Notes on the bending of beams

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1 Introduction

When a beam is subject to a load, it deforms. The purpose here is to determine the shape of a deformed beam from its geometry and material properties as well as the loads applied to the beam.

As it deforms, a beam develops stresses within it that keep each section of the beam in equilibrium. Both shear and normal stresses are present. In these notes, I assume that the loads are such that the deformation of the beam is small and that vertical planes through the cross-section of the beam remain planes, though they are slightly rotated from their original vertical orientation. The normal stresses are such that they exert a torques on any section of the beam. These torques are balanced by other torques caused by the supports and the loads.

In the following sections the shape of the beam's deflection is found from considerations of the state of stress and strain in the beam and the requirements of equilibrium of any section of the beam. The analysis starts with a calculation of the strain in the beam by considering any small length of the beam to be a small arc of a circle. The radius of that circle at any point, also called the radius of curvature, will be used to compute the axial strain of the beam at that point. There are both compressions and extensions of the material. One consequence of there being both compressions and extensions is that there is a surface within the beam, called the neutral surface, that does not change its length and so remains unstrained.

The normal stress on any face of the beam can be found from the strain at that face. Because the net horizontal force on any section of the beam vanishes, the neutral surface will be seen to pass through the center of gravity of any face of the beam. The net torque exerted on a face of the beam by the normal stresses acting there is determined not only by the radius of curvature at that point and the Young's modulus of the material, but also by the shape of the cross-section of the beam through its "areal moment," a quantity that not only depends on the area, but also its shape.

If one can find the torque due to normal stresses on any face of the beam, one can find the radius of curvature of the beam at that face. When the deflection of the beam is small, the reciprocal of the radius of curvature of the beam, called simply the curvature, is very nearly the second derivative of the deflection of the beam with respect to the distance along the beam. This equation can be used to find the deflection of the beam. At the end I find the deflections for four cases, a beam supported at both ends and a cantilever supported at one end, both under a point load and a uniformly distributed load.

2 State of stress and strain of the beam

2.1 Elementary considerations

Consider a beam of length L as shown in Fig. 1, which supports a load P at its center. If we look at the free body diagram of a section of the beam from the left end to a point a distance x from that end, as depicted in Fig. 2, we find that there not only must be a shear stress downward on the rightmost face of that section of the beam, but that there also must be a normal stress on that rightmost face of the section of the beam in order to provide a counterbalancing torque to the torque caused by the support at the end. To compute that normal stress, we must look at the deformation of the beam and the strain induced by it.

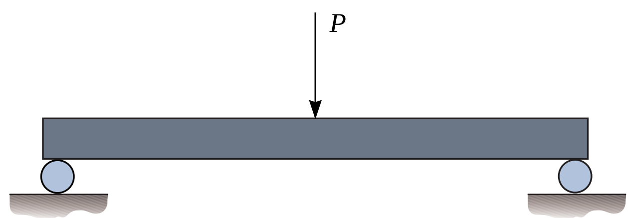

Figure 1 A beam supported at both ends subjected to a point load P at its center.

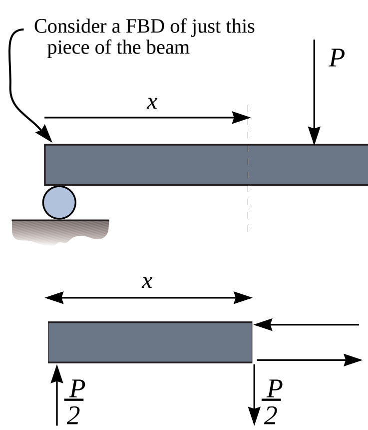

Figure 2 The free body diagram of the leftmost part of the beam shows that there must not only be a shear stress on the face of the section, but also a normal stress in order to supply a balancing torque.

2.2 Strain

By considering a very short piece of the beam, we can arrive at the correct strain in the beam. Figure 3 shows the piece of the beam as a greatly exaggerated arc of a circle of radius R. The figure is greatly exaggerated in that we are actually considering that the radius R is very large and the angle θ is very very small so that the arc length $R\theta$ of the section of the beam is also very small, much smaller than a centimeter.

We already know that for the case shown in Fig. 1, the lower edge of the beam is under a state of tension, while the upper edge of the beam is under a state of compression. Because the sign of the strain changes from top to bottom, there must be a surface through the beam that does not stretch or compress. This surface is the **neutral surface**. We will see that the neutral surface passes through the center of gravity of the cross-sectional face of the beam.

If z is the vertical distance of a point in the cross-sectional face of the beam, the strain at that point is

$$\varepsilon = -\frac{z}{R}. \tag{1}$$

We can see that the strain given in Eq. (1) has the correct units by noting that it is unitless, as strains must be. It also increases in magnitude linearly in distance from the neutral surface. It is not difficult to prove Eq. (1) by looking at each section of the small piece of beam at a fixed value of z as the arc of a circle. The radius of the circle corresponding to the neutral surface is R. The uncompressed arc length of the surface at a given z in the beam is the same as the length of the arc of the neutral surface,

$$\ell_0 = \theta R. \tag{2}$$

A distance z above the neutral surface the radius of the arc is $R - z$, so that the change in length of that arc from its uncompressed length is

$$\Delta\ell = \theta(R - z) - \theta R = -\theta z, \tag{3}$$

so that the strain becomes

$$\varepsilon = \frac{\Delta\ell}{\ell_0} = -\frac{\theta z}{\theta R} = -\frac{z}{R}. \tag{4}$$

2.3 Normal Stress

The normal stress at that point is simply Young's modulus (also called the Elastic Modulus) times the strain.

$$\sigma = Y\varepsilon = -\frac{Yz}{R}. \tag{5}$$

Equation (5) shows that the normal stress vanishes at the neutral surface and increases linearly with vertical distance from the neutral surface. The line where the neutral surface meets the face of the section of beam is called the **bending axis**. Figure 4 shows the cross-section of the beam and the normal stress on the face.

The net horizontal force on the beam must vanish. To compute the net horizontal force, we break up the face into small areas on which the stress is approximately constant and then we sum up the resulting forces, which are stresses σ times areas ΔA. In the limit that the

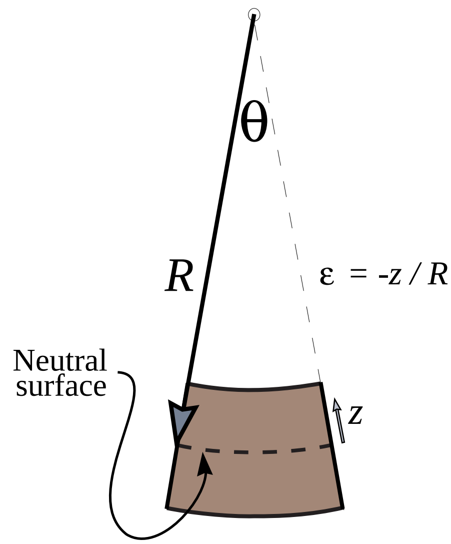

Figure 3 A small section of the beam may be considered to be a short arc of a circle, shown here as highly exaggerated. There is one part of the beam that has an unchanged length, called the "neutral surface." Above the neutral surface the beam is compressed, and below it, stretched.

small areas become vanishingly small – and their number becomes correspondingly great – we are left with an integral of the stress over the cross-sectional area of the face.

$$\text{net horizontal force} = \sum_{\text{face}} \sigma_i \, \Delta A_i \longrightarrow \int_{\text{face}} \sigma \, dA = 0. \tag{6}$$

When we substitute Eq. (5) into Eq. (6), the resulting equation (7) implies that the neutral surface must pass through the center of gravity of the cross-sectional face of the beam.

$$\int_{\text{face}} z \, dA = 0. \tag{7}$$

It may not be obvious at first sight that Eq. (7) implies that the neutral surface must pass through the center of gravity of the cross-sectional face. If one were to treat the small areas ΔA as having a mass proportional to their areas, then the vertical position of the center of gravity would be

$$z_{\text{C.M.}} = \frac{\sum_{\text{face}} z_i \Delta A_i}{\sum_{\text{face}} \Delta A_i} = \frac{\int_{\text{face}} z \, dA}{\int_{\text{face}} dA} = 0. \tag{8}$$

As we are measuring the value of z from the neutral axis, the center of gravity is at $z = 0$, or at the neutral surface.

2.4 Net torque on a cross-sectional face of the beam

We see from Fig. 4 that the normal stress exerts a net torque on the cross-sectional face of the beam, just as we surmised should happen in Fig. 2.

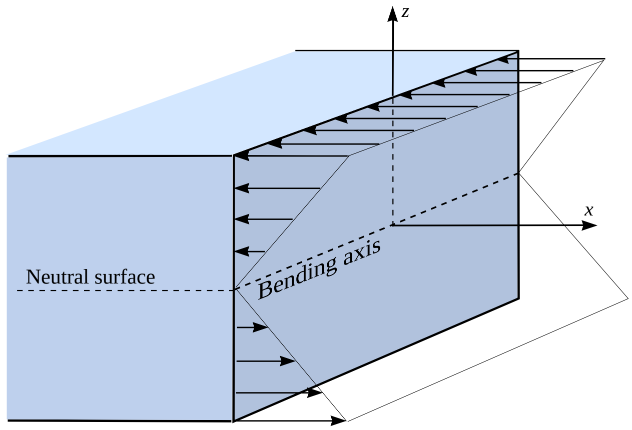

Figure 4 The normal stresses on the cross-section of the beam vanish at the neutral surface and increase linearly in magnitude away from it.

The net torque about the bending axis is simply the force on a small area of the cross-section times the distance z of the small area from the bending axis, which is the lever arm. The geometry of the torques is shown in Fig. 5. The net counterclockwise torque can be expressed as

$$\text{net torque} = \tau = \sum_{\text{face}} z\sigma\,\Delta A \longrightarrow \int_{\text{face}} z\sigma\,dA. \tag{9}$$

Substituting the expression (5) in for the stress σ, we find that the net counterclockwise torque is

$$\tau = \int_{\text{face}} z\sigma\,dA = \frac{Y}{R}\int_{\text{face}} z^2\,dA \equiv \frac{Y\,I_A}{R}. \tag{10}$$

The integral of the square of the distance from the bending axis over the cross-sectional area,

$$I_A = \int_{\text{face}} z^2\,dA, \tag{11}$$

is known as the **areal moment**, which you can find tabulated for variously shaped beam cross-sections, but is also fairly easy to compute as an integral using the methods of Calculus II. The areal moments of a rectangular and circular cross-sectional area are given in Table 1.

Shape	I_A
Rectangular $w \times h$	$w\,h^3/12$
Circular	$\pi r^4/4$
Hollow Rectangle $w \times h$, wall thickness t	$t(h^3 + 3w\,h^2)/12$

Table 1 Areal moments for three different cross-sectional areas.

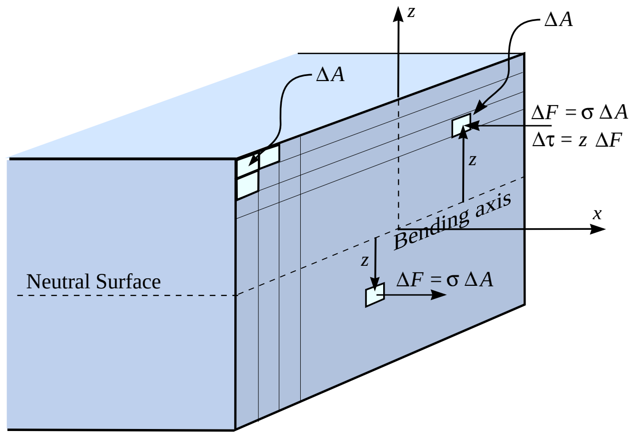

Figure 5 To compute the forces and torques on the cross-sectional face of the section of beam, the area is broken up into small rectangular regions ΔA over which the stress is relatively constant. The amount of force on that small region is $\Delta F = \sigma \Delta A$. The contribution of that force to the total torque about the bending axis (or any other axis) is $\Delta\tau = -z\sigma\Delta A$. The minus sign comes about because the force is taken positive to the right, out of the surface.

3 Bending

3.1 Bending Equation

Curiously, Eq. (10) is the key to finding the shape of a deflected beam. We can rearrange the equation to find the reciprocal of the radius of curvature of the beam, otherwise known as the curvature of the beam,

$$\frac{1}{R} = \frac{\tau}{Y I_A}. \tag{12}$$

The curvature is the key to finding the shape, as it is related to the deflection of the beam, $y(x)$, as function of position x along the beam. Figure 6 shows the meaning of the deflection as a function of position along the beam. In the limit of small deflections, when the slope of the deformed beam, dy/dx, is much smaller than 1, the curvature is just the second derivative of the deflection with respect to position.

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left(1 + (\frac{dy}{dx})^2\right)^{3/2}} \approx \frac{d^2y}{dx^2}. \tag{13}$$

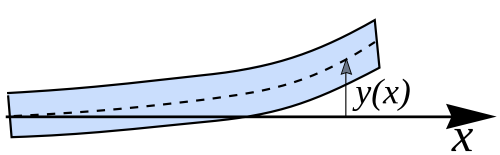

Figure 6 The vertical deflection of the neutral surface from the horizontal at a distance x from the left end is denoted by $y(x)$.

To find the shape of the beam under load, one looks at a free body diagram of the leftmost part of the beam which is x long and finds the required torque $\tau = \tau(x)$ on the rightmost face for equilibrium. This torque then is placed into the combined equations (12) and (13),

$$\frac{d^2y}{dx^2} = \frac{\tau(x)}{YI_A}, \tag{14}$$

and the resulting Eq. (14) is then solved for the deflection $y(x)$, subject to physically reasonable assumptions on the displacement. These physically reasonable assumptions are known as **boundary conditions**. Typically, the assumptions are that the beam does not deflect where it is supported, or if the beam is a cantilever supported at one end by being held in a larger structure, such as a wall, that does not allow the beam to rotate at its point of support, the beam is further assumed to be horizontal at the point of support.

4 Four examples

4.1 Bending of a beam by a point load

Here we consider the bending of a beam supported at each end by a point load at the center of the beam, first shown in Fig. 1. The first step is to find the torque on a face of the beam that is a distance x from the leftmost end. We draw a free body diagram of a section of the beam and compute torques about the lower right hand corner of the beam section, shown in Fig. 7. The net counter-clockwise torque, which must vanish, is

$$\tau - Px/2 = 0. \tag{15}$$

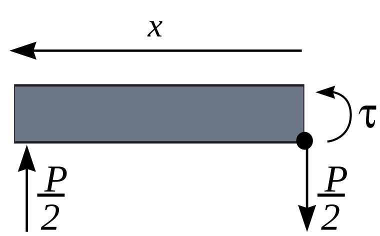

Figure 7 Free body diagram of the leftmost section of the beam.

The shape of the deflected beam is thus determined by the solution to the bending equation, Eq. (14), specialized to this case:

$$\frac{d^2y}{dx^2} = \frac{Px}{2YI_A}. \tag{16}$$

Equation (16) can be solved by integrating it twice with respect to x, yielding

$$y(x) = \frac{Px^3}{12YI_A} + C_1x + C_2, \tag{17}$$

where C_1 and C_2 are constants of integration, which must be determined by other considerations. The reasonable assumptions are that the beam does not deflect over its supports, $y(0) = 0$ and that it reaches its maximum deflection underneath the point load, which is at the center of the beam, or $x = L/2$. This last consideration is equivalent to $dy/dx = 0$ at $x = L/2$.[1] These conditions determine that

$$C_1 = -\frac{PL^2}{16YI_A}, \tag{18}$$

$$C_2 = 0. \tag{19}$$

The deflection of the beam between $x = 0$ and $x = L/2$ is then

$$y(x) = \frac{Px}{YI_A}\left(\frac{x^2}{12} - \frac{L^2}{16}\right). \tag{20}$$

As the beam is symmetric, and is loaded symmetrically, the deflection beyond $x = L/2$ is just the reflection through the midpoint of the deflection to the left of the load. The maximum deflection of the beam occurs at its center, $x = L/2$; this maximum deflection is

$$y_{\max} = -\frac{PL^3}{48YI_A}. \tag{21}$$

The exaggerated shape of the deflection of the beam is accurately shown in Fig. 8.

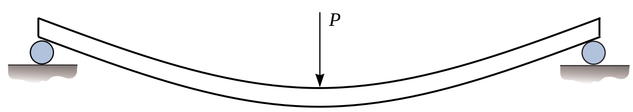

Figure 8 The exaggerated shape of a beam of uniform cross-section supported at its ends and loaded at its center.

4.2 Bending of a beam by a uniformly distributed load

Figure 9 shows a beam supported at the ends under a uniform load. Such a beam is slightly more difficult to analyze. The analysis proceeds as before but with a slightly different free body diagram, which is shown in Fig. 10.

The downward uniformly distributed load is Px/L. Because this load is evenly distributed, like weight, it produces a net torque that is the same as if it were acting in the middle of the free body. The net torque about the bottom right corner on the free body is

$$\tau + \left(\frac{Px}{L}\right)\frac{x}{2} - \left(\frac{P}{2}\right)x = 0, \tag{22}$$

which leads to the torque on the right face of the free body

$$\tau = \frac{P}{2}\left(x - \frac{x^2}{L}\right). \tag{23}$$

[1] One cannot instead use the condition that the deflection at $x = L$ is zero because none of our free bodies has had in it the load P at the center of the beam.

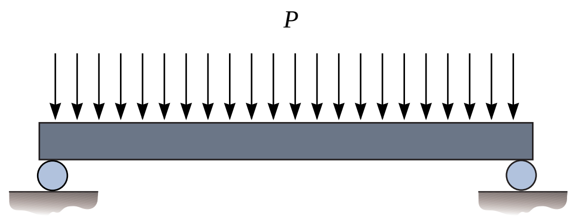

Figure 9 A beam of uniform cross-section, supported at the ends and subject to a uniformly distributed load P.

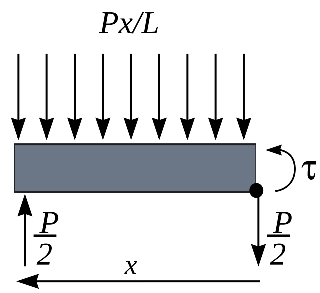

Figure 10 Free body diagram of the leftmost section of the beam.

The bending equation,

$$\frac{d^2y}{dx^2} = \frac{\tau}{YI_A} = \frac{P}{2YI_A}\left(x - \frac{x^2}{L}\right), \tag{24}$$

when integrated twice leads to the deflection

$$y(x) = \frac{P}{2YI_A}\left(\frac{x^3}{6} - \frac{x^4}{12L}\right) + C_1 x + C_2. \tag{25}$$

The physically reasonable conditions on the deflection are again that it vanish at the support at $x = 0$, and that it reach its maximum deflection in the center[2], $x = L/2$. We have $y(0) = 0$, and $dy/dx = 0$ at $x = L/2$, which lead to the values

$$C_1 = -\frac{P}{2YI_A}\frac{L^2}{12}, \tag{26}$$

$$C_2 = 0, \tag{27}$$

and the shape of the deflected beam becomes

$$y(x) = \frac{P}{2YI_A}\left(\frac{x^3}{6} - \frac{x^4}{12L} - \frac{L^2 x}{12}\right). \tag{28}$$

The maximum deflection of the beam, the value of y at $x = L/2$, is

$$y_{\max} = -\frac{5}{384}\frac{PL^3}{YI_A}. \tag{29}$$

The exaggerated shape of the beam is accurately shown in Fig. 11.

[2]In this case, we could instead have taken the condition that the deflection vanish at the right end of the beam, or $y(L) = 0$, instead of $dy/dx = 0$ at $x = L/2$

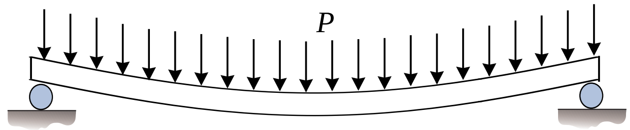

Figure 11 The exaggerated shape of a beam of uniform cross-section supported at its ends and uniformly loaded.

4.3 Bending of a cantilever loaded at the end

The analysis of a cantilever is a bit more involved than a beam supported simply at each end. In the case of the cantilever, there is a torque exerted on the fixed end of the cantilever that serves to keep it from twisting to the ground. Once this torque is taken into account, the analysis proceeds as before.

In the case of a cantilever loaded at its end, shown in Fig. 12, the torque at the wall counteracts the torque of the load,

$$\tau_{\text{Wall}} = PL. \tag{30}$$

Figure 12 A cantilever held by a wall at one end and loaded at its free end.

As before, we consider a free body diagram of a piece of the cantilever, x long, shown in Fig. 13. The net torque is

$$\tau + PL - Px = 0, \tag{31}$$

which leads to the value of the counter-clockwise torque on the right face,

$$\tau = P\,(x - L)\,. \tag{32}$$

It is important to note that the torque (32) is negative, meaning that the torque caused by the normal stress is actually a clockwise torque, and therefore the top of the beam is under tension, and the bottom is under compression, exactly opposite to the case of a beam supported at both ends.

The bending equation (14) becomes

$$\frac{d^2y}{dx^2} = \frac{P}{YI_A}\,(x - L)\,, \tag{33}$$

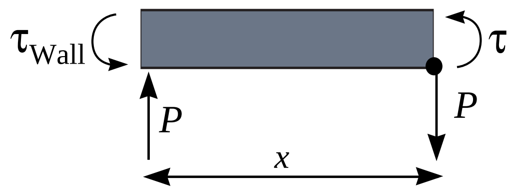

Figure 13 Free body diagram of the leftmost section of the cantilever. Torques are computed about the lower righthand corner.

which, upon two integrations, yields the general solution

$$y(x) = \frac{P}{YI_A}\left(\frac{x^3}{6} - \frac{Lx^2}{2}\right) + C_1x + C_2. \tag{34}$$

The physically reasonable conditions on the deflection are that the deflection vanish at the wall, $x = 0$, and that the beam remain horizontal at the wall. Mathematically expressed, these conditions are $y(0) = 0$, and $dy/dx = 0$ at $x = 0$, which implies that

$$C_1 = 0, \tag{35}$$
$$C_2 = 0. \tag{36}$$

The shape of the deflection becomes

$$y(x) = \frac{Px^2}{6YI_A}\left(x - 3L\right). \tag{37}$$

It is intuitively clear that the maximum deflection occurs at the end, or at $x = L$. This maximum deflection is the $y_{\text{max}} = y(L)$, or

$$y_{\text{max}} = -\frac{PL^3}{3YI_A}. \tag{38}$$

The exaggerated shape of the cantilever is accurately shown in Fig. 14.

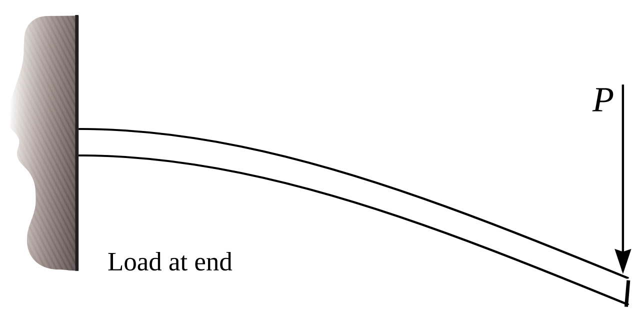

Figure 14 The exaggerated shape of a cantilever of uniform cross-section supported at one end and supporting a load at the other.

4.4 Bending of a uniformly loaded cantilever

The case of a uniformly loaded cantilever, shown in Fig. 15, is the most complex case that we will consider, though the reader is now equipped to tackle cases quite a bit more complex than even this one. The analysis is the same as the three previous cases and proceeds from the free body diagram of a section of the beam x long. The free body diagram of the leftmost

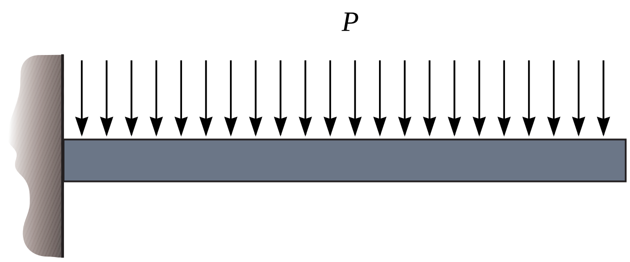

Figure 15 A cantilever held by a wall at one end and loaded uniformly along its length.

section of length x of the cantilever is shown in Fig. 13. The torque exerted by the wall on the whole cantilever counteracts the net torque $-PL/2$ from the distributed load

$$\tau_{\text{Wall}} = \frac{PL}{2}. \tag{39}$$

The net counter-clockwise torque about the lower righthand corner of the free body then is

$$\tau + \frac{PL}{2} - Px + \left(\frac{Px}{L}\right) x = 0, \tag{40}$$

which can be used to determine the torque τ on the right face of the section,

$$\tau = Px - \frac{P}{2}\left(L + \frac{x^2}{L}\right). \tag{41}$$

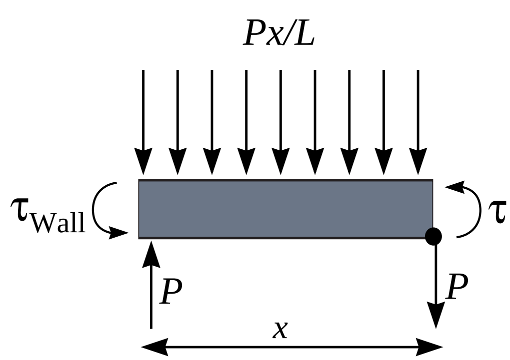

Figure 16 Free body diagram of the leftmost section of the cantilever. Torques are computed about the lower righthand corner.

As before, the torque (41) determines the shape of the cantilever's deflection from the bending equation, Eq. (14), which becomes

$$\frac{d^2y}{dx^2} = \frac{P}{2YI_A}\left(2x - L - \frac{x^2}{L}\right). \tag{42}$$

Upon two integrations, Eq. (42) yields the general solution

$$y(x) = \frac{P}{YI_A}\left(\frac{x^3}{6} - \frac{Lx^2}{4} - \frac{x^4}{24L}\right) + C_1x + C_2. \tag{43}$$

Just as for the cantilever loaded at the end, the physically reasonable conditions on the deflection are that the deflection vanish at the wall, $x = 0$, and that the beam remain horizontal at the wall. Mathematically expressed, these conditions again are $y(0) = 0$, and $dy/dx = 0$ at $x = 0$, which implies that the boundary conditions are the same as Eqs. (35),

$$C_1 = 0, \tag{44}$$
$$C_2 = 0. \tag{45}$$

The shape of the deflection becomes

$$y(x) = \frac{P}{YI_A}\left(\frac{x^3}{6} - \frac{Lx^2}{4} - \frac{x^4}{24L}\right). \tag{46}$$

Again, it is intuitively clear that the maximum deflection occurs at the end, or at $x = L$. This maximum deflection is the $y_{\max} = y(L)$, or

$$y_{\max} = -\frac{PL^3}{8YI_A}. \tag{47}$$

The exaggerated shape of the deflected cantilever is accurately shown in Fig. 17.

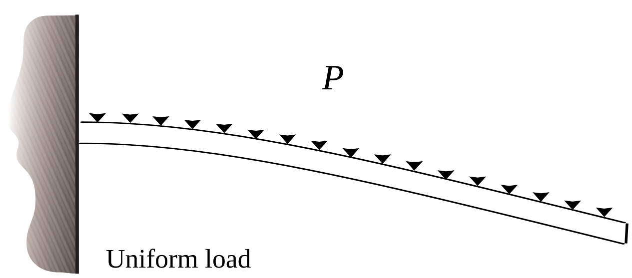

Figure 17 The exaggerated shape of a cantilever of uniform cross-section supported at one end and supporting a uniform load.

5 Discussion and Conclusions

The bending of beams is determined by the specific loads that the beam supports, how the beam is supported, the Young's modulus of the beam, and the shape of the cross-section of the beam. The product YI_A is sometimes called the "stiffness" of the beam. The larger the stiffness, the less the deflection. There are two ways to increase the stiffness of the beam. One is to make the beam from a material with a larger Young's modulus. The other is to make the cross-section of the beam such that its areal moment I_A is larger. For a beam that has to support a load that will always be in one direction, say a vertical load, the "I-beam" cross-section is has the largest stiffness for a given amount of material. If the load can be in an arbitrary direction, an annular cross-section has the largest stiffness an any direction for a given amount of material.

While it is necessary to solve the bending equation, Eq. (14), to predict the actual magnitude of the maximum deflection, we nonetheless notice that all of the maximum deflections (21), (29), (38), (47) are proportional to the quantity PL^3/YI_A:

$$y_{\max} \propto \frac{PL^3}{YI_A}. \tag{48}$$

This is not a coincidence, but is rather a consequence of dimensional analysis. If we look at the units of the load P, the beam length L, the Young's modulus Y, and the areal moment I_A, we find

$$\begin{aligned} [P] &= \text{Mass} \cdot \text{Length} \cdot \text{Time}^{-2}, \\ [L] &= \text{Length}, \\ [Y] &= \text{Mass} \cdot \text{Length}^{-1} \cdot \text{Time}^{-2}, \\ [I_A] &= \text{Length}^4, \end{aligned} \tag{49}$$

so that the combination $\frac{P}{YI_A}$ has dimensions of

$$\left[\frac{P}{YI_A}\right] = \text{Length}^{-2}. \tag{50}$$

To get a maximum deflection, which has dimensions of length, a factor of the remaining variable, L, to the third power is necessary.

Because of the fact expressed in Eq. (48), we can make a few of observations. First, just as for a spring, the maximum deflection of a beam is proportional to the applied force. If the force is doubled, the maximum deflection is doubled. Second, a beam that is made to span a distance that is twice as long as its original design length will have a deflection that is eight times as great as its maximum original design deflection. Third, if a beam's weight can be neglected, for the same load, the maximum deflection of a beam that is twice as thick in both cross-sectional directions is only 1/16th as great, because I_A increases by a factor of $16 = 2^4$. Last, if the load supported by a beam is negligible in comparison with the beam's weight, a beam that is twice as thick in both cross-sectional directions would sag only half as much, because the load (the beam's weight) will increase by a factor of $8 = 2^3$, while the areal moment will increase by a factor of $16 = 2^4$.

A more advanced observation is that if we wish to find the shape of a deflection of a beam that has a cross-sectional shape or Young's modulus (or both) that depends on distance along the beam, we can still solve Eq. (14), but this time with the variables Y and I_A as functions of x:

$$\frac{d^2y}{dx^2} = \frac{\tau(x)}{Y(x)I_A(x)}. \tag{51}$$

Some other things that can be worked out are the relationship of the shear stresses to the torques and the applied loads, the analysis of statically indeterminate cases, such as that of a beam that is held at both ends by walls that can each exert a torque as in the case of the cantilevers, and the maximum stresses on the beam for a given load and support type. The analysis of a beam under load can be pushed further and solved for more general cases, but this gives the flavor of the analysis.