

# Physics 150 Laboratory Manual

Hobart & William Smith Colleges



HWS Physics Department

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# General Instructions For Physics Lab

Come to lab prepared — study the week’s experiment in this lab manual beforehand and bring:

- your quadrille-ruled, bound lab notebook;
- this lab manual;
- a scientific calculator;
- a pen and a fine-lead pencil; and
- optionally, a laptop computer and a spreadsheet program—some of the experiments will go more smoothly if at least one lab partner is able to use a spreadsheet for data analysis.

## RECORDING DATA

Each student must record all data and observations in ink in their own lab notebook. Mistaken data entries should be crossed out with a single line, accompanied by a brief explanation of the problem, not thrown away. Do not record data on anything but your lab notebooks. If you are using a spreadsheet for data analysis, your data should be first recorded in your notebook and then, only after all data are recorded, entered into the spreadsheet.

Put the date, your name, and the name of your lab partner(s) at the top of your first data page. Your data and observations should be neat and well organized. Your laboratory notebook will be examined by your instructor and will count towards part of your laboratory grade. Your laboratory notebook must be legible and comprehensible.

## LAB REPORTS

A written or oral lab report is required of all experiments. Your instructor will tell you ahead of time which form is required in each lab. Specific advice on written reports is given at the end of each experiment's description, but a laboratory report will generally consist of an introduction describing the purpose of the experiment and the theory behind the experiment, your data and observations, your analysis of your data, a summary of your experimental and calculational results, and a discussion of your conclusions including a summary of what you learned by doing the experiment. A written lab report should be written with a word processor and a spreadsheet and printed and handed in, not emailed to the instructor. An oral report should be of the same form, with the various sections described above being presented by different lab partners, but you should also show your instructor the neat summaries of your results and example calculations in your lab notebooks when you present your results. Your instructor may have more specific instructions on how to write your laboratory report, which supercede the instructions given here and elsewhere in this manual.

Your lab group must turn in its attendance sheet to your instructor as soon as you are all certain that you have finished the lab. If your lab instructor calls for written lab reports, your group will be checked out for data completeness and correctness of data analyses before you leave. If the lab instructor asks for oral reports, lab groups will give their reports in the order that they finished. Lab groups may not leave the lab until they have been checked out or given their oral reports.

It is important that a written lab report be well organized, clear and legible since your lab instructor will not be able to ask for clarifications before assigning a grade. If several measurements are to be compared and discussed, they should be entered into a table located near the discussion. Your data and results should always include appropriate units.

Occasionally your instructor may give some additional instructions in class.

Repetitive calculations should not be written out in your report; however, you should include representative examples of any calculations that are important for your analysis of the experiment.

## GRAPHING

Graphs should be drawn with a fine-lead pencil either in your lab notebook or on graph paper provided by the physics department that is later taped into your notebook. A complete graph will have a title, clearly labeled axes with units, and will show error bars on all data points for

which you have legitimate uncertainty values. You may scan, photocopy, or take clear, careful, readable large pictures of any graphs from your lab notebook to include in a written lab report. Alternatively, you may re-do the graph in a graphing program and include the resulting graph as a figure in your report.

Note that when you are asked to plot " $T^2$  versus  $M$ ," that means that you are asked to put  $T^2$  on the vertical axis and  $M$  on the horizontal axis.

Do not choose awkward scales for your graph; *e.g.*, 3 graph divisions to represent 10 or 100. Always choose 1, 2, or 5 divisions to represent a decimal value. This will make your graphs easier to read and plot. When finding the slope of a best-fit straight line drawn through experimental points:

1. Use a best-fit line to your data. You may do this either by hand or using a graphing program, such as the one with Microsoft Excel, though it is recommended that you do your graphs by hand.
2. Do not use differences in data point values for the rise and the run. You should use points that are on your best-fit line, not actual data points, which have larger errors than your best-fit line.
3. Do use as large a triangle as possible that has the straight line as its hypotenuse. Then use the lengths of the sides of the triangle as the rise and the run. This will yield the most significant figures for the value of your slope.

Please study the *Notes on Graphing* appendix for more detailed information.

## REPORTING OF EXPERIMENTAL QUANTITIES

Use the rules for significant figures to state your results to the correct number of significant figures. Do not quote your calculated results to eight or more digits! Not only is this bad form, it is actually dishonest as it implies that your errors are very small! Generally you measure quantities to about 3 significant figures, so you can usually carry out your calculations to 4 significant figures if calculating by hand, or to as many significant figures as the machine uses if you are using a calculator or spreadsheet, and then round off to 3 figures at the final result. After you calculate the uncertainties in your results, you should round off your calculated result values so that no digits are given beyond the digit(s) occupied by the uncertainty value. The uncertainty value should usually be only one significant figure, perhaps two in the event that its most significant figure is 1 or 2. For example:  $T = 1.54 \pm 0.06$  s, or  $T = 23.8 \pm 1.5$  s. If you report a quantity without an explicit error, it is

assumed that the error is in the last decimal place. Thus,  $T = 2.17$  s implies that your error is  $\pm 0.01$  s. Because the number of significant figures is tied implicitly to the error, it is very important to take care in the use of significant figures, unlike in pure mathematics.

## COLLABORATIVE AND INDIVIDUAL WORK

You may collaborate with your partner(s) and other students in collecting and analyzing the data, and discussing your results, but for a written report each student must write their own summary and discussion their own words.

## GENERAL ADMONITION

When written lab reports are called for, don't take your data and then leave the lab early. It should be rare that anyone leaves the lab before 3:30 PM. If you have completed data collection early, stay in the lab and begin your analysis of the data — you should be able to complete most or all of the analysis before leaving. This is good practice because the analysis is easier while the work is still fresh in your mind, the instructor is able to answer any questions about the analysis immediately and perhaps most importantly, you may discover that some of the data are faulty and you will be able to repeat the measurements immediately.



# Random Error & Experimental Precision

## GOALS & GENERAL APPROACH

We will measure the period of a pendulum repeatedly, observe the random errors, treat them statistically, and look to see if the measurements distribute themselves as expected. Then we will attempt to determine experimentally whether or not the period of a simple pendulum depends on the amplitude of its motion.

## APPARATUS

A simple pendulum (a metal ball suspended from a string), a stop watch, and a photogate timer.

## PROCEDURE I: TIMING A SINGLE PERIOD

1. You and your partner(s) should *each* independently make twenty manual measurements (40 measurements altogether if there are two lab partners) of the time taken by a single swing of the pendulum. For each measurement, draw the pendulum to one side about 20 degrees and release it in order to start it swinging. Try to draw it as closely as possible to the same distance to the side in every case, so that you will be measuring the period for the same amplitude in all cases. Allow the pendulum to complete at least one swing before beginning your period measurement, so as to allow it to settle down a bit. Then, using the stop watch, measure the time taken by a *single complete* swing. Give some thought as to how best to accomplish this. You might start and stop the watch when the ball reaches the top of its swing, or you may prefer marking time as it passes over a fixed point on the table. In the latter case be sure that the ball is be moving in the same direction when you stop your watch as it was when you started your watch

so that the measurement is for a complete swing. You may want to practice several ways before deciding which one to employ. **Do not measure the time for multiple swings.**

- Now read the *Notes on Experimental Errors* appendix at the end of this manual before proceeding to the analysis section.

## ANALYSIS I

- Create a separate histogram of pendulum period data for each data set (one for your 20 data points and one for each of your partners's). Do this by plotting the position of each of your time measurements on a time axis as shown in Figure 1. It will be most instructive if the two graphs use the same time axis, with one plotted above the other, each with its own individual plotting mark (say "x" for you and "o" or "+" for your partner(s)). This makes for easier comparison.

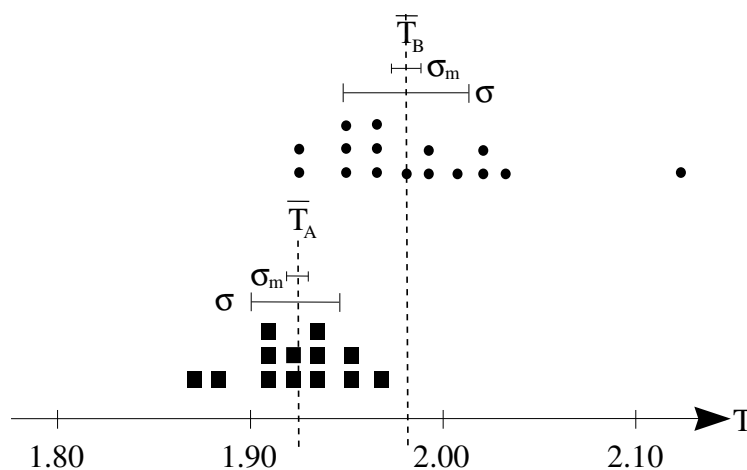



Figure 1

- Calculate the mean (average) and standard deviation for each of your data sets. You may set up the calculation either by hand, or in a spreadsheet, but *do not use a programmed function on a calculator or spreadsheet standard deviation function to perform these calculations*. For each calculation, set up a table or spreadsheet with columns for the measured values, the difference between each measured value and the average, and the squares of these differences. At the bottom of the  $T_i$  and  $(T_i - \bar{T})^2$  columns enter the average value of the numbers in that column. The mean is the average of the  $T_i$  column and the standard deviation is the square root of the average of the last column. *Be sure to keep all digits in the intermediate calculations. Do not round!* It is not necessary that each of you do all the calculations. You may divide up the work between or among all lab partners as you will.

Measurement # ( $i$ )	$T_i$	$T_i - \bar{T}$	$(T_i - \bar{T})^2$
1			
2			
3			
$\vdots$			
Average			

In a spreadsheet, the notation  $A1$  refers to the entry in the cell in column  $A$ , row  $1$ . If you copy and paste into another cell a formula involving  $A1$ , it will be rewritten to refer to the new cell. To prevent that from happening, place a dollar sign,  $\$$  before the column, the row, or both. If you want to paste a formula, such as the difference between the value in the current cell, say  $A1$ , and a fixed cell, say  $A22$ , type the formula  $= A1 - \$A\$22$  into the new cell(s). If you are completely unfamiliar with spreadsheets but want to try to use one, ask your instructor for help to get started.

3. Create a table giving the average,  $\bar{T}$ , the uncertainty in the average,  $\sigma_m$ , and the standard deviation,  $\sigma$ , for each of your data sets. It is a common mistake to think that the standard deviation *is* the uncertainty in the average. *It is not*. If you thought so, go back and read the notes on error again more carefully.
4. On your time graphs mark with vertical dotted lines the average value for each data set,  $\bar{T}$ , and then draw horizontal bars centered on each average value, representing  $\pm \sigma$ , the standard deviation, and  $\pm \sigma_m$ , the uncertainty in the average (see Fig. 1).

## NOTES ON PRECISION & ACCURACY

The precision of a measurement refers to its reproducibility. How close together do repeated measurements of a quantity lie? This is measured by the standard deviation. Precise measurements have small random uncertainties reflected in a small standard deviation. Precision is to be distinguished from accuracy. An accurate measurement not only has small random errors, but is also free from significant systematic errors. Thus a measurement may be very precise, in that repeated measurements give values lying very close together, but it may be very inaccurate if those values in fact grossly misrepresent the value of the quantity being measured. This would be the case, for example, if the frequency of the oscillator in your stop watch differed significantly from its design specifications.

## QUESTIONS I

1. Are your and your partner(s)'s values for the period in agreement? Justify your answer. Here it is crucial that you take the uncertainty in the average,  $\sigma_m$ , into account.
2. Do you think you can detect any difference in the precision with which you and your partner(s) measured the period of the pendulum?
3. For each of your data sets, how many of the individual measurements lie within  $\pm$  one standard deviation of the mean? Compare that with what the theory predicts, namely two thirds of the measurements.

## PROCEDURE II: DEPENDENCE OF PERIOD ON AMPLITUDE

1. Try to determine experimentally whether or not the period of the pendulum depends on the amplitude of its motion, using only single-swing stop watch measurements of the sort employed in Procedure I. Choose two rather different amplitudes ( $5^\circ$  and  $20^\circ$  are suggested) and make multiple measurements for each one. If you use  $20^\circ$ , you may use your data from Procedure I and only measure the period at  $5^\circ$ , but you'll need to make as many measurements at the new amplitude as you did in Procedure I.
2. Repeat the amplitude experiment using the photogate timer. Your laboratory instructor will show you how it works. Note that the results from the photogate timer may be highly reproducible. If you find many measurements of exactly the same result, the error is not zero even though the standard deviation is. In this case, the correct error to use is the so-called "least count" error. For example, if the least significant digit is 0.001 second, then use that for the value of  $\sigma$ , the error in any given measurement, and not the standard deviation.

## ANALYSIS II

Perform the necessary calculations to allow you to answer the amplitude question for both the stopwatch and photogate experiments. Use the statistical function on your calculator or in a spreadsheet to perform the standard deviation calculations rather than performing the laborious calculations by hand. Numerically, it is instructive to calculate the *difference* in the periods, along with the error in the difference to see if the difference in the periods is consistent with zero. See Table 1 in the appendix *Notes on Propagation of Errors* for how to calculate the error in a difference from the errors in the two quantities being subtracted. It may also be instructive to create a histogram of all the measurements at both amplitudes to get a feel for how well they agree or disagree.

## REPORT

For Procedure I you need only display the results asked for in the accompanying analysis section, and answer the three questions asked in the Questions section. For Procedure II, write a brief description of your experimental procedure, display clearly in a table the numerical results of your measurements, and use them to answer the question under consideration. Also, write a short paragraph explaining the difference between accuracy and precision.



# Instantaneous Velocity

## GOALS & GENERAL APPROACH

This experiment will help you to understand the concept of instantaneous velocity. You may want to read the relevant sections in your text before embarking on the experiment. Intuitively we think of instantaneous velocity as the velocity that an object has at a particular instant in time and at a particular point in space. Now in physics, the definition of a concept must contain within it, either implicitly or explicitly, directions for measuring it. But how does one go about measuring a velocity at a point in time?

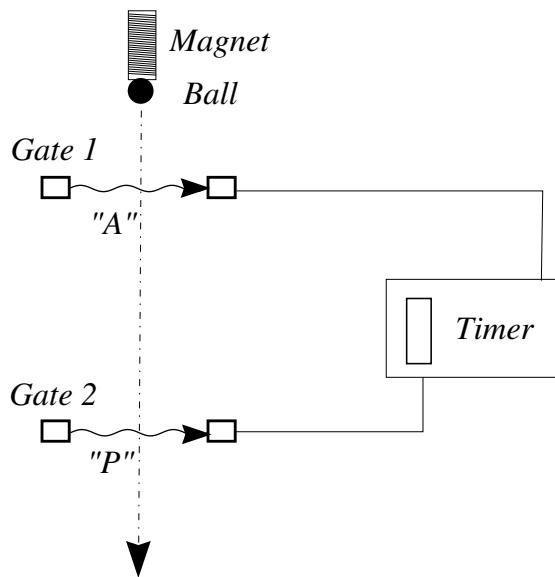
We know that average velocities are measured by dividing a distance traversed by the time taken. An instant in time presumably has no duration, and a point in space presumably has no extension. Thus velocity at an instant seems to defy our powers of measurement, and perhaps also our powers of imagination. (The ancient philosophers struggled mightily with this problem, and you will be rewarded by reading about some of their efforts.) So we will have to enlarge our intuition to include the notion of instantaneous velocity.

We will define instantaneous velocity in terms of the experiment that you are going to do today so as to be as clear and concrete as we can. Fig. 1 will help you to follow the explanation. You will drop a steel ball from the ceiling and attempt to measure its instantaneous velocity at point  $P$  along its path. We propose to do that in the following way.

We establish a higher point  $A$  and measure the distance between  $A$  and  $P$ , as well as the time it takes for the ball to fall from  $A$  to  $P$ . Dividing the distance by the time will give us the average velocity between these two points. We suspect that this average velocity will be less than the velocity at  $P$ , since the velocity of the ball is presumably increasing as it falls. (Note: When we use the word velocity without a modifier, we are referring to *instantaneous* velocity.)

Now choose a new point, call it  $A'$ , which is not as high as the first, but still above point  $P$ . Measure the average velocity between  $A'$  and  $P$  using the same technique. Once again one expects this average velocity to be

less than the velocity at  $P$ , but greater than the average velocity between  $A$  and  $P$ , because it samples velocities closer to  $P$ . Repeating this procedure for additional points, each successively closer to point  $P$ , results in a series of generally increasing average velocities, all of which we suspect to be less than the velocity at  $P$ . Now we are in a position to offer a practical definition of the instantaneous velocity at  $P$ .



**Figure 1** Schematic diagram of the apparatus.

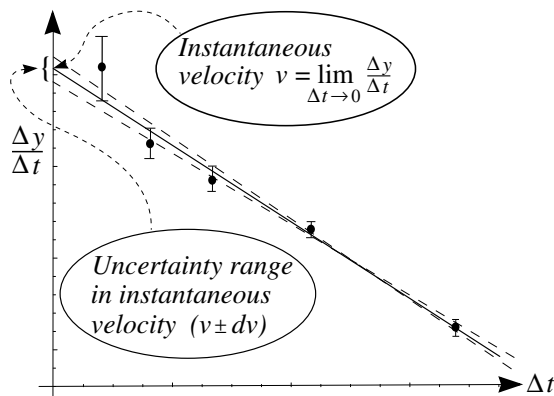
**Definition** Plot the average velocity values that you measured, as a function of the corresponding time intervals. Draw a smooth curve through the data points and extrapolate it to zero time interval. The velocity value given by this curve for zero time interval (the  $y$ -intercept) is defined to be the instantaneous velocity at  $P$  (see Fig. 2). More formally and mathematically one says this as follows (see your text book): the instantaneous velocity at a point is the limit that the average velocity between that point and a neighboring point approaches, as the neighboring point is brought successively closer to it. Or,

$$v = \lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t} . \quad (1)$$

You no doubt recognize this as the definition of the derivative of the position  $y$  with respect to the time  $t$ . So the instantaneous velocity is the derivative of the position with respect to time, evaluated at the point of interest.

We will also use a second method for determining the instantaneous velocity at point  $P$ . This second method relies upon the assumption that the ball is undergoing constant acceleration. We first choose a point  $A$  just a small distance above point  $P$ . We will measure the time the ball takes in dropping between points  $A$  and  $P$  and then, keeping the point  $A$





**Figure 2** Graph of the average velocity  $\Delta y/\Delta t$  versus time interval  $\Delta t$ . The intercept of the “best fit” straight line through the data is the instantaneous velocity. The dashed straight lines are reasonable alternative “best fits” that are used to find the uncertainty in the instantaneous velocity.

fixed, we will find another point, call it  $B$ , such that the time the ball takes falling between  $A$  and  $B$  is *exactly* twice the time taken between points  $A$  and  $P$ . If the ball is accelerating uniformly, the average velocity over the interval from  $A$  to  $B$  is equal to the instantaneous velocity of the ball at a *time* halfway between the times at which the ball is at the points  $A$  and  $B$ , or, the time at which the ball is at point  $P$ . This situation is depicted in Fig. 3.

Note that because the ball is accelerating, it covers less distance in the first half of the time between  $A$  and  $B$  than it does in the second half of the time, so the point  $P$  is *not* halfway in distance between points  $A$  and  $B$ , but is closer to point  $A$ .

## APPARATUS

The steel ball is initially held near the ceiling by a magnet, and can be released by activating a switch that interrupts the current to the magnet. Two photogates are mounted on a track down which the ball falls (See Fig. 1). Each gate consists of a light source, focusing lenses, and a light detector. (The light is infrared so you will not be able to see it.) **The photogate timer should be set to the pulse mode.** As the ball falls through each light beam, the light is momentarily interrupted and the photo-detector sends out a signal to the interval timer. When the upper light beam is interrupted, the interval timer is turned on, and when the lower light beam is interrupted, the interval timer is turned off. Thus, when the ball is dropped through the gates, the interval timer automatically registers the time elapsed during the ball’s fall between the two gates.

## PROCEDURE I: INSTANTANEOUS VELOCITY

1. Measure the distance from the floor to the bottom of the magnet (use a 1-meter and 2-meter stick taped together with the sliding pointer mounted on the top stick). Measure the diameter of the steel ball with a micrometer. These two numbers will enable you to find the distance from the bottom of the ball to the floor when the ball is hanging from the magnet. **REMINDER: Don't forget that all measurements need to have an uncertainty value attached to them before they are complete.**
2. Set photogate 2, the location of the point "P" at which we want to measure the velocity of the ball, on the track not more than a meter above the floor, make sure it is oriented perpendicular to the track, and then tighten it very securely to the track. **Photogate 2 must remain in this position throughout Procedure I. Be careful not to disturb it.** Use a 2-meter stick with sliding pointer to measure the height of the light beam above the floor. Do this by resting the end of the meter stick on the floor and sliding the slider down (the way the ball will be moving) until the flat edge interrupts the light beam and starts the timer. Then read the position of the slider on the meter stick. Be sure that you have the meter stick oriented correctly. Take 2 or 3 independent measurements of this position in order to estimate your uncertainty.
3. Make a theoretical calculation of the expected velocity of the ball at gate 2 from the distance the ball has fallen and the value of the acceleration of gravity,  $g = 9.81 \text{ m/s}^2$ . You'll need this velocity value to make your plot in the following step.
4. Place photogate 1 at various positions above point  $P$ , and at each position measure the height of its beam above the floor (using the same procedure that you used for gate 2) and the time it takes for the ball to fall through the two photogates (**be sure that your timer is set to 0.1 msec and pulse mode**). Do this for not fewer than five such positions. Begin with a position near the top of the track and work down to a final position as close to gate 2 as you can get (when their brackets touch — but be careful not to disturb the position of gate 2). Don't use equal distances between the gate 1 positions, as that will make the elapsed times change by rather unequal amounts. Rather, change the position less when gate 1 is high, and change it more as it gets closer to gate 2. Plot your average velocity values versus  $\Delta t$  as you take the data. This will help you in choosing the best positions for gate 1 as you go along. In making your plot, the vertical axis should go to values 15% or so higher than the theoretical value you calculated in the previous step.

The elapsed time  $\Delta t$  should be measured several (at least three) times for each position in order to check for reproducibility. If slightly different times are measured, an average can be taken and an uncertainty

assigned sufficient to span the range of the individual readings. If the times are significantly different (more than a few tenths of a millisecond), something is wrong. If the times are identical, take the uncertainty to be one in the least significant figure.

Likewise measure the beam heights more than once in order to obtain an estimate of the uncertainty in those values.

## ANALYSIS I

1. Calculate the average velocity of the ball between the two gates for each gate 1 position. In doing this, make a neat table such as the one shown below, displaying the distance between gates,  $\Delta y$ ; the elapsed time  $\Delta t$ ; and the average velocity for each case. Be sure to leave enough space to record your uncertainties as well. *If you are using a spreadsheet, place the values and their uncertainties in separate columns. At the same time, while you are taking and recording this data, plot your average velocity values against the corresponding times  $\Delta t$  in your lab book. Don't wait to do this after all of the velocities have been measured. Don't start the velocity axis (vertical axis) scale at 0 m/s, but rather at a value close to the smallest of your average velocity values. This will magnify the apparent slope of the line.*

	$\Delta y$ (m)	$\Delta t$ (s)	$\bar{v}$ (m/s)
1	$1.752 \pm 0.001$	$0.7759 \pm 0.0001$	$2.258 \pm 0.001$
2			
3			
4			
5			

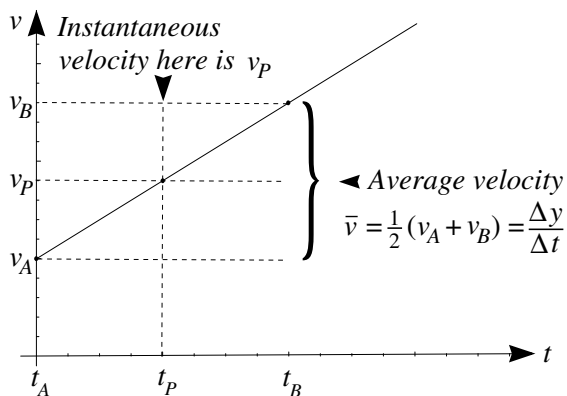
2. **Re-read the appendix** *Notes on Propagation of Error* and then calculate the uncertainty (error) for each calculated velocity value. Place these uncertainty values next to their corresponding velocities in the table you have created. Include the uncertainties in distance and time on this table as well.
3. Using the uncertainties calculated above, place vertical error bars on each of your plotted velocity values. **Read the appendix** *Notes on Graphing*. Find the best straight line that will fit your set of data points (being mindful of the error bars) and locate its intersection with the velocity axis. This intercept value is the limit of the average velocities and is by definition the instantaneous velocity at the position of gate 2.
4. To obtain an uncertainty for this experimental instantaneous velocity value, draw another straight line that is steeper, but which still manages to pass through most or all of the error bars. The  $y$ -intercept

of this line provides another possible value for the instantaneous velocity. You can use  $\pm$  the difference between these two values as the uncertainty for the measured instantaneous velocity.

In subsequent experiments, we will want to measure a whole set of instantaneous velocities. That will be very time consuming if we have to use the procedure employed in this experiment. A simpler procedure would be to measure the average velocity over a very small interval centered on the point of interest. But would this average value be sufficiently close to the instantaneous value that we want? Let's check it out against the data we have just taken.

Before starting Procedure II, complete your graph from the plot you made in Procedure I and make sure that the results are reasonable. Once you begin Procedure II, it will be difficult to restore your apparatus to the Procedure I configuration.

## PROCEDURE II: AVERAGE VELOCITY



**Figure 3** For constant acceleration, the average velocity equals the instantaneous velocity at a time halfway between the endpoints. Thus the *average* velocity between points *A* and *B* is equal to the *instantaneous* velocity at the point *P*. You cannot simply use  $\bar{v} = (v_A + v_B)/2$ , since you do not have the instantaneous velocities at *A* or *B*, but you *can* use  $\Delta y/\Delta t$ .

1. Your two photogates should still be in their last positions (as close as possible). Re-measure the elapsed time for this position to make sure that it hasn't changed. **Photogate 1 will remain fixed throughout Procedure II. Be careful not to disturb it.**
2. Leaving gate 1 fixed, drop gate 2 (this should be the first time that gate 2 has been moved) until the elapsed time between gates 1 and 2 is exactly twice its previous value (this requires some trial and error), ensuring that the time the ball spends falling from gate 1 (point *A*) to point *P* (the original position of gate 2) equals the time the ball spends

falling from point  $P$  to point  $B$  (the new position of gate 2.) With gates 1 and 2 in these positions, we are sampling velocities above and below the velocity at  $P$  for equal times, and the average should then be the instantaneous velocity at  $P$ . Figure 3 shows a graph of the velocity versus time.

3. Measure carefully the height of the two light beams above the floor.
4. Calculate the average velocity of the ball between the two gates. Also calculate the estimated uncertainty (error) in this value.

## ANALYSIS II

1. Compare the values you have obtained for the instantaneous velocity of the ball at point  $P$  from the first procedure and the second procedure. Are they in agreement?
2. You have taken sufficient data to allow you to calculate the distance the bottom of the ball has fallen from rest (at the magnet) to point  $P$  (photogate 2). From the theory of freely falling bodies, and the accepted value for the acceleration of gravity, calculate the velocity that the ball is predicted to have after falling that distance, and use the uncertainty in the distance fallen to calculate an uncertainty in your calculated velocity value. Does this calculated value agree with your two experimental values?

## REPORT

Make sure to give a clear presentation of the Analysis sections in your report.



# Force Table

## GOALS AND GENERAL APPROACH

In this experiment we will attempt to prove for ourselves that forces are vectors. By this we mean that if we represent a set of forces acting on an object by vectors, and then add the vectors together, the resultant vector accurately represents a force which is equivalent to the original set of forces. By equivalent we mean that this single force would have the same effect on the object's motion as the original set of forces. In addition we will get some practice at adding vectors, both graphically and analytically.

## APPARATUS

The apparatus for today's experiment is called a force table, and consists of a horizontal circular table with a removable peg at its center. An angular scale is provided around its edge, where pulleys may be clamped. A string passes over each pulley, one end of which is connected to a ring at the center of the table and the other to a weight hanger. A set of weights and protractors will also be provided.

## PROCEDURE

The ring is acted on by three forces, whose magnitudes depend on how much weight is placed on each weight hanger, and whose directions depend on the directions assumed by the strings. If the three forces do not add up to zero, the ring will tend to move in the direction of the net force, and it will move until it reaches a new position where the new forces (new because the directions have changed) do add up to zero. We call this the equilibrium position, and say that the ring is in equilibrium because it has no tendency to move away from this point.

1. You will be assigned a set of three angles. Set your pulleys at those angles.

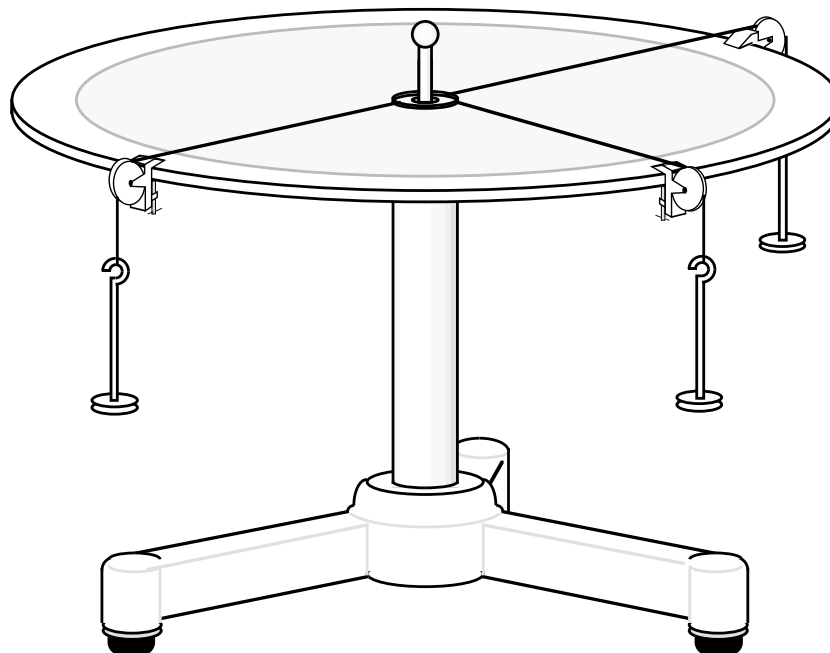


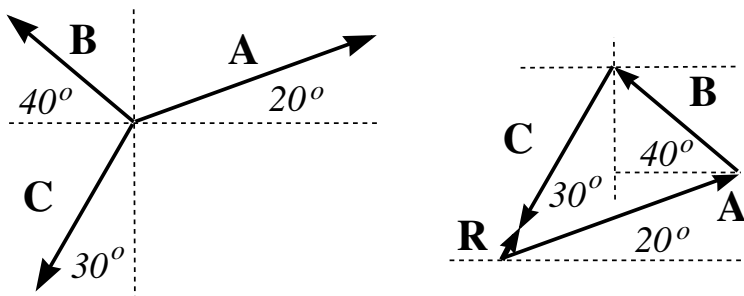
Figure 1 Force table apparatus used in this experiment.

2. With the pin in place in the center of the table, place some weights on the weight hangers. (The pin keeps the ring from flying off the table while you are getting the weights in place.) **When choosing weights in this experiment avoid choosing very light ones, but do not exceed about 500 grams.** Adjust the amount of weight on each hanger until you succeed in stabilizing the ring exactly at the center of the table (the pin will be at the center of the ring). Jiggle the strings after each weight change to aid the system in coming to equilibrium.
3. After achieving ring equilibrium at the center of the table, make small changes in each of the weights and observe how large a change is necessary in order to produce a noticeable change in the equilibrium position of the ring. These just noticeable weight changes are the uncertainty values that you will assign to each weight value.
4. Record the final best weights, weight uncertainties, and angles. (Don't forget to include the weight of the hanger in your weight value.) This information provides you with the magnitude and direction of each force.

## ANALYSIS

Since the ring remains at rest, the sum of the forces on it must be zero (Newton's 1st & 2nd Laws). Represent each of the three forces by a vector whose length is determined by the weight applied to the string,





**Figure 2** The leftmost figure shows the forces  $\vec{A}$ ,  $\vec{B}$ , and  $\vec{C}$  acting on a point. The rightmost figure shows the graphical method of adding the vectors to produce their resultant  $\vec{R} = \vec{A} + \vec{B} + \vec{C}$ .

and whose direction is along the direction of the string. Then find the sum of the three vectors and see how close it comes to being zero. The summation should be performed in two different ways — graphically and analytically.

### Graphical Addition of Forces

1. Draw two *rough* sketches in your notebook, one representing the forces as they act on the ring and the other representing the same forces laid head to tail for addition. Choose the positive  $x$ -axis to correspond to zero degrees of angle. Make the length of each vector proportional to the magnitude of the corresponding force and make the angles representative of the actual physical angles. Your sketches will look something like Figure 2.
2. Calculate the angles that your vectors make with the  $x$ -axis and label the angles on your sketches accordingly. Also label each vector with the weight in grams.
3. Now use a protractor and ruler to produce a very careful and accurate version of your second sketch on good graph paper. Again, the positive  $x$ -axis should correspond to zero degrees. Work carefully and use a sharp pencil. You will need to choose a scale for your diagram, on which a distance will represent a force. Let each centimeter represent some number of grams such that the scale is easy to use (multiples of 10, 20, or 25 gm) and such that diagram is as large as your paper will allow. (Of course grams are not units of force; they must be divided by 1000 and multiplied by  $9.8 \text{ m/s}^2$  to obtain the corresponding weight in Newtons. However, since the mass is proportional to the weight, *we can enjoy the convenience of using gram units without affecting our analysis and conclusions.*)
4. Draw the resultant  $\vec{R} = \vec{A} + \vec{B} + \vec{C}$  of the three forces on your diagram and measure its magnitude and direction (an angle with respect to some convenient axis). Remember that the resultant is the vector

drawn from the tail of the first of your three vectors to the head of the third.

### Analytical Addition of Forces

1. Make a table in your notebook with two columns, the first for  $x$ -components and the second for  $y$ -components. Label the rows A, B, and C to represent your three force vectors.
2. Use trigonometry to calculate the  $x$  and  $y$  components of each vector (your first rough sketch will assist you at this). Be sure to include the correct algebraic sign in each case, which you can determine by inspection of the figure. Do each calculation neatly in your notebook. Enter these component values (in grams) into the table you have created.

Force	$x$ -component	$y$ -component
A		
B		
C		
Total		

3. Add a fourth row to your table labeled "Total" and find the  $x$  and  $y$  components of the vector sum by adding the components in each column and recording the result in the "Total" row.
4. Use the Pythagorean Theorem and trigonometry to find the magnitude and direction of the "Total" vector from its  $x$  and  $y$  components (again all values expressed in grams).

### QUESTIONS

1. Construct a small table giving the magnitude and direction of the vector sum by each method. In each case do you think that the magnitude of your vector sum came as close to zero as one might reasonably expect? (Because the sum vectors are so nearly equal to zero, their directions may differ wildly.) A reasonable expectation would be that the magnitude of the resultant should not be much larger than the uncertainties in the magnitudes of the individual forces. Uncertainties in the angles will also affect the magnitude of the sum, but it is rather difficult to calculate their effects, and so we will not be able to treat them.

If the sums are zero to within the experimental uncertainty, you have demonstrated that forces behave as vectors and can be added accordingly.

2. You can obtain very good values for the  $x$  and  $y$  components of each vector by reading them directly from the careful scale graph you created in doing the graphical summation. Compare the values obtained this way with the values you obtained by the analytical method. Do they seem to be equal to within the limitations of the scale used in the graph?

## REPORT

The analysis section of your report should contain a clear presentation of all of the graphs, tables, and calculations called for under the Procedure section, and your answers to the two questions.



# Newton's 2nd Law

## GOALS & GENERAL APPROACH

In this experiment we will exploit the frictionless surface of an air track, and computer interfaced electronic timing technology to study the motion of a mass acted upon by a constant force. Figure 1 is a schematic diagram of the apparatus which will aid you in following this discussion.

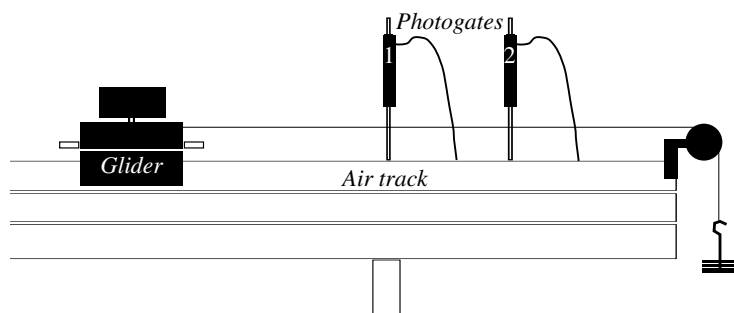


Figure 1 Apparatus used in the experiment.

The air track gliders are free to move with very little friction on the air tracks. The tracks are equipped with two photogates that each send and receive a beam of infrared light across the track. When the shutter that is mounted on top of each glider interrupts one of the light beams, a signal is sent to the computer through an interface box. The computer can measure the elapsed time during which any particular gate is blocked as well as the times of arrival of the gating signals. Computer software allows one to control which times are measured and to perform calculations on those time values. Your instructor will walk you through the instructions you need to operate the software.

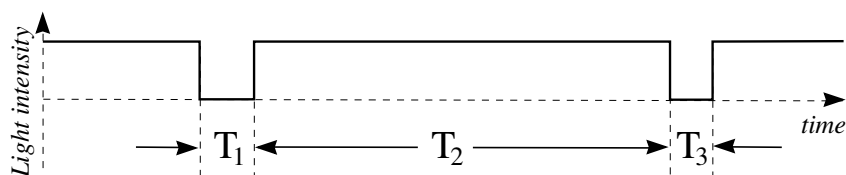
You will employ two methods for applying a constant, unbalanced force to the glider:

- attaching the glider to a hanging weight via a pulley (see Fig. 1); and
- tilting the air track from horizontal slightly.

Under these conditions you can use the photogates and computer to quickly measure the resulting accelerations and verify that:

- the value of the acceleration does not depend on the value of the initial velocity; and
- the expression  $F = ma$  gives the correct value of the acceleration for a given net force  $F$  and mass  $m$ .

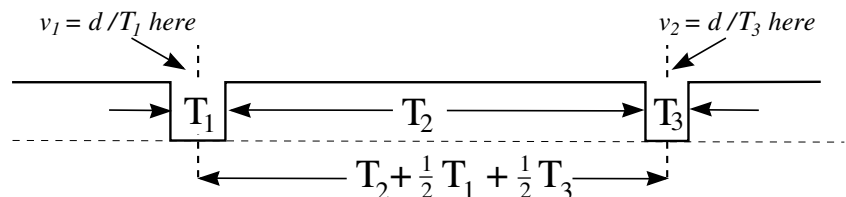
Here is how the computer goes about measuring the acceleration for you. As the glider passes through the two photogates, the computer measures three different times:  $T_1$  and  $T_3$  are the amounts of time for which gates 1 and 2 are blocked.  $T_2$  is the time between the unblocking of gate 1 and the blocking of gate 2. The following diagram will make this much clearer.



**Figure 2** Light intensity seen by the photogates as a function of time.

The computer calculates the velocity of the glider as it passes gate 1 by dividing the length of the glider's shutter,  $d$ , by  $T_1$ , and similarly uses  $T_3$  and  $d$  to calculate the velocity of the glider as it passes gate 2. If the acceleration is a constant, it can be evaluated by dividing the change in the velocity by the elapsed time between the two velocities. What value should we use for this elapsed time?

The velocities that have been calculated are average velocities over the time intervals  $T_1$  and  $T_3$ . However, if the acceleration is constant, the average velocity over a time interval is equal to the instantaneous velocity at the middle of the time interval (remember our earlier experiments with the drop towers). So the change in the instantaneous velocity is equal to  $v_2 - v_1$ , and the time it took for this change to occur is  $T_1/2 + T_2 + T_3/2$ . The following diagram will help to make this clearer.



**Figure 3** The computation of the acceleration from the instantaneous velocities and the elapsed time between them.

The difference in velocities is  $\Delta v = (d/T_3) - (d/T_1)$  and the elapsed time between them is  $\Delta t = (T_1/2) + T_2 + (T_3/2)$ . Thus, the average acceleration,

which equals the instantaneous acceleration if the acceleration is constant, is

$$a = \frac{\frac{d}{T_3} - \frac{d}{T_1}}{\frac{T_1}{2} + T_2 + \frac{T_3}{2}} \quad (1)$$

## APPARATUS

Air track, gliders, two photogates, pulley, weights, weight hanger, wood blocks, caliper, triple beam balance, computer, and computer interface.

Note that the gliders, despite looking old and scuffed up, have very flat faces that ride on the air track. Be very careful with the gliders. A dropped glider may be rendered permanently damaged if its faces become dinged or bent. Because the faces are so flat, these gliders are relatively expensive and can be difficult to repair even if only slightly damaged.

## PROCEDURE I: WEIGHT DRIVEN GLIDER

**Note:** For this experiment you need not attempt to estimate measurement uncertainties and calculate errors for your experimental results. Such an analysis would be difficult in this case. If you are ambitious and want to try anyway, it is not forbidden.

1. Measure the length of the shutter on your glider. Do this first with a meter stick as a first estimate and a check on the next measurement. Because of the shape of the light beams produced by the photogates, this physical measurement may not correspond to the effective light blocking distance of the shutter. To accurately measure the latter follow the following procedure.

Choose "**Photogate Status Check**" from the main menu. The monitor will now indicate whether each gate is blocked or unblocked (if your photogate has an LED indicator light mounted on it, you can use it as the indicator). With the air to the track off, slowly slide the glider through a gate and observe the position of one of its edges at the point where the gate is first blocked, and then again at the point where the gate is first unblocked. Use the scale that is mounted to the track for this measurement. To do this you will need to lay a small ruler along one end of the glider to bridge the gap between glider and scale, and then read the position of the ruler relative to the scale. Slide the glider through in the same direction it will travel during the experiment. This measurement needs to be made with great care as it can introduce considerable error into your results. Do it for both gates, and give the computer an average of the two values if they are only slightly different (0.5 mm or less). *If they are significantly different,*

*check with your instructor.* In this case you will have to calculate the accelerations by hand in order to obtain reliable results.

2. For the measurements you will make today you need to choose “**Miscellaneous Timing Modes**” from the menu, and then “**Gate and Pulse (2 gates)**” from the subsequent menu. Each time the glider passes through the gates, entries are added to the table on the monitor for  $T_1$ ,  $T_2$ , and  $T_3$ . Copy these times into your lab notebook, and into a spreadsheet if you have one available. When you are ready to evaluate the accelerations, choose “**Special Options**” from the menu and provide the length of your shutter in meters when prompted for it.
3. Verify that the track is level by observing whether or not a stationary glider will drift away from its original position when released. Do this at several points on the track. It is impossible to have it perfect in this regard, but the small drifts ought at least be in different directions at different points along the track.
4. Verify that the track is level by giving the glider a push, allowing it to move freely through the two photogates, and observing the computer measured value of the acceleration. It should be very close to zero for a level track. Record these values with the remainder of your data in your notebook.
5. Allow the weight hanger by itself (5 grams) to pull the glider through the photogates and measure the acceleration. **Catch the base of the glider softly before it hits the end of the track.** Do this more than once, allowing the glider to begin its motion at different distances from the first gate. In this way you will be able to see that the acceleration remains the same for very different gate velocities. **When you pull the glider back for each new measurement, lift it off the track and bring it back around the first photogate so that it is not triggered on the way back.** For each trial record the velocity at the first photogate and the measured acceleration.

Try starting the glider out between the two gates and giving it a push that will send it backwards through the first gate. It will slow down, stop, and then come back through both gates as before. (It takes a little practice to accomplish this without pushing the glider so hard as to cause the weight hanger to crash into the pulley.) In this case the initial and final velocities will have different algebraic signs, and so the computer calculated acceleration will be incorrect. You will have to calculate the acceleration by hand using the formula discussed under Goals and General Approach, the computer provided times, and being careful to use the correct algebraic sign for each velocity.

6. Repeat the measurements of part 5 for two more weights. Do not exceed 30 grams. Again, try to catch the glider softly each time before it hits the end of the track.
7. Use one of the balances to measure the mass of your glider.



## ANALYSIS I

1. For one of your trials, display a sample calculation of the acceleration using the raw times and the length of the shutter. Compare it with the computer calculated value.
2. Using two free body diagrams, apply Newton's Laws to the glider, pulley, string, and hanging mass system and derive a formula for the acceleration of the glider in terms of the mass of the glider, the mass of the hanging weight, and the acceleration of gravity. This derivation should be clearly written out as part of your report.
3. For each hanging weight studied, display clearly in a table the velocities at the first photogate and the measured accelerations. Also calculate and display the values of the acceleration that you predict from the formula derived in part 2.

## QUESTIONS I

What can you conclude from your experimental results about the following questions? Give clear explanations in each case, showing how your answer is based on the experimental results.

1. Does the measured acceleration depend on the measured velocity with which the glider enters the photogate region? (What do the data say?)
2. Do you find agreement between the accelerations you measure and the accelerations you predict from Newton's Laws? (Without honest error bars on your acceleration values, you of course cannot give a scientifically justified answer to this question. But you can at least comment on whether or not they seem to be within reasonable experimental uncertainty of each other.)

## PROCEDURE II: INCLINED PLANE

1. You will be given wood blocks with which to elevate the far end of the track. Measure the thickness of the blocks with a caliper.
2. Measure the distance between the single leg at one end of the track and a line joining the two legs at the other end of the track.
3. Raise one end of the track with the blocks. From the measurements of parts 1 and 2 calculate the angle, and the sine of the angle, that the track now makes with the horizontal.

4. Let the glider coast down the track (*no string and hanging weight in this experiment*) and through the gates. Do this several times for different glider starting points. In each case record the velocity at the first photogate and the measured acceleration. Start the glider between the two gates and give it a push that will send it backwards through the first gate, after which it will stop and return on a regular course. Once again for this case the acceleration will have to be calculated by hand from the computer time values.
5. Choose a more massive glider and repeat the measurements of part 4.

## ANALYSIS II

1. Using a free body diagram, apply Newton's Laws to this situation and derive an equation for the acceleration of the glider in terms of the mass of the glider and the acceleration of gravity.
2. For each glider, display clearly in a table the velocities at the first gate and the measured accelerations. Also calculate and display the values of acceleration that you predict from the formula just derived for this case.

## QUESTIONS II

What can you conclude from your experimental results about the following questions? Give clear explanations in each case, showing how your answer is based on the experimental results.

1. Does the measured acceleration depend on the measured velocity with which the glider enters the photogate region? (What do the data say?)
2. Does the acceleration you measure depend on the mass of the glider? (What do the data say?)
3. Do you find agreement between the accelerations you measure and the accelerations you calculate from Newton's Laws? (Without honest error bars on your acceleration values, you of course cannot give a scientifically justified answer to this question. But you can at least comment on whether or not they seem to be within reasonable experimental uncertainty of each other.)

## REPORT

In the analysis portion of your report, give a clear presentation of the data and the analysis called for in the Procedure and Analysis sections. Write out clear readable answers to the five questions.

# Simple Harmonic Motion

When an object in stable equilibrium is displaced, a force acts to return it to its equilibrium position. If the restoring force is proportional to the displacement (a “linear” force), disturbing the object will result in a sinusoidal motion, also called “simple harmonic motion (SHM).” Simple harmonic oscillations are important because even when restoring forces are not linear with respect to displacement, they approach linearity as displacements become smaller and smaller. Therefore, small amplitude oscillations are usually simple harmonic oscillations even when oscillations of larger amplitude are not. Today’s SHM experiment will employ a mass suspended from a spring, a common example of simple harmonic motion.

## GOALS AND GENERAL APPROACH

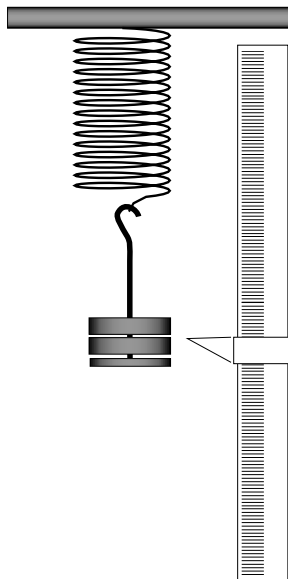
1. To determine the extent to which the frequency of the mass/spring system oscillations is dependent on the amplitude of the oscillations. (Recall your exploration of this question for the simple pendulum earlier in the term. The frequency is completely independent of amplitude for a true harmonic oscillator.)
2. To test the dependence of the period of the oscillations on mass and spring constant, as summarized by the equation

$$T = 2\pi\sqrt{m/k}, \quad (1)$$

where  $k$  is the spring constant,  $m$  is the oscillating mass, and  $T$  is the period of the oscillator.

## APPARATUS

A weight hanger suspended from a vertically hanging spring, an assortment of weights, a meter stick with sliding pointer, a triple beam balance, and a stop watch.



**Figure 1** The apparatus used in this experiment consists of a spring, a hanger with masses and a meter stick with a sliding pointer.

### PROCEDURE I: EFFECT OF AMPLITUDE ON PERIOD

1. Measure the period of the oscillations for two very different amplitudes but the same mass, using a mass somewhere between 200 and 400 grams. The large amplitude should not be more than about 12 cm, and the small amplitude can be as small as 1 or 2 cm. To increase the precision of your measurements you will want to measure the total time needed for several oscillations and then divide that number by the number of oscillations (unlike your pendulum measurements where we insisted on your measuring the period of individual swings). Check the reproducibility of your measurements by making several trials. (The spring should always hang with its small end at the top.) The uncertainty values that you assign to your measurements here will be very important.

### ANALYSIS I

1. Compare the periods for the different amplitudes along with their uncertainties. Do the periods differ by more than can be accounted for by experimental error?

### PROCEDURE II: STATIC DETERMINATION OF SPRING CONSTANT

Hang various masses on the spring and measure the corresponding positions of the bottom of the weight pan with a vertically held meter stick

and sliding pointer. Be sure that the small end of the spring is at the top. Take enough data to make a good plot of applied force,  $F$ , versus position. **Do not hang more than 350 grams on the spring.**

## ANALYSIS II

1. Make a careful plot of applied force versus the position of the bottom of the weight pan. The force must be expressed in Newtons and the position in meters. Include error bars on your data points.
2. The data should fall along a straight line. This may not be true for the smallest masses, but should become true as the masses get larger and the displacements become greater. You will obtain your value for  $k$  from the slope of the straight part of your graph. Estimate the uncertainty in your value for  $k$  by drawing more than one reasonable straight line through the data points with their error bars, and measuring their slopes. If the data points all lie so close to a straight line, and the data points are all so small, that this is not feasible, then estimate by how much the slope calculation would change if the outer data points were changed by amounts equal to their uncertainty values.

## PROCEDURE III: DYNAMIC DETERMINATION OF SPRING CONSTANT

1. Measure the period  $T$  for at least four different hanging masses. Use small amplitude oscillations for these measurements. **Do not hang more than 350 grams on the spring.**
2. Measure the mass of the spring,  $m_s$ .

## ANALYSIS III

We can write the relationship between period and mass in the following form:

$$T = 2\pi\sqrt{\frac{M + m}{k}} .$$

or, equivalently,

$$T^2 = 4\pi^2 \frac{M + m}{k} = \left(\frac{4\pi^2}{k}\right) M + \left(4\pi^2 \frac{m}{k}\right) , \quad (2)$$

where  $M$  is the mass hung from the spring (including the pan), and  $m$  is some additional mass that must be added to the hung mass in order to take into account that the spring itself is moving and it possesses mass as

well. We cannot add all of the mass of the spring because not all of the spring is moving the same amount. The very top of the spring does not move at all, while the very bottom of the spring moves just as much as the hanging mass. So  $m$  will be some fraction of  $m_s$ , the total mass of the spring.

According to this equation, a plot of  $T^2$  versus  $M$  should result in a straight line ( $y = ax + b$ ). Can you identify the groups of constants in Eq. (??) which correspond to  $a$  and  $b$ ?

1. Construct such a plot, and from the slope ( $a$ ) and  $y$ -intercept ( $b$ ) calculate values for  $k$  and  $m$ . Include your data from the small amplitude trial of Procedure I as well. Once again, estimate the uncertainty in the value of the slope by putting error bars on your plotted points and drawing more than one reasonable straight line through the data points with their error bars. From the uncertainty in the slope calculate the uncertainty in your value of  $k$ .
2. Compare this value of  $k$  with that obtained in procedure II, the force versus position curve. In order to do this, of course, you will need an uncertainty value for each of your  $k$  values. Are they in agreement?
3. From your values for  $m_s$  and  $m$  calculate the *fraction* of the spring's total mass that must be added to the hanging mass in this experiment in order to obtain the correct value of the period from Eq. (??)?

## QUESTIONS

1. Explain in your own words (don't use equations), using the laws of physics, why the period of an oscillator should increase as the oscillating mass increases.
2. Doesn't it seem counterintuitive that the period *does not* depend on the amplitude of the oscillations? After all, the mass has farther to travel. Can you give a convincing explanation of this phenomenon?

## REPORT

In the analysis section of your report, give a clear presentation of the data and the analysis called for in the Procedure and Analysis sections. Write out clear, readable answers to the two questions.

# Standing Waves

## GOALS AND GENERAL APPROACH

Standing waves occur when waves of equal velocity,  $v$ , and wavelength,  $\lambda$ , pass over one another while moving in opposite directions. This situation commonly occurs when a wave moving in one direction is reflected back on itself by a barrier. There is not time here to give a general treatment of this phenomenon, so you should consult your text on these matters if you need to.

In this experiment you will set up standing waves on a string and on a column of air, observe the relationship between their frequencies, and check the values of the observed wave velocities against theory and handbook values.

## STRINGS

Strings are most usually held firmly at each end, as for example in stringed musical instruments, so that the ends of the string are nodes of the standing wave pattern (where no motion occurs). The points midway between the nodes, where maximum motion occurs, are called antinodes. Typical standing wave patterns for a string fixed at both ends are shown Fig. 1.

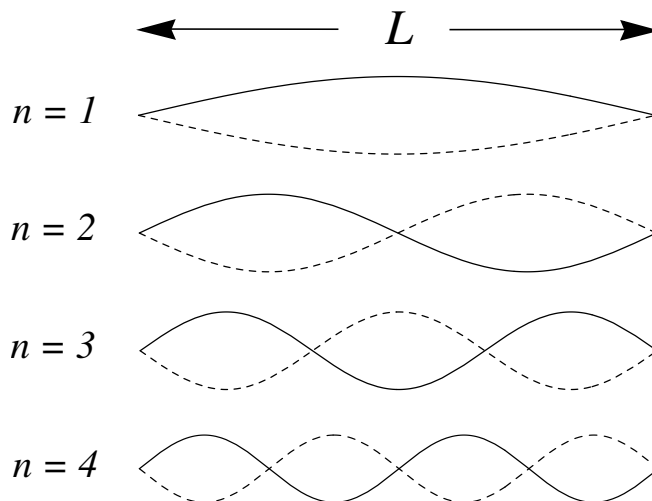
From the diagrams it should be clear that each complete "loop" is  $\lambda/2$  long, and some integer number of such loops fit exactly on the string with length  $L$ . From this observation we can write the condition for a standing wave as:

$$n(\lambda/2) = L, \quad (1)$$

where  $n$  is any positive integer. This means that in principle there is an infinite number of possible standing waves, but in practice only a finite number are achievable. You will be able to observe about ten.

Now for any wave, the wavelength,  $\lambda$ , is related to the frequency,  $f$ , and velocity,  $v$ , in the following way

$$f\lambda = v.$$



**Figure 1** The first four modes of a standing wave that has nodes at the ends.

Solving this equation for  $\lambda$  and substituting the resulting expression into Eq. (??) we obtain

$$n(v/2f) = L, \quad \text{or} \quad f = n(v/2L). \quad (2)$$

Equation (??) tells us that the frequencies of the various standing waves on a string of length  $L$  with wave velocity  $v$ , are integer multiples of a fundamental frequency whose value is  $v/2L$ . This fundamental frequency is also the lowest standing wave frequency for the string. Another common name for these various standing waves is “normal modes.”

Each such normal mode (standing wave) motion is a natural motion for the string. If we try to drive the string into motion, it will respond very strongly if our driving force varies with one of these frequencies. We say that each of the normal mode frequencies is a “resonant” frequency for the string.

Notice that by measuring the resonant frequencies of the string and its length  $L$ , Eq. (??) allows us to calculate the velocity of waves on that string. It must have the same value no matter which normal mode is excited. Theory predicts that the velocity of a wave on a string depends on its mass density and the tension applied to the string in the following way

$$v = \sqrt{\frac{T}{\mu}}, \quad (3)$$

where  $T$  is the tension in Newtons and  $\mu$  is the linear density of the string in kg/m.



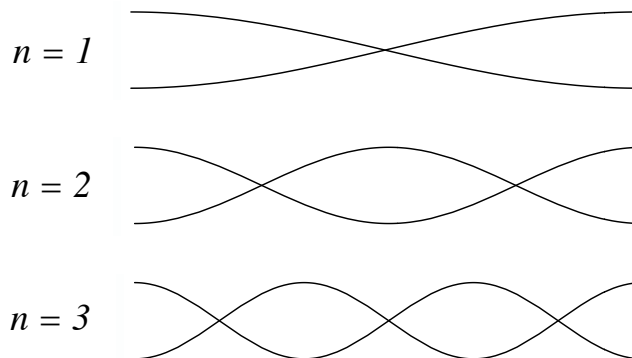
## AIR COLUMNS

An air column can be formed by a tube of any cross section, inside of which the air molecules vibrate in a direction parallel to the long axis of the tube when a sound wave passes through it. Thus we have longitudinal waves in the case of a column of air, in contrast to the string which supports transverse waves (string motion is perpendicular to the direction of wave motion).

When a sound wave traveling through a tube reaches the end of the tube, most of the wave is reflected back down the tube if the end of the tube is closed off. Surprisingly enough, even if the end of the tube is open, some of the wave is reflected back down the pipe. So whether the ends of the tube are open or closed, a wave is reflected back and forth between the ends, generating a standing sound wave (or normal mode). In practical applications at least one end of the tube must remain open to let the sound out, and so the most common situations are both ends open, or just one end open. Brass and reed instruments are examples of the former, while flutes are an example of the latter. Organ pipes can be of either the open (both ends open) or closed (one end closed) variety.

### Open Pipes

Consider first a tube that is open at both ends. It can be shown that the molecules at the ends must vibrate the maximum amount, and therefore the antinodes of the standing wave occur at the ends of the tube. Thus the allowed normal modes for an air column open at both ends can be represented by the following diagrams. Bear in mind that the standing waves in this case are longitudinal, even though the diagrams which represent them are patterned after those for transverse waves. There is no other clearer way to represent a longitudinal standing wave.



**Figure 2** The first three modes of a standing wave in an air column that is open at both ends.

From the diagrams it should be clear that each complete “loop” is  $\lambda/2$  long, and some integer number of such loops fit exactly on the air column

with length  $L$  (in this case we have a half loop at each end with complete loops in between). Therefore the derivation of the formula for the frequencies of the normal modes is identical to that for the string, resulting in

$$f = n(v/2L) , \quad (4)$$

where  $n$  is any positive integer.

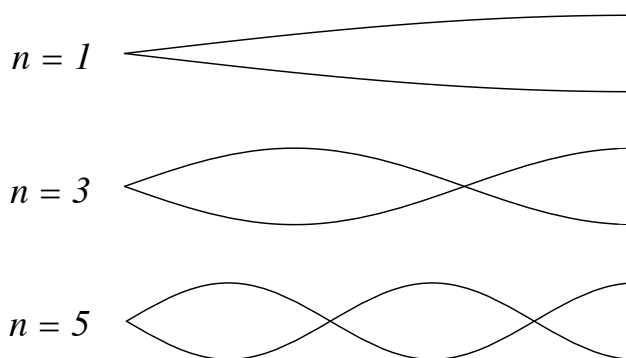
Equation (4) tells us that the frequencies of the various standing waves in an open air column of length  $L$  with wave velocity  $v$ , are integer multiples of a fundamental frequency whose value is  $v/2L$ . This fundamental frequency is also the lowest standing wave frequency for the air column. In this case  $v$  is the velocity of sound in air.

Now  $L$  is not exactly the same as the length of the tube that enclosed the air column. It can be shown that the antinode that occurs at the open end of a tube occurs at a point slightly beyond the end. (It takes a little while for the wave to realize that the tube has come to an end.) How far beyond the tube the antinode extends depends on the diameter of the tube. For a tube of circular cross section, the antinode occurs approximately  $0.6r$  beyond the end, where  $r$  is the radius of the cross section. So for a tube open at both ends,

$$L = L_{\text{tube}} + 2 \times 0.6r . \quad (5)$$

### Closed Pipes

If the pipe is closed at one end and open at the other, there must be a node at the closed end and an antinode at the open end. In this case the allowed normal modes of vibration can be represented by the following diagrams.



**Figure 3** The first three modes of a standing wave in an air column that is closed at one end.

From the diagrams it should be clear that in this case, an odd number of half loops must fit exactly on the air column of length  $L$ , each half loop

being  $\lambda/4$  long. From this observation we can write the condition for a standing wave as

$$n'(\lambda/4) = L, \quad (6)$$

where  $n'$  is any positive odd integer.

Again, solving  $f\lambda = v$  for  $\lambda$  and substituting the resulting expression for  $\lambda$  into this equation, one obtains the result

$$f = n'(v/4L), \quad (7)$$

where again  $n'$  is any positive odd integer.

Equation (7) tells us that the frequencies of the various standing waves in a closed air column of length  $L$  with wave velocity  $v$ , are positive odd integer multiples of a fundamental frequency whose value is  $v/4L$ . This fundamental frequency is also the lowest standing wave frequency for the air column. Here again  $v$  is the velocity of sound in air, but  $L$  is the length of the tube  $+ 0.6r$ ,

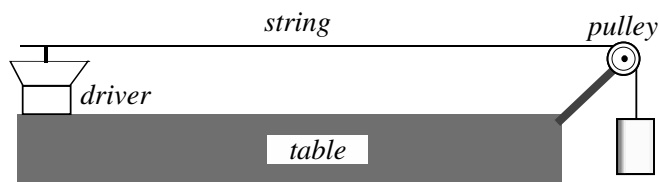
$$L = L_{\text{tube}} + 0.6r. \quad (8)$$

## APPARATUS

A function generator, mechanical wave driver, banana plug wire leads, loudspeaker, meter stick, calipers, glass tube, cork dust, cord, pulley, weight hanger and weights.

## PROCEDURE I: STRINGS

1. Connect the function generator to the mechanical wave driver using banana plug wire leads. The cord should be anchored to the post clamp at the end of the mechanical wave driver, pushed down into the slot of the drive rod, and passed over the pulley at the other end of the table. Use the weight hanger to hang a 500 gram weight on the end of the cord.



**Figure 4** A string fixed at both ends is driven by a small speaker to produce standing waves. A mass at the end provides tension.

2. If the function generator is on and not set to sine wave mode, set the function generator wave form button to sine wave, otherwise, turn the function generator on; its default waveform mode is sine wave. Adjust the frequency control knob to 0.1 Hz. (The device will start up set to 1000 Hz.) Set the voltage control to 2.00 Vp using the voltage adjust knob.
3. Adjust the continuous frequency control knob until you achieve a clear standing wave pattern. It is sufficient to adjust to the nearest 0.1 Hz, though you might try getting to the nearest 0.01 Hz. You are looking for a "resonance," which means that the amplitude of the vibrations will reach a maximum when the frequency matches the resonance frequency of the mode. Carefully tune the frequency until the maximum amplitude of vibration is achieved. This will be the frequency of this mode.
4. Before looking for another normal mode (standing wave pattern), detune the oscillator and re-tune it for a maximum again. You will want to do this more than once. In this way you will be able to estimate the uncertainty in your measured frequency values.
5. Adjust the frequency controls to observe yet other normal modes and their frequencies. To observe the lowest mode, the multiplier switch will have to be set on 1. Measure the frequencies (along with uncertainties) for as many as you can. Always adjust the amplitude to the smallest possible value necessary to observe the resonance.
6. Record the frequencies you observed along with the integer number for each mode. This integer will be one less than the total number of nodes (including the nodes at the ends); it is also equal to the number of loops (or antinodes) on the string. Also include a simple diagram of the string for each mode.
7. Carefully measure the length of the string between the two end nodes.
8. Measure the total mass and total length of your string (along with uncertainties). You'll have to find a way to measure the length of string in each loop without taking it apart because the string has loops fastened into it with wire clips or nylon wire ties to hold weights or hold to a metal bar.

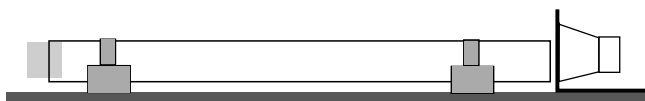
### ANALYSIS I

1. Plot the frequency of each mode against the mode integer. Include error bars on your points.
2. According to Eq. (??) the slope of this line is equal to  $v/2L$ . Use Eq. (??) to calculate a value for  $v$  (and its uncertainty). Use your measured slope  $v/2L$  along with its uncertainty, to calculate a value for the speed  $v$  and its uncertainty.

## QUESTIONS I

1. Does your graph confirm the dependence of frequency on mode number predicted by Eq. (??)? Explain.
2. Are your two values for the velocity of the wave in agreement within experimental uncertainties?

## PROCEDURE II: AIR COLUMNS



**Figure 5** The modes of a tube are excited with a small speaker. The tube can be corked or uncorked, depending on whether a closed or open tube is desired.

1. Connect the output of the function generator to the loudspeaker. Set the output frequency to 0.100 Hz and the output voltage to 3.00 V<sub>p</sub>.
2. Hold the loud speaker close to the end of your glass tube and turn up the frequency until you hear a low level tone. Now adjust the frequency, and with your ear close to the end, listen for a noticeable increase in the volume and noticeable vibration pattern in the cork dust at the resonant frequencies. If the far end of the tube is corked off, the first resonance should occur somewhere between 50 and 100 Hz. If both ends of the tube are open, the first resonance should occur somewhere between 100 and 200 Hz. The resonance volume increase will be more noticeable if you keep the sound level of the loudspeaker quite low.

If you are unsure that you have a resonance, you can check for it by moving the speaker back and forth in front of the tube in a direction perpendicular to the axis of the tube. The amplitude of sound should noticeably decrease as the loud speaker moves away, and increase again as it is brought back to its position in front of the mouth of the tube.

3. Find as many normal mode frequencies as you can, both with one end closed off with the cork, and with both ends open. Don't forget to measure the uncertainties in those frequencies as well. When you locate each frequency by ear using a low level tone, then turn the amplitude of the driving sound up high and observe the action of the cork dust lying along the bottom of the tube. It will make the standing wave behavior of the gas molecules visible by bouncing about where the gas is vibrating at large amplitude, and lying still where the gas is relatively quiescent (nodes).

4. Record the frequencies, the appropriate integer, and the amplitude diagram for each normal mode. The integer and diagram are to be obtained from the cork-dust pattern. Remember, only odd integer modes exist for this case
5. Try to find some even integer modes for the corked tube and convince yourself that they don't exist.
6. Measure the length of the glass tube (from end to end for the open ends case, but from end to cork barrier for the closed end case). Also measure the inside radius of the cross section of the tube.

## ANALYSIS II

1. Plot frequency versus mode integer for each of the two cases (open and closed). These two curves should be plotted on the same graph.
2. From the slopes of the plotted lines and the measured value of  $L$  (don't forget the end correction) calculate the velocity of sound in air.

## QUESTIONS II

1. Do your graphs confirm the dependence of frequency on mode number predicted by equations (??) and (??)? Explain.
2. Compare your values for the velocity of sound in air with each other and with the professionally measured value, which is 344 m/s at 20°C. The velocity of sound rises 0.6 m/s for each degree centigrade temperature rise, so you can correct this value for the temperature of the laboratory. There is a thermometer on the barometer, located near the doorway.

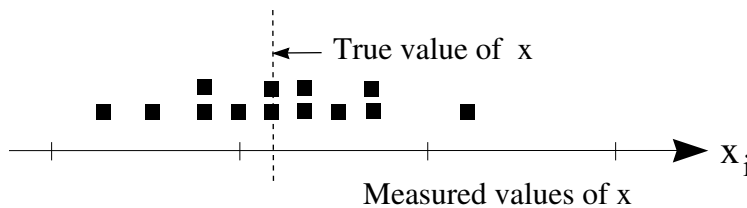
## REPORT

In the analysis section of your report, give a clear presentation of the data and the analysis called for in the Procedure and Analysis sections. Write out clear, readable answers to the four questions.

# Notes on Experimental Errors

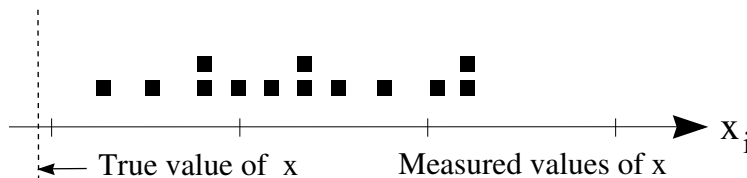
## EXPERIMENTAL ERROR AND UNCERTAINTY

We generally speak of two broad classes of errors – random and systematic. The term random error refers to the random and unpredictable variations that occur when a measurement is repeated several times with non-identical results, although the results do cluster around the true value. The stopwatch or photogate time measurements that you have performed are very good examples of this.



**Figure 1** A set of measurements that has a fair amount of random error, but little, if any, systematic error.

Systematic errors are those errors in a measurement that are regular and consistent in the sense that the measurements are consistently too large or too small, and usually by about the same amount for each measurement. The manual measurement of a time period would be an example of systematic error if the person is consistently late in stopping the clock.



**Figure 2** A set of measurements that has a fair amount of random error, as well as significant systematic error.

Other examples of sources of systematic error would be a stop watch that ran consistently fast or slow, and a meter stick that was mistakenly manufactured too long or too short.

Any actual measurement will ordinarily be liable to both random and systematic errors. The random errors reveal their presence much more readily, and can usually be observed by repeating the measurement several times and observing the differences in the values obtained. The measured values will be observed to scatter themselves in some random way about an average value.

Systematic errors are usually much more difficult to detect and measure. We are often unaware of their existence. If one is measuring something whose value is already thought to be known, and consistently obtains a value that is significantly different, one is certainly led to suspect the presence of one or more systematic errors. (The alternative is that the experiments that led to the accepted value contained the systematic error.) However, if one is measuring something that has never been measured before, any systematic error present will not be obvious, and great care must be taken to investigate and eliminate every source of systematic error that one can think of. Experience and ingenuity are the valuable assets in such an investigation. Obviously sometimes things are overlooked and values published that later turn out to be wrong.

The remainder of these notes will confine themselves to a discussion of random errors. There are several reasons for doing this.

- A discussion of systematic error depends critically on the experiment being performed, and cannot easily be generalized.
- Random errors are present and significant in most experiments.
- Random errors are susceptible to a general mathematical treatment by means of which their effects can be reduced.

Before going on, it would be well to comment briefly on the use of the word "error." Random error might better be called random uncertainty. It is the amount by which our measured values uncontrollably fluctuate, and thus provides a measure of our lack of certainty about the value of the quantity we are attempting to measure. The word error connotes mistake, and random error should not be thought of in those terms. The word error is more appropriate in the term, systematic error, where something more like a mistake is actually being made. Nevertheless, we will continue to use the word "error" to denote random uncertainties, because it is so firmly entrenched in our customary usage.

We will frequently use the term experimental error. *Experimental error is simply total uncertainty in any measured quantity, or any quantity calculated from measured quantities. It is not the amount by which your value differs from the generally accepted value.* Experimental error will generally have both systematic and random components, and one of the most important tasks of the experimenter is to measure, estimate, and assess the experimental "uncertainty" of all experimental and experimentally derived quantities.



How does one go about doing that? Consider first, directly measured quantities like length or time. One may repeat the experiment several times, taking care to make each measurement independent of the others. The variation in these values will enable us to measure the random “error” in the measurement, using a method to be discussed later in these notes. Many measurements will involve reading a scale and interpolating between the marks on the scale, as well as zeroing the scale at one end or the other. Such interpolations cannot be perfect and one is often content to record as the experimental error simply the uncertainty in one’s mind when making the interpolation. For example, consider measuring the length of a solid body. If a meter stick is used to measure this length and every measurement yields the same value, we would say that the “error” in the measurement is the “least count” of the meter stick, namely 1 mm. So, we might quote the error as  $\pm 1$  mm. Sometimes the uncertainty will be determined by consulting the specifications for the equipment being used, where the manufacturer will state the limits of accuracy of the equipment. All of the possibilities cannot be covered here, and you will have to use your common sense and ingenuity as particular situations arise.

In any event, *no experimental measurement is complete until a value of the uncertainty, or experimental error, is determined.* Thus every entry in the data pages of your lab book must be accompanied by a value of its uncertainty. This is usually expressed in the following form:

$$x = 35.7 \pm 0.3 \text{ cm} .$$

If the experimental uncertainty is the same for a whole series of data entries, it would be sufficient to record the uncertainty for one entry and indicate that it is the same for all of the others.

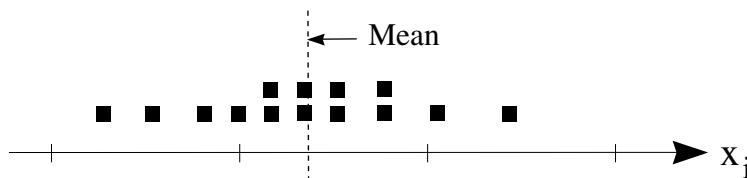
One never needs more than two significant figures to specify an experimental error, and usually a single significant figure is sufficient. The value of the measured quantity should be carried out as many decimal places as necessary so as to include the decimal places used in the experimental error — no more and no less. The following examples illustrate this point.

<i>Correct</i>	<i>Incorrect</i>
$3.4423 \pm 0.0002 \text{ s}$	$3.54073 \pm 0.12 \text{ m}$
$3.54 \pm 0.12 \text{ m}$	$3.5 \pm 0.12 \text{ kg}$
$3.5 \pm 0.1 \text{ kg}$	$3 \pm 0.1 \text{ m/s}$

**As a final note, NEVER use the term “human error” in discussing your experimental error. It is meaningless; which is why so many people are tempted to use it. If you know the origin of the error, say so specifically. If you do not know, it is better to simply say so.**

## MEAN & STANDARD DEVIATION

Very often an experimentalist will measure a quantity repeatedly, and then use an average of the measured values as the final result. Why is that done? what is the justification for it? and what is the uncertainty or error associated with that average? This is the classic example of random error and its treatment. If we plot each individual measurement on an axis with appropriate scale and units, it might look something like this.



**Figure 3** A set of measurements and its mean.

Such a plot provides a clear and graphic description of the uncertainty, or experimental error, associated with this measurement. If the errors in these measurements are truly random, then one would expect that measured values would be larger than the “true value” just as often as they would be smaller, and by similar amounts. This in turn implies that the “true value” lies somewhere in the center of the distribution of measured values. You might choose a best value by simply using your eye to find the center of the distribution. A mathematical procedure that everyone would agree upon would of course be better. There is such a procedure, and it is called calculating the mean, or average. You already know how to do that. In what sense does that find the center of the distribution?

Let us call the value of the center of the distribution  $\bar{x}$ , and define it so that if we add up all the differences between it and the measured data points, we will get zero. (A data point to the left of  $\bar{x}$  will produce a negative difference while a data point to the right of  $\bar{x}$  will produce a positive difference.) This can be expressed mathematically in the following way.

### Definition of the Mean

The mean,  $\bar{x}$ , is defined by the relation

$$\sum_{i=1}^N (x_i - \bar{x}) = 0, \quad (1)$$

where the  $x_i$  are individual measurements.

Since the order in which we do the additions and subtractions does not

affect the outcome, this can be rewritten as

$$\sum_{i=1}^N x_i - \sum_{i=1}^N \bar{x} = 0, \quad \text{but} \quad \sum_{i=1}^N \bar{x} = \bar{x} \sum_{i=1}^N 1 = N\bar{x}$$

$$\sum_{i=1}^N x_i = N\bar{x}, \quad \text{which implies} \quad \bar{x} = \frac{1}{N} \sum_{i=1}^N x_i.$$

This last equation is just the ordinary prescription for calculating an average. Thus we have shown that the familiar average value of a set of measurements coincides with a reasonable definition of the center of the distribution of measured values.

Now the question arises, what experimental error should one associate with each of the measured values in our example? A crude way of doing this would be to choose the uncertainty value to give a range of values just as wide as that covered by the actual measurements. But this would probably overestimate the uncertainty, since a single widely divergent measurement would make the spread quite large, and would not be truly representative of most of the values.

Someone might suggest finding, on average, how much each individual value differs from the mean. But we have already seen that the sum of all the deviations from the mean is zero, and therefore such an average would be zero — not a very helpful experimental error value. This method could be saved by finding the average of the absolute values of the deviations from the mean, thus eliminating all of the negative signs from the negative deviations. Although this method could be used, another method has been practically universally accepted; that of the standard deviation.

### Definition of the Standard Deviation

The standard deviation of the distribution is calculated in the following way.

- a) *Square* each deviation. (This insures that we have only positive quantities to deal with.)
- b) Find the *mean* of these squared deviations.
- c) Take the *square root* of this mean. (We want a measure of the deviations, not their squares.)
- d) The result of these calculations is the standard deviation,  $\sigma$ , also called the root mean square deviation (r.m.s. deviation) for obvious reasons.

Mathematically we can represent this as follows:

$$\sigma = \sqrt{\frac{1}{N} \sum_{i=1}^N (x_i - \bar{x})^2} \quad (2)$$

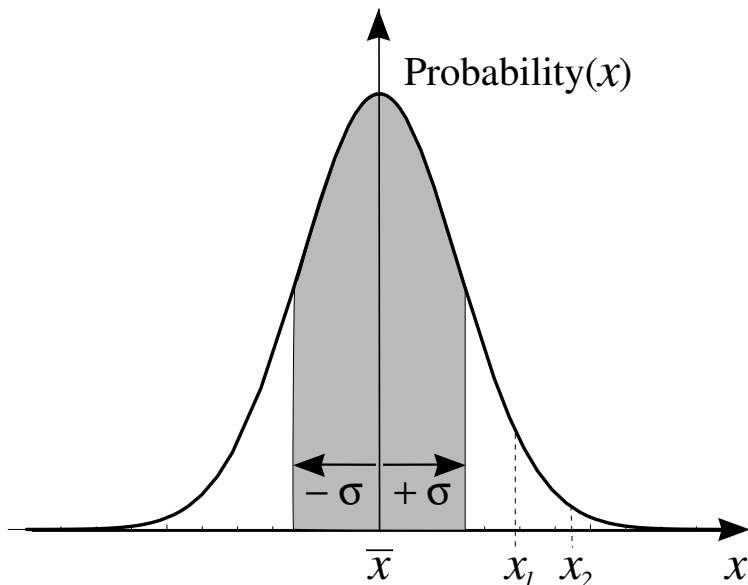
We will use the standard deviation as our best estimate of the random experimental error associated with each of the individual measurements. *That is, each of the individual measurements is uncertain by an amount  $\pm \sigma$ , which we express by writing the value of the  $i$ -th measurement as  $x_i \pm \sigma$ .*

What justification do we have for using the standard deviation in this way? It has been found that the random errors associated with many types of measurements follow what is known as a normal probability distribution, which is illustrated in Fig. 1. The height of the curve above any point on the  $x$ -axis is proportional to the probability of obtaining an experimental result very close to that value of  $x$ . This curve meets our crude expectations that the likelihood of obtaining any measured value will grow less the more the value differs from the average value that lies at the center of the probability curve. The left-right symmetry of the curve insures that measurements are equally likely to be too small or too large.

More precisely, the area under the probability curve between any two  $x$  values,  $x_1$  and  $x_2$ , gives the probability that a measurement will result in a value lying between  $x_1$  and  $x_2$ . The total area under the curve out to  $\pm\infty$  must equal 1, since the probability that any measured value will lie between  $-\infty$  and  $+\infty$  must be 1. But what does this have to do with the standard deviation?

It can be shown that the area under the normal distribution between  $\bar{x} - \sigma$  and  $\bar{x} + \sigma$  is approximately 0.6827. This means that any individual measurement has a probability of about 68% of lying within one standard deviation from the average of many measurements. Similarly it can be shown that any individual measurement (which obeys the normal distribution) has a probability of 95% of lying within two standard deviations from the average and a probability of 99.7% of lying within three standard deviations from the average. Thus by using the standard deviation to represent the uncertainty in our measured values, we are providing some very specific information about how much this value is likely to depart from the true average value.

Now the *average of all our measured values* is certainly less uncertain than the individual measurements. What error shall we associate with the average? It can easily be shown (see notes on propagation of error) that the error to be associated with the average is approximately given by the standard deviation of the measurements divided by the square root of the number of measurements included in the average calculation. We will call this uncertainty for the average  $\sigma_m$ . We can summarize our conventions as follows:



**Figure 4** A normal probability distribution curve. The total area under the curve = 1. Two-thirds of the area lies between  $\bar{x} - \sigma$  and  $\bar{x} + \sigma$ , which implies that  $2/3$  of the measurements should fall within  $\pm\sigma$  of the mean. The probability of obtaining a value between  $x_1$  and  $x_2$  equals the area under the curve between  $x_1$  and  $x_2$ .

If we have  $N$  independent measurements,  $x_i$ , of a quantity  $x$ , then the important quantities and their uncertainties are:

- $x_i \pm \sigma$ , where  $x_i$  is any individual measurement,
- and  $\bar{x} \pm \sigma_m$ , where  $\sigma_m = \sigma/\sqrt{N}$ .

We note that whether or not the individual measurements are normally distributed, the *Central Limit Theorem* of probability theory guarantees us that if we take a very large number of independent measurements, the mean of those measurements will tend toward a normal distribution as the number of measurements increases.

Two words of caution are in order. First, it is a common mistake to think that the standard deviation  $\sigma$  represents the uncertainty in the average. *It does not.* Rather, it represents the uncertainties associated with each of the individual measurements. Second, these statistical methods work well only when  $N$  is large, say 10 or more, and the larger  $N$  is, the better they work. Occasionally we use them for smaller values of  $N$  because we have no other choice. In such cases we must take our conclusions with a grain of salt.

## AGREEMENT OF EXPERIMENTAL RESULTS

Why do we lay such stress on measuring and treating our experimental uncertainties? This becomes quite clear when we go to compare two experimentally derived numbers. The two numbers might represent two different measurements of the same quantity, taken from the same experiment; or they might represent the values arrived at by two different pairs of students in the laboratory; or they might represent your experimentally determined value and a value measured and published in the literature by a professional physicist. The two numbers will almost invariably be different. What does that mean? They don't agree? Does "agreement" mean identical? You will realize immediately that this cannot be the case. Then what does it mean for values to be in agreement?

For our purposes, the following crude criteria will be sufficient to draw conclusions about the agreement (or lack thereof) of two measured values.

1. **Agreement:** Two values are in agreement if either value lies within the error range of the other.
2. **Disagreement:** Two values disagree if their respective error ranges do not overlap at all.
3. **Inconclusive:** No clear conclusion can be drawn if the ranges overlap, but neither value lies within the range of the other.

More correctly, what one should check is that the difference between two values is consistent with zero. This requires that one use propagation of error to find the error for the difference of two values. *If that error is greater than the difference of the values, then the results are said to be in agreement.*

The following examples and diagrams will help to make this idea clearer.

### Values in agreement:

$$\begin{aligned}x_A &= 5.7 \pm 0.4 \\x_B &= 6.0 \pm 0.2\end{aligned}$$

Here  $x_B$  lies within the uncertainty range of  $x_A$ , though  $x_A$  does not lie within the uncertainty range of  $x_B$ . Note that

$$x_B - x_A = (6.0 - 5.7) \pm \sqrt{(0.4)^2 + (0.2)^2} = 0.3 \pm 0.4$$

is consistent with zero.

**Values in disagreement:**

$$\begin{aligned}x_A &= 3.5 \pm 0.5 \\x_B &= 4.3 \pm 0.2\end{aligned}$$

The two uncertainty ranges do not overlap. Note that

$$x_B - x_A = (4.3 - 3.5) \pm \sqrt{(0.5)^2 + (0.2)^2} = 0.8 \pm 0.5$$

is not consistent with zero.

**Inconclusive:**

$$\begin{aligned}x_A &= 8.3 \pm 0.3 \\x_B &= 8.8 \pm 0.3\end{aligned}$$

Neither value lies within the uncertainty range of the other, although the two uncertainty ranges do overlap. Note here that

$$x_B - x_A = (8.8 - 8.3) \pm \sqrt{(0.3)^2 + (0.3)^2} = 0.5 \pm 0.4$$

is not consistent with zero, but not by much.

So we see that no experimental measurement can be complete until its experimental uncertainty is determined. Otherwise, the measurement is of little or no use to us.





# Notes on Propagation of Errors

We have learned something about measuring and estimating the uncertainties in experimentally measured values. We know that this is important because it allows us to compare two different measurements of the same quantity. It is also important because it allows us to calculate the uncertainties in other quantities which are calculated from the directly measured quantities. In what follows we will determine how to calculate the errors in a function  $G(x, y)$  of the experimentally measured values  $x$ , and  $y$ .

## SUMS OF MEASUREMENTS

For example, suppose that  $G(x, y) = x + y$ , where  $x$  and  $y$  are experimentally measured quantities with uncertainties  $dx$  and  $dy$  respectively. What is the uncertainty,  $dG$ , in the value of  $G$  that is calculated from the experimentally measured values for  $x$  and  $y$ ? The largest and smallest values for  $G$  would be:

$$\begin{aligned} G_{\max} &= (x + dx) + (y + dy) = x + y + (dx + dy) \quad \text{and} \\ G_{\min} &= (x - dx) + (y - dy) = x + y - (dx + dy) . \end{aligned}$$

In this case we could write:  $dG = (\pm)(dx+dy)$ . Graphically the uncertainty in  $G$  could be expressed as follows:

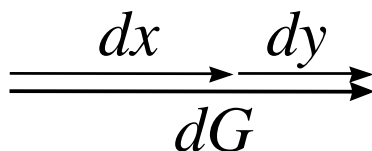


Figure 1

However, in order for the value of  $G$  to be this far off, both  $x$  and  $y$  would have to be in error by their maximum amounts and in the same sense; i.e.,  $dx$  and  $dy$  both positive or both negative. If the errors in  $x$  and  $y$  had

opposite signs something like this would happen:

$$G = (x + dx) + (y - dy) = x + y + (dx - dy) .$$

Here the error in  $G$  is smaller than in the previous case, and could be zero if  $dx = dy$ . This case can be represented graphically as follows:

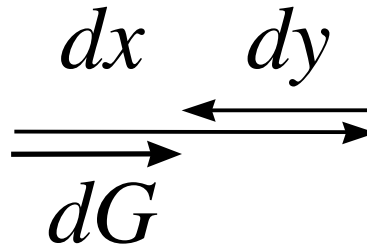


Figure 2

Statistically speaking, it is unlikely that the error in  $G$  will be either as large as the first assessment or as small as the second. The truth must lie somewhere in between. Our rule for quantities calculated from sums (and also differences by the way) will be:

$$dG^2 = dx^2 + dy^2 .$$

Graphically this can be represented as “error vectors” being added at right angles to each other rather than parallel or anti-parallel:

**If  $G(x, y) = ax \pm by$ , then the error in  $G$  is given by:**

$$\boxed{dG^2 = (a dx)^2 + (b dy)^2} . \quad (1)$$

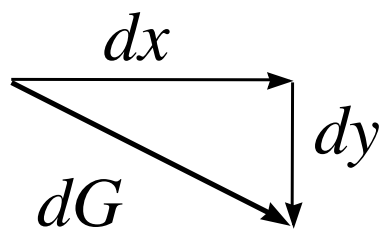


Figure 3

## PRODUCTS AND POWERS OF MEASUREMENTS

Suppose that  $G(x, y) = Kx^p y^q$ , where  $p$  and  $q$  can be any positive or negative numbers. For example, if we were considering the relation

$$v = R\sqrt{\frac{g}{2h}} ,$$

which could be rewritten as  $v = (\sqrt{g/2})R^1h^{-1/2}$ , where  $R$  and  $h$  are the experimentally measured quantities (corresponding to  $x$  and  $y$ ), then  $p = 1$ ,  $q = -1/2$ , and  $K = \sqrt{g/2}$ .

The rule in this case will be quite similar to that for sums of measurements, but in this case it is the *fractional* uncertainties in  $G$ ,  $x$ , and  $y$  ( $dG/G$ ,  $dx/x$ , and  $dy/y$ ) that will be important.

**When  $G(x, y) = Kx^py^q$ , then the fractional uncertainty in  $G$  is given by**

$$\boxed{\left(\frac{dG}{G}\right)^2 = \left(p \frac{dx}{x}\right)^2 + \left(q \frac{dy}{y}\right)^2} \quad (2)$$

### An Example Computation

As an example of the application of these rules, we consider finding the uncertainty in the average velocity over an interval when we know the uncertainties in the length and the time.

Suppose that the length of the interval is  $x = 1.752 \pm 0.001$  m and the time is  $t = 0.7759 \pm 0.0001$  s. The average velocity is  $\bar{v} = x/t = 2.25802$  m/s, which we must truncate to four significant figures, or  $\bar{v} = 2.258$  m/s. The uncertainty is calculated from Eq. (2) above:

$$\begin{aligned} \left(\frac{d\bar{v}}{\bar{v}}\right) &= \sqrt{\left(\frac{dx}{x}\right)^2 + \left(\frac{dt}{t}\right)^2} \\ \left(\frac{d\bar{v}}{2.25802 \text{ m/s}}\right) &= \sqrt{\left(\frac{0.001 \text{ m}}{1.752 \text{ m}}\right)^2 + \left(\frac{0.0001 \text{ s}}{0.7759 \text{ s}}\right)^2} = 0.000585146. \end{aligned}$$

Thus, the uncertainty in the velocity is  $dv = 0.000585146 \times 2.25802$  m/s = 0.00132127 m/s. **Keeping only the correct number (four) of significant figures and correctly rounding the uncertainty**, we would report the average velocity as  $2.258 \pm 0.001$  m/s.

### An Application: Uncertainty in the Mean

Both rules, for sums and products, are easily generalized to cases with more than two independent variables, i.e.,  $x$ ,  $y$ ,  $z$ , ... etc.

The rule for  $dG$  when  $G$  is a sum or difference has a nice application to the uncertainty in an average value of several measurements. Suppose that  $G_{\text{ave}} = (g_1 + g_2 + g_3 + \dots + g_N)/N$ . Then by Eq. (1) the uncertainty in

$G_{\text{ave}}$  is given by:

$$(dG_{\text{ave}})^2 = \frac{1}{N^2} (dg_1^2 + dg_2^2 + dg_3^2 + \dots + dg_N^2) . \quad (3)$$

But  $(1/N)(dg_1^2 + dg_2^2 + dg_3^2 + \dots + dg_N^2)$  is the mean square deviation,  $\sigma^2$ , of the  $g$  measurements, or the square of the standard deviation of the  $g$  measurements. Using this identification, we can transform Eq. (3) to:  $(dG_{\text{ave}})^2 = \sigma^2/N$ , and therefore the uncertainty in the average,  $dG_{\text{ave}}$ , which we sometimes call  $\sigma_m$ , is

$$dG_{\text{ave}} = \frac{\sigma}{\sqrt{N}} . \quad (4)$$

This confirms our feeling that the uncertainty in an average value should be less than the uncertainties in the individual measurements (otherwise, why take averages on repeated measurement?), and confirms the formula for the uncertainty in a mean which was given without proof in the Notes on Experimental Errors.

We summarize these notes with a table showing the propagation of errors for a few simple cases.

**Table 1** Propagation of Error for a few simple algebraic expressions.

Expression	Error in Expression
$ax \pm by$	$\sqrt{(a \, dx)^2 + (b \, dy)^2}$
$xy$	$\sqrt{(x \, dy)^2 + (y \, dx)^2}$
$\frac{x}{y}$	$\frac{x}{y} \sqrt{\left(\frac{dx}{x}\right)^2 + \left(\frac{dy}{y}\right)^2}$
$\sqrt{ax}$	$\frac{1}{2} \sqrt{ax} \left  \frac{dx}{x} \right $
$x^2$	$2  x \, dx $

# Notes on Graphing

## GRAPHICAL REPRESENTATION AND ANALYSIS

Graphs are used scientifically for two purposes. The first is simply to represent data in a form that can be understood visually. That is, a graph shows schematically how one variable depends on another. This is the usage that is most familiar to most people. Perhaps the most common graph of this kind is that of the Dow Jones Industrial Average versus time. One might look at such a graph to see whether stock prices were falling throughout the day or to see how much volatility there was in the Dow. Beyond these general trends, most people do not look further, except perhaps to see if the low or high for the day reached or exceeded some particular level. The size of such a graph does not particularly matter for such purposes so newspaper editors generally print the graph small to save space.

The second and less common use of a graph is to analyze data. This is a more sophisticated use of graphing and more stringent criteria apply to making and using such graphs. In a graph of the Dow Jones Industrial Average versus time, it is not so important to be able to read the value of the Dow to great precision. This is not the case with a scientific graph with which one will analyze data. Just as most students are reluctant to leave out any of the ten digits in a calculated result (even though they must according to the rules of significant figures!), a good scientist ought to be reluctant to lose any precision in analyzing data graphically.

## LINEARIZING THE DATA

In analyzing experimental data, we most often will be interested in finding a representation of the data that results in a **linear** relationship between the quantities. This is because our visual system is acutely sensitive to lines and fairly bad at distinguishing other curves. To the eye, the crest or trough of a sine curve looks identical to a parabola until they are laid on top of one another. In addition, it is much more difficult to find the curvature of a parabola, for instance, than it is to find the slope of a line!

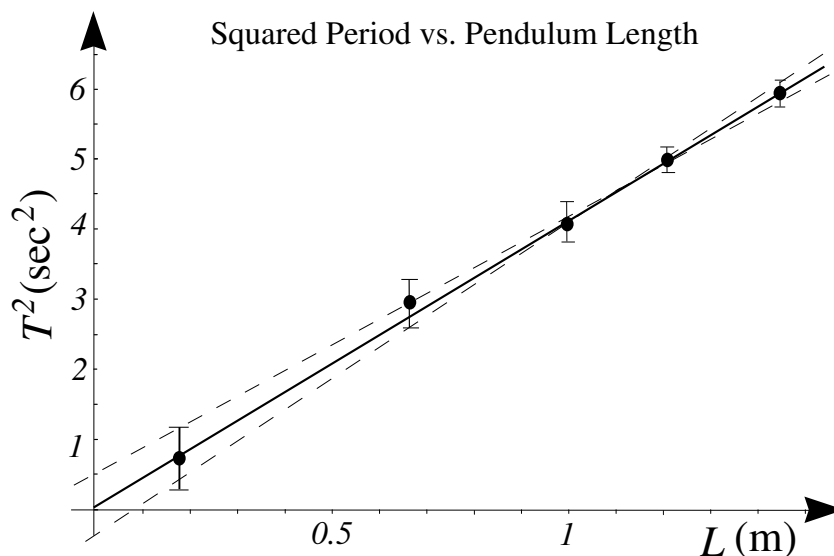
The two values that are of interest in a linear graph are the slope and the  $y$ -intercept. For instance, the period of a pendulum,  $T$ , and its length  $L$  are related theoretically by

$$T = 2\pi\sqrt{L/g}$$

when the amplitude of the motion is very small. If we were to test this relationship experimentally by graphing data, we would square the equation to get

$$T^2 = \left(\frac{4\pi^2}{g}\right)L. \quad (1)$$

Eq. (1) indicates that if we were to graph  $T^2$  versus  $L$ , we should find a line with slope  $4\pi^2/g$ . In fact, this would be a reasonable way of measuring the acceleration of gravity,  $g$ . (Recall that when we say a graph of  $y$  versus  $x$ , we mean a graph with  $y$  along the vertical axis and  $x$  along the horizontal axis.) According to Eq. (1) we should also find an intercept of zero: when  $L$  is zero,  $T^2$  should also be zero. However, we might find that there is a non-zero intercept, which would indicate that perhaps we had not measured the length of the pendulum correctly: we may have a non-zero offset.



**Figure 1** A well-drawn graph of data for the squared period of a pendulum versus length of the pendulum. The solid line is the best-fit line. The dashed lines are lines of good fit. Note that the line and the data points fill the whole range of the graph.

## MAKING GOOD GRAPHS

The first rule of scientific graphing is to make the graph easy to read accurately and precisely. This means that **graphs must be made as large as practicable**. In practice you should make the graph as large or nearly as large as your sheet of graph paper will allow, i.e., the the whole or nearly the whole sheet. Included in this is the requirement to make the range of the axes no larger than is necessary; you want your line to fill up the whole range. Just as it is more difficult to measure very small objects, it is difficult to read very small graphs accurately. In addition, graphs made very small lack the precision of larger graphs simply due to the size of the mark made by even a sharp pencil. In addition to being as large as practicable, a graph must be labeled in such a way as to be easy to read quickly and accurately. This means that **the axes must be labeled in a simple and regular fashion. It is best to use 1, 2, or 5 divisions to represent a decimal value.** That way the graph is easiest to read and to make; one need not guess where a value of 1.50 lies between marks at 0.85 and 1.70!

MAKE LARGE GRAPHS

NUMBER AXES SIMPLY

In addition to being of good size and having well labeled axes, to be complete graphs must have a descriptive title and the quantities being graphed must be labeled on the axes with their proper units.

TITLE GRAPHS,  
LABEL AXES,  
USE UNITS

Figure ?? shows a good graph with all the features necessary to a good graph. These good features include

ATTRIBUTES OF A  
WELL-DRAWN GRAPH

- Large size;
- Range of the axes just covers that of data;
- Well numbered axes;
- Short and Clear Title;
- Axes labeled with the appropriate quantity;
- Units on the axes;
- Data points with uncertainties (error bars);
- A best-fit line;
- One or two alternate good-fit lines.

## MEASURING THE SLOPE OF A LINE

In most of the graphs that we will be using in the introductory laboratory the most important piece of information will be the slope of the “best-fit line.” There are many ways of obtaining this slope. One way is to use

the formulas at the end of these notes into which one may substitute the coordinates  $(x_i, y_i)$  and errors  $\sigma_i$  in  $y_i$  of the data points. This is best done with a spreadsheet as it is tedious and error-prone to do by hand. If the errors  $\sigma_i$  are all of the same size, one might use Microsoft Excel to calculate the least-squares best-fit slope and intercept. (Using Excel's best-fit line function is not recommended in general because it is not correct if there are large error bars on some data points and not on others.) Last, one may use a hand-drawn graph that is carefully made. On the hand-drawn graph there should be a best-fit line to the data. It is this line whose slope we wish to calculate. Note that the best-fit line does not necessarily go through any of the data points so it is not correct to use the data points to calculate its slope.

### Why use widely separated points?

CHOOSE POINTS FAR  
APART TO MINIMIZE  
ERROR IN MEASURING  
THE SLOPE

The slope of the line is given by the familiar rule "rise over run." Now, mathematically, it does not matter which two points you choose on a line to calculate slope; any two points will give the same answer. However, **we cannot read the coordinates of the points to arbitrary precision, hence there is inherent error in measuring the slope from an actual graph.** The rise is difference in the heights of the two points,

$$\Delta y = y_2 - y_1 ,$$

so the error in the rise is found from the uncertainty in the  $y$ -coordinates themselves,

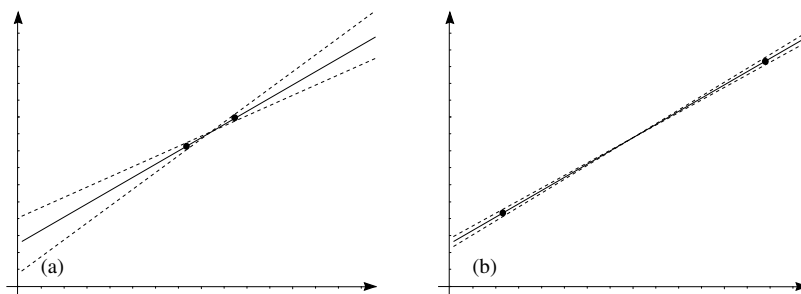
$$\sigma_{\text{rise}} = \sigma_{\Delta y} = \sqrt{\sigma_{y_2}^2 + \sigma_{y_1}^2} ,$$

and is independent of  $\Delta y$ . **The uncertainty in the rise depends only on the uncertainty in reading the points on the line, which is minimized by making the graph as large as practicable.** Similar results hold for the run,  $\Delta x$ . The slope, being the ratio of rise to run, has an error

$$\frac{\sigma_{\text{best fit slope}}}{\text{best fit slope}} = \sqrt{\left(\frac{\sigma_{\text{rise}}}{\text{rise}}\right)^2 + \left(\frac{\sigma_{\text{run}}}{\text{run}}\right)^2} . \quad (2)$$

It is useful to pause here to think about Eq. (2). To minimize the error in the slope, we should minimize the errors in the rise and run, but we should also **maximize the rise and the run themselves**. The rise and run are maximized when the points on the line taken to compute the slope are **as far apart as possible** while still being on the graph. In practice, one will not choose the points to be way at the ends of the line, but rather one will choose points quite far apart but lying near the crossing of two lines on the graph paper so that their coordinates may be read accurately. A graphical illustration of the Eq. (2) is shown in Fig. ??.





**Figure 2** The separation of the points chosen on a line for the computation of the slope has a strong effect on the error in the computed slope. In figure (a) the points, whose widths are exaggerated for clarity, are close together, resulting in a large error in the slope. In figure (b) the points are far apart, resulting in a more precise determination of the slope.

### Finding the range of slopes for lines of good fit

It is important to point out that the uncertainty given in Eq. (??) is the uncertainty in measuring from your graph the actual slope of the best-fit line. This is logically different from quoting a range of values for lines of good fit! In other words, even if all the points were to lie perfectly along a line, there is a limitation in finding the slope of that line due to the fact that graphing is a physical process and reading the graph has an uncertainty associated with it. In practice, however, the points generally will not lie along a line and there may be many lines that have a reasonable fit to the data. We'd like to find the range of the slopes for these lines that fit the data reasonably well. The procedure in this case is to draw two more lines in addition to the best-fit line. One line will have the largest possible slope and still fit the data reasonably well and the other will have the smallest possible slope and still fit the data reasonably well. Example lines are drawn Figure ?? . To find the quoted error in the slope of the best-fit line, take half the difference in the slopes of the two outlying dashed lines. For example, if the slope of the best-fit line in the figure is  $4.00 \text{ sec}^2/\text{m}$ , the slope of the steeper dashed line is  $4.16 \text{ sec}^2/\text{m}$ , and the slope of the shallower dashed line is  $3.90 \text{ sec}^2/\text{m}$ , you would quote the slope as  $4.00 \pm 0.13 \text{ sec}^2/\text{m}$ , because  $(4.16 - 3.90)/2 = 0.13$ . In a pinch, you could draw just one other good-fit line and then the error in the slope is just the difference between the best-fit line slope and the slope of the good-fit line.

ERROR IN REPORTING  
THE SLOPE OF THE  
BEST-FIT LINE

### NUMERICALLY FINDING INTERCEPTS ON UNFORTUNATE GRAPHS

Sometimes students spend a great deal of time making a careful graph only to find that they forgot to start their horizontal axis at zero so that they could find the  $y$ -intercept. In these cases, it is necessary either to redraw that carefully drawn graph or to use algebraic means to find the intercept.

Usually both the slope and the intercept are needed for the analysis. Here we assume that two well-separated points,  $A$  and  $B$ , on the best-fit line have been used to find the slope of the best-fit line. The equation of the line then says that any point on the line, when used with either of the points  $A$  or  $B$  will give the same slope:

$$\frac{y - y_A}{x - x_A} = \frac{y_B - y_A}{x_B - x_A} = \text{slope}. \quad (3)$$

Eq. ?? can be rearranged to yield

$$y - y_A = \frac{y_B - y_A}{x_B - x_A} (x - x_A), \quad (4)$$

or

$$y = \left( \frac{y_B - y_A}{x_B - x_A} \right) x + \left( y_A - \left( \frac{y_B - y_A}{x_B - x_A} \right) x_A \right). \quad (5)$$

Thus the intercept of the best-fit line can be found by computing

$$\text{intercept} = y_A - (\text{slope}) x_A, \quad (6)$$

where  $(x_A, y_A)$  are the coordinates of any point on the best-fit line.

## LEAST SQUARES METHOD

If graphing seems too imprecise to you or you just like algebra better, there is a method that will give you the slope and the intercept of the best-fit line directly without graphing. Of course, you will forgo the benefits of seeing the relationship between the data points and the visual estimate of the goodness of fit, but you will also avoid the artistic perils of graphing.

The idea is that the best-fit line ought to minimize the sum of the squared vertical deviations of the data from the line. That is, you want a line  $y = mx + b$  that has the least value for

$$\Delta(m, b) \equiv \sum_{i=1}^N (y_i - (mx_i + b))^2.$$

If the data points have vertical error bars that are  $\pm \sigma_i$  for data point  $(x_i, y_i)$ , then the correct quantity to minimize is chi-squared, or

$$\chi^2 \equiv \sum_{i=1}^N \left( \frac{y_i - (mx_i + b)}{\sigma_i} \right)^2. \quad (7)$$

The conditions for a minimum are the simultaneous equations

$$\frac{\partial \chi^2}{\partial m} = 0, \quad \frac{\partial \chi^2}{\partial b} = 0. \quad (8)$$

The algebra is not difficult to do and the result of minimizing with respect to both  $m$  and  $b$  yields values for the slope and the  $y$ -intercept as

$$m = \frac{S_1 S_{xy} - S_x S_y}{S_1 S_{x^2} - S_x^2}, \quad (9)$$

$$b = \frac{S_y S_{x^2} - S_x S_{xy}}{S_1 S_{x^2} - S_x^2}, \quad (10)$$

where the quantities  $S_1$ ,  $S_x$ ,  $S_y$ ,  $S_{xy}$ ,  $S_{x^2}$ , and  $S_{y^2}$  are defined by

$$\begin{aligned} S_1 &= \sum_{i=1}^N \sigma_i^{-2}, & S_x &= \sum_{i=1}^N \sigma_i^{-2} x_i, \\ S_y &= \sum_{i=1}^N \sigma_i^{-2} y_i, & S_{xy} &= \sum_{i=1}^N \sigma_i^{-2} x_i y_i, \\ S_{x^2} &= \sum_{i=1}^N \sigma_i^{-2} x_i^2, & S_{y^2} &= \sum_{i=1}^N \sigma_i^{-2} y_i^2. \end{aligned} \quad (11)$$

The errors in the slope and intercept are given by<sup>1</sup>

$$dm \approx \sqrt{\frac{S_1}{S_1 S_{x^2} - S_x^2}}, \quad (12)$$

$$db \approx \sqrt{\frac{S_{x^2}}{S_1 S_{x^2} - S_x^2}}. \quad (13)$$

The goodness of fit can be found by evaluating the correlation coefficient

$$R = \frac{S_1 S_{xy} - S_x S_y}{\sqrt{(S_1 S_{x^2} - S_x^2)(S_1 S_{y^2} - S_y^2)}}, \quad (14)$$

in the notation of Eqs. (??). When the  $R = 0$ , there is essentially no linear relationship present, while a value of  $R = 1$  occurs for perfect correlation, or all points lying on a line.

A spreadsheet implementing these formulas can be found at <http://people.hws.edu/tjallen/Physics150/BestFitLine.xlsx>

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<sup>1</sup>P.R. Bevington and D.K. Robinson, *Data Reduction and Error Analysis for the Physical Sciences*, 3rd Ed., (McGraw-Hill, 2003).



