, the transmitted intensity, after the analyzer is $I_0 \sin^2(\phi_{on} / 2)$.

b. Suppose that the first linear polarizer is followed by a $\lambda/4$ plate with its fast axis in the $\hat{y}$ direction. Show that the intensity transmitted through the analyzer, when the variable plate is turned off, is $I_0 / 2$. Now when a time varying electric field is applied to the modulator the intensity will be modulated up and down around this value. By detecting the light with a photodiode and using a scope set to AC we can pick out only the oscillating parts of the signal, i.e. we can selectively look only at changes in intensity as electric field is switched on and off. Show that as the retardation is switched between 0 and $\phi_{on}$ the change in the transmitted intensity after the analyzer is $(I_0 / 2)\sin(\phi_{on})$.

c. For small electric fields the change in phase shift $\phi_{on}$ will be small. Determine, for $\phi_{on} \ll 1$, which arrangement (a or b) gives a bigger change in intensity for a given $\phi_{on}$.

4. **Optical Activity.** An “optically active” material is one in which both RHC and LHC light propagate with no change in polarization state but which presents a different index of refraction for the RHC ($n_R$) than for the LHC ($n_L$). The sugar solution that you looked at in lab has this property, as does a piece of quartz cut along a particular crystal axis. As you observed in lab, when linearly polarized light travels through an optically active material, it remains linear but its plane of polarization is rotated. The key to understanding this effect is to represent the linear light as a superposition of RHC and LHC each of which propagates through the material...
retaining its polarization state, but with a relative phase determined by $n_R$ and $n_L$. You can read the details in Bennet section 9.7. Be sure you take a few minutes and follow through his calculations.

A beam of light (unit intensity) initially vertically polarized passes through a piece of optically active quartz of length $L$ as shown in Bennett fig 9.21

a. Find the output state of the light (hint: feel free to follow the argument in the text. The output should end up being linearly polarized).

b. If you were to monitor the output polarization in the usual way (by using a linear polarizer as an “analyzer,” like you did in lab) what is the minimum thickness of quartz ($L > 0$) for which you would find the polarization state to be unchanged (i.e. returned to being “vertical.”). Use the numerical data given below 9.39 to get an actual thickness.

c. “The” index of refraction of quartz is about 1.54 at a wavelength of 589 nm (which is the wavelength for which the data following 9.39 applies). At this wavelength, what is the fractional index difference $(n_L - n_R) / n$?

5. What is it? Bennett’s table of Jones matrices includes “a rotator.” The name indicates what this device might do, but one should also be able to infer its properties from the Jones matrix by “experimenting” with its effect on various input states. Assume $\alpha \neq 0$.

a. To start with, let’s pursue the idea that this might be some kind of birefringent device. What effect does this device have on light linearly polarized in the vertical direction? Can the effect be described simply by an index of refraction? How does the output intensity relate to the input intensity (the thought being that a drop in intensity might suggest that the device involves a linear polarizer)?

b. What effect does this device have on light linearly polarized in the horizontal direction? Can the effect be described simply by an index of refraction?

c. Let’s think about the possibility that this is a birefringent device, but perhaps with axes that are not aligned horizontal & vertical. Consider a linearly polarized input beam with polarization angle $\theta$ as described by eqn. 9.15. Is there a choice of polarization angle $\theta$ for which linear polarized light passes through unchanged in angle? Does the intensity of the beam change when going through the device?

d. Let’s try another tack. What are the eigenvectors and eigenvalues of $M$? Do those eigenvectors correspond to any polarization state we are familiar with?

e. You have seen this kind of behavior before. What kind of a device or material would be described by matrix $M$?
6. Rotating Polarizers and Wave plates. The table of Jones Matrices gives values for a few orientations of the wave plates and polarizers. Here we look at how to find the Jones matrix for an arbitrary orientation. Imagine that we have some mystery wave plate-like or polarizer-like device marked with an axis of some sort. The axis is initially set to be horizontal. Imagine that one sends in first horizontal linear light and then vertical linear light and (somehow) in each case determines the output Jones vector.

a. Suppose the output Jones vectors are:

\[
\begin{pmatrix} a \\ b \end{pmatrix} \quad \text{for horiz and} \quad \begin{pmatrix} c \\ d \end{pmatrix} \quad \text{for vert}
\]

What is the Jones matrix \( M \) for this device (with its axis horizontal)?

b. Continuing on, one could rotate the mystery device so that the axis makes an angle \( \theta \) to the x-axis, and remeasure the output Jones vectors to find a new matrix \( M' \). However one could also achieve an equivalent arrangement by inserting two rotators, one before the mystery device and one after. To what angles should the two rotators be set in order to achieve exactly the same output Jones vector as rotating the mystery device itself by \( \theta \)?

c. As a test of your theory, starting with the known Jones matrix for a linear polarizer with pass axis horizontal, derive the Jones matrices for a polarizer with pass axis at +45 and at -45. (They should, of course, agree with ones in Table 9.1).

Epilogue. Here you have derived by physical arguments the formula 9.25, arrived at more abstractly in the text.

7. Light intensity transmitted/reflected at normal incidence. Consider a lightwave going from a block of material of index \( n_i \) into one with index \( n_t \). As long as the light wave hits the boundary at normal incidence the expressions we found for the reflected and transmitted voltage between two sections of transmission line (Smith page 361) also apply to the electric field (see the table on page 360 for the translation between the two systems). Assume that both materials are non-magnetic, i.e. \( \mu = \mu_0 \) everywhere. The intensity of the incident light wave is \( \left< S_i \right> = \frac{1}{2} \varepsilon_i c_i \left| E_i \right|^2 \) (with \( c_i = 1/\sqrt{\mu_0 \varepsilon_i} \), \( n_i = \sqrt{\varepsilon_i/\varepsilon_0} \)).

a. Find the intensity of the reflected light (in terms of the indices \( n_i \) and \( n_t \)).

b. Find the intensity of the transmitted light (in terms of the indices \( n_i \) and \( n_t \)).

c. Show that energy is conserved (in the sense that all of the incident intensity is either transmitted or reflected)

d. When light goes from air into a piece of glass \( (n = 1.5) \) what fraction of the intensity is reflected?

Epilogue. You can check your answers against Smith pg. 366-367. These equations for reflection and transmission are known as the Fresnel equations. As you know from lab, when the angle of incidence is not zero the behavior is considerably more complicated! Bennett actually does out the general case in Chapter 3.3, but we leave that for Physics 405.
8. **Multiple interfaces.** This is an extension of the string system discussed in Smith 10.1 and 10.2. Consider a (semi-infinite) light string connected to a length $L$ of heavy string connected to another (semi-infinite) light string thereby defining three regions as shown in the figure below. In regions I and III (the light string) the wave speed is the same. In region II (the heavy string) the wave speed is different. Hence for a given frequency $\omega$ the wavenumber in regions I and II will be the same ($k_1$) while in region II it will be something different ($k_2$). A wave (amplitude $A_i$) is incident from the left (in region I). Let’s call the reflected wave amplitude $B_i$. I have indicated in the diagram the various reflected and transmitted waves one needs to consider (i.e. in region II, there is a wave to the right ($A_{II}$) and one to the left ($B_{II}$) and in region III one to the right ($A_{III}$).)

![Diagram of wave speeds and regions]

a. Consider the specific case where $k_2 L = 2\pi$. By imposing the boundary conditions (page 345) at both $x = 0$ and $x = L$, find the amplitude of the transmitted wave $A_{III}$. What is $|A_{III}/A_i|$? (Hint: you might find the answer surprising). What is the amplitude of the reflected wave, i.e. $|B_i/A_i|$?

b. Consider the case $k_2 L = \pi$. What is $|A_{III}/A_i|? |B_i/A_i|$?

**Epilog:** We looked at a couple of very special values of $L$. Typically, for most values of $L$, $|A_{III}/A_i| < 1$ (as you might have expected). The optical version of this system, in which the two interfaces are two partially reflective mirrors, is an important optical instrument, the Fabry-Perot interferometer. More about this shortly.

9. **Impedance matching.** Smith 10.12. The “second order reflection effects” that he says to ignore are what you investigated in the previous problem. For special choices of $L$ and $k$ they can (as he notes and you have shown) lead to large transmission.