THE SINGULAR MECHANICS OF PARTICLES AND STRINGS

Thesis by

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In memory of my father Arnon R. Allen

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Abstract

The quantum mechanics of singular systems is a topic of considerable importance for all the theories of elementary particle physics in which gauge invariance is a universal attribute. This is especially true for string theories which are gauge theories *par excellence.*

This thesis begins with a brief exposition of singular Hamiltonian mechanics. This tool is applied principally to manifestly supersymmetric particle and string theories. The Dirac particle and the bosonic particle and string are briefly examined. In particular, a method is shown for quantizing the point superparticle in four and ten dimensions. The two actions proposed for describing the manifestly supersymmetric string are shown to be essentially equivalent. The problems of their quantization are briefly discussed.

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Table of Symbols

$:=$	The left side is defined as
$=:$	The right side is defined as
$\approx$	Weakly equals
$\equiv$	Is identically equal to
$\{f,g\}_{DB}$	The Dirac bracket of f and g
$\{f,g\}_{PB}$	The Poisson bracket of f and g
$\Box$	$\nabla^2 - \frac{d^2}{dt^2}$
$\lvert A\rvert$	The Grassmann parity of A
b,$\bar{b}$	Ghost momenta
c,$\bar{c}$	Ghost coordinates
χ_a	A generic second-class constraint
$\delta(\cdot)$	The Dirac delta function
$\delta'(\cdot)$	The derivative of the delta function
D	The supersymmetry covariant derivative
$\int \mathcal{D}q\,\mathcal{D}p$	Functional integral
f^a_{bc}	Structure functions
Γ	The canonical phase space
Γ_2	The phase space satisfying second-class constraints
Γ_1	The subspace of Γ_2 satisfying first-class constraints
Γ_ξ	The subspace of Γ_1 satisfying $\xi \approx 0$
γ^μ	Dirac gamma matrices
gh	Ghost number
H_0	The canonical Hamiltonian
H_{BRS}	The BRS invariant Hamiltonian
λ^a	The canonical Lagrange multiplier

Ω	The BRS generator
$\not{P}$	"P slash" $= P_\mu \gamma^\mu$
$\Pi^{A\alpha\beta}$	Two-dimensional (anti)self-duality projector
π_a	The momentum conjugate to λ^a
ψ_a	A generic first-class constraint
$\|phys\rangle$	A physical quantum state
Q	Supersymmetry generator
sdet	Superdeterminant
str	Supertrace
$\theta(\cdot)$	The Heaviside theta function
θ,$\overline{\theta}$	The spinorial coordinates of superspace
Υ^a_A,$\Upsilon^a_{\dot{A}}$	Auxiliary superspace coordinates
V^a_b	Structure functions
ζ	The momentum conjugate to θ

1

Introduction

Non-relativistic mechanics, both quantum and classical, lies at the foundations of physics. Both its Hamiltonian form and its Lagrangian form are necessary knowledge for anyone who would understand any part of the modern edifice.

Relativistic mechanics, the language of particle physics, is found almost exclusively in its Lagrangian paradigm, because the Hamiltonian paradigm must, of necessity, obscure Lorentz invariance.

However, the Hamiltonian paradigm is more appropriate in many ways. Its connections with quantum mechanics are more immediate and its richer structure translates directly into quantum mechanics. For example, unitarity is manifest in the Hamiltonian paradigm; all one needs is a quantum Hamiltonian which is Hermitian. This is an advantage of the Hamiltonian path integral over the Lagrangian path integral, which must be checked for unitarity explicitly.

The Hamiltonian treatment of relativistic systems is complicated by their common tendency to be singular in the sense of Dirac. The cause of this on the one hand is the symmetric treatment of both time and space and the consequent irrelevance of which time coordinate one chooses to be *the* time. On the other hand, manifest Lorentz invariance often requires the addition of extra variables to the system whose dynamics are pure gauge. An example of the first difficulty is the theory of General Relativity, whose Hamiltonian vanishes identically because of general coordinate invariance. An example of the latter is electrodynamics where A_0 (say) is an irrelevant variable.

Relativistic systems usually have phase spaces which are smaller than one would naively think them to be, just because of the existence of irrelevant variables. In this sense, singular systems are ubiquitous in relativistic physics. The importance of singular systems extends to string theories as well. In fact, for string theories singular mechanics is important in two ways. Not only should the spacetime mechanics of string theories be singular (because of the multitude of symmetries possessed by the theory) but the two-dimensional first-quantized mechanics of the string is singular.

First-quantized particle and string theories form the main subject of this thesis. Because we are studying the worldline or worldsheet mechanics, the spacetime Poincaré invariance of the theories will be manifest and the main objection to the use of Hamiltonian methods will be avoided.

This thesis is organized into three chapters. The first chapter is a brief exposition of singular mechanics. The general theory of singular mechanics proceeds from Dirac's generalized Hamiltonian mechanics [21,32]. Batalin, Vilkovisky and Fradkin [3,4,5,6,7,26,27], building on the foundation laid down by Dirac, have enlarged and clarified the role of singular mechanics in quantum systems, culminating in the BFV path integral formulation of quantum mechanics.

The second chapter starts from the example of the bosonic particle and continues with the examination of the Dirac electron and the various manifestly supersymmetric particle theories. In particular it is shown that the superparticle theories of Brink and Schwarz [16] and Siegel [52] are inequivalent. The Brink-Schwarz superparticle is shown to be the Wess-Zumino scalar multiplet in four spacetime dimensions. The ten dimensional Brink-Schwarz superparticle, which has resisted covariant quantization since 1981, is reduced to a system which is covariant and should be quantizable in a covariant form. This is accomplished through the introduction of extra variables into superspace.

The third chapter opens with a discussion of the bosonic string and reproduces the gauge-fixed Polyakov path integral [48] from the BFV path integral. From there the Green-Schwarz [30,31] and Siegel [53] actions for the superstring are analyzed and shown to be essentially equivalent. Finally, the problems of covariant quantization of the theories are discussed and a formal quantization is proposed.

I. Singular Mechanics

1.1 Singular Systems

When passing from a Lagrangian description to a Hamiltonian description of a dynamical system, one must perform a Legendre transformation on the Lagrangian, which is a function of the generalized velocities to obtain the Hamiltonian, which is a function of the momenta. First one defines the momenta as

$$p_i := \partial L / \partial \dot{q}^i \tag{1.1}$$

and then inverts the relation (1.1) to obtain the velocities in terms of the momenta and the coordinates. The Hamiltonian is defined by the Legendre transformation

$$H_0(q,p) = p_i \dot{q}^i(p,q) - L(q, \dot{q}(p,q)). \tag{1.2}$$

In some dynamical systems it is impossible to invert the relation (1.1) defining the momenta. A system in which the relation (1.1) is not invertible is said to be singular. A singular system has the non-invertible relations of (1.1) as *constraints*.

Generally, the non-invertible relations of (1.1) can be expressed as vanishing functions of the positions and momenta:

$$\phi_n(p_i, q^j) \approx 0, \quad n = 1, ..., N. \tag{1.3}$$

(Here the curly equals sign, read as *weakly equals*, is a reminder that the equations (1.3) are not identities and should not be set to zero when they appear inside Poisson brackets.) The constraints (1.3) define a subspace of the full phase space in which the system's evolution takes place. The Hamiltonian for the evolution of the system is ambiguous on the constraint surface given by equations (1.3). One may add to the Hamiltonian H_0 an arbitrary combination of the constraints ϕ_n, since all such Hamiltonians are equal on the constraint surface and are therefore physically equivalent.

Thus the most general Hamiltonian one can write is of the form

$$H_\lambda = H_0 + \lambda_n \phi_n, \tag{1.4}$$

where H_0 is the Hamiltonian of (1.2) and the λ_n are, as yet, arbitrary. Another way of stating that the evolution of the system takes place in the submanifold defined by the constraints is that the evolution of the system must conserve the constraints:

$$\dot{\phi}_n = \{\phi_n, H_\lambda\}_{PB} \approx 0. \tag{1.5}$$

The PB denotes the Poisson bracket and the ϕ_n are assumed not to be explicitly time dependent so that all of the time dependence is implicit in the dependence on the phase space variables.

There are three possible ways that the conservation of the constraints (1.5) might be satisfied. The first is that it might be identically satisfied. The second is that the λ^n are not arbitrary but must be specified functions of the dynamical variables:

$$\lambda_n = \Lambda_n(p, q). \tag{1.6}$$

If this is the case, there may be some arbitrariness in the solutions (1.6). If there are any solutions to the conditions

$$Z_m \{\phi_n, \phi_m\}_{PB} \approx 0, \tag{1.7}$$

then the solutions (1.6) are arbitrary up to the addition of some linear combination of the solutions of (1.7). That is, $\lambda_n = \Lambda_n(p, q)$ is obviously equivalent to

$$\lambda_n = \Lambda_n + \lambda'_i Z_n^{(i)} \tag{1.8}$$

with λ'_i arbitrary and $Z_n^{(i)}$ the i^{th} solution to equation (1.7).

The third possibility for satisfying the conservation of the constraints in equation (1.5) is to impose additional constraints, sometimes called secondary constraints. If it is necessary to impose secondary constraints, the system's phase space is the subspace satisfying both the primary constraints of (1.3) and the additional secondary constraints.

Secondary constraints are treated in the same manner as the primary constraints. The Hamiltonian is again unique only up to the addition of these further constraints. All physical Hamiltonians must preserve the constraints.

In general this procedure is iterative. Each constraint may be added to the Hamiltonian and the constraint must be preserved. This procedure must terminate with some set of constraints ϕ_m, $m = 1, ..., M$ and a Hamiltonian which is unique up to the addition of some constraints

$$H = H_0 + \Lambda_m \phi_m + \lambda_k Z_m^{(k)} \phi_m. \tag{1.9}$$

Furthermore, the constraints will all be conserved under time evolution

$$\dot{\phi}_m = \{\phi_m, H\} = V_m^n \phi_n \approx 0. \tag{1.10}$$

Repeated indices are understood to be summed in the above equations.

Once a full set of conserved constraints is found and the most general Hamiltonian is determined, the detailed dynamics may be examined.

1.2 Constraints

Among the constraints there is one important distinction to be made. This distinction separates the constraints into two classes. Roughly put, the constraints of the first class generate gauge symmetries while those of the second class are irrelevant degrees of freedom which must be removed from the dynamics. A constraint is said to be *first-class* if its Poisson brackets with *all* the constraints of the system vanish weakly. A *second-class* constraint is any constraint which is not first-class.

Until it is stated otherwise, the constraints are assumed to be irreducible. That is, there is no linear combination of the constraints which is either zero or another constraint except the trivial combination.

It is easily shown that the Poisson bracket of any two first-class constraints is also of the first class by the use of the Jacobi identity for Poisson brackets. Thus the first-class constraints by themselves form a Lie algebra.

Let us denote the first-class constraints by ψ_a and the second-class constraints by χ_a. The algebra satisfied by the first-class constraints is

$$\{\psi_a, \psi_b\}_{PB} = g^c_{ab}(p, q)\psi_c. \tag{1.11}$$

Naturally, the second-class constraints do not form an algebra. It is easy to see that, in fact, the determinant of the matrix of Poisson brackets of second-class constraints must not vanish, even weakly.

$$\det|\{\chi_a, \chi_b\}_{PB}| \not\approx 0. \tag{1.12}$$

If the determinant in (1.12) did vanish then there would be an eigenvector of $\{\chi_a, \chi_b\}$ with a zero eigenvalue: $\{\chi_a, \chi_b\}v^b \approx 0$. Equivalently, $\{\chi_a, \chi_b v^b\} \approx 0$, which contradicts the assumption that all of the χ_a are second-class because the linear combination $\chi_b v^b$ is evidently first-class.

An illuminating example of this classification of constraints is the following. Assume that there is a dynamical system which yields the constraints

$$\begin{aligned} q^i &\approx 0\ , \quad i = 1, ..., N+M, \\ p_j &\approx 0\ , \quad j = 1, ..., N. \end{aligned} \tag{1.13}$$

The constraints $q^i \approx 0, i = N+1, ..., M$, have vanishing Poisson brackets with all other constraints. They are thus first-class. Any p_j has unit Poisson bracket with a corresponding q^j so that the p_i and $q^i, i = 1, ..., N$ are second-class.

In this simple example the second-class constraints are superfluous degrees of freedom. No dynamical quantity depends on them. However, in order to obtain a consistent quantization, one may not set them to zero naively without modifying the dynamical system. Consistency will require that the commutator of two quantum mechanical constraint operators either vanish or be another constraint.

Second-class constraints do not have Poisson brackets which vanish on the constraint surface. Because Poisson brackets become commutators after quantization, the Poisson bracket must be modified so that it ignores the degrees of freedom corresponding to the second-class constraints. In this simple example it is clear that one should just drop the variables q^i and p_i, $i = 1, ..., N$ from the sum in the original Poisson bracket

$$\{A, B\}_{PB} = \sum_{i=1}^{D} \left\{ \frac{\partial A}{\partial q^i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q^i} \right\}. \tag{1.14}$$

This modified Poisson bracket, sometimes called a *Dirac bracket*, can be written for the system with constraints (1.13) as

$$\{A, B\}_{DB} := \sum_{i=N+1}^{D} \left\{ \frac{\partial A}{\partial q^i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q^i} \right\}. \tag{1.15}$$

The Dirac bracket has the property that the bracket of any second-class constraint with any other dynamical variable vanishes identically. This is crucial for quantum mechanics while it may be merely convenient for a classical system. The reason for this is that one must define quantum mechanical states which are "physical." That is, one must define states which correspond to the classical configurations which satisfy the constraints. One way of implementing the constraints is to impose them as operatorial equations, or in other words, to require that the quantum operators which represent the constraints annihilate physical states. If two operators annihilate a physical state then so too must their (anti)commutator. If that (anti)commutator is neither zero nor another constraint, then there are no physical states at all.

$$0 = [\hat{\chi}_a, \hat{\chi}_b]|phys\rangle = \Delta_{ab}|phys\rangle \quad \Rightarrow \quad |phys\rangle = 0. \tag{1.16}$$

1.3 The Reduced Phase Space

Suppose that one is given a dynamical system having a set of dynamical variables $\Gamma = \{(q^i, p_i)|i = 1, ..., D\}$, a set of first-class constraints $\{\psi_j|j = 1, ..., f\}$, a set of second-class constraints $\{\chi_k|k = 1, ..., s\}$ and a Hamiltonian which preserves the constraints. One can reduce this system to an equivalent system containing only first-class constraints by introducing the Dirac bracket and setting the second-class constraints to zero identically. Just as for the simple example of second-class constraints which was exhibited earlier, the Dirac bracket must have the property that any second-class constraint has vanishing bracket with any dynamical quantity. The general definition of the Dirac bracket [21,32] is

$$\{A, B\}_{DB} := \{A, B\}_{PB} - \{A, \chi_i\}_{PB}\Delta^{ij}\{\chi_j, B\}_{PB} \tag{1.17}$$

with Δ^{ij} denoting the inverse matrix of the Poisson bracket of all second-class constraints χ_i:

$$\begin{aligned} &\Delta^{ij}\Delta_{jk} = \delta^i_k, \\ &\Delta_{ij} := \{\chi_i, \chi_j\}_{PB}. \end{aligned} \tag{1.18}$$

The inverse exists because $\det|\{\chi_i, \chi_j\}| \neq 0$, as was argued earlier.

It is obvious that (1.17) has the property that $\{A, \chi_i\}_{DB} \equiv 0$ for all A and all i. The second-class constraints may now be taken as identities. After excising the second-class constraints one is left with a phase space $\Gamma_2 \subseteq \Gamma$ in which the second-class constraints are satisfied, a Hamiltonian H which conserves the constraints,

$$\dot{\psi}_i = \{\psi_i, H\}_{DB} = V_i^j(p, q)\psi_j \approx 0, \tag{1.19}$$

and an algebra of first-class constraints

$$\{\psi_i, \psi_j\}_{DB} = f_{ij}^k(p,q)\psi_k. \tag{1.20}$$

It was remarked earlier that the Hamiltonian is actually one of an f parameter set of equivalent Hamiltonians. That is, all Hamiltonians

$$H_\lambda = H + \lambda_i \psi_i \tag{1.21}$$

are physically equivalent. The difference of any two physical Hamiltonians must generate an unphysical change in a dynamical variable. The Hamiltonians $H + \lambda_i\psi_i$ and $H + \lambda_i'\psi_i$ generate flows in the reduced phase space

$$\begin{aligned} \frac{dG}{ds} &= \{G, H + \lambda_i\psi_i\}, \\ \frac{dG}{ds'} &= \{G, H + \lambda_i'\psi_i\} \end{aligned} \tag{1.22}$$

whose difference is a flow in an unphysical direction.

$$\frac{dG}{d(\delta s)} = \{G, \delta\lambda_i\psi_i\}.$$

The first-class constraints generate unphysical or "gauge" transformations in any dynamical variable. This means that a physical phase space may be chosen as a subspace of $\Gamma_1 = \{(p,q) \in \Gamma_2 | \psi_i(p,q) = 0\}$ which cuts the flows generated by the first-class constraints. One specifies any such space, Γ_ξ, by imposing exactly f additional arbitrary constraints, ξ_i, subject only to the requirement that

$$\det|\{\xi_i, \psi_j\}_{DB}| \not\approx 0. \tag{1.23}$$

These additional gauge fixing constraints, together with the first-class constraints, may be thought of as being second-class.

The original phase space had $2D$ dimensions. Eliminating the s second-class constraints reduced the dimension to $2D - s$. Each first-class constraint effectively eliminates two degrees of freedom. Each first-class constraint itself eliminates a degree of freedom and the arbitrary gauge-fixing constraint eliminates a second. Thus the number of physical degrees of freedom in a constrained system is $2D - s - 2f$.

1.4 Canonical Quantization

Upon quantization the dynamical variables go over to operators whose commutation relations are given by the classical bracket relations. For non-singular systems one makes the operatorial transcriptions

$$\begin{aligned} p \rightarrow \widehat{p} &:= -i\hbar \frac{\partial}{\partial q}, \\ q \rightarrow \widehat{q} &:= q, \\ i\hbar\{q, p\} &\rightarrow [\widehat{q}, \widehat{p}]. \end{aligned} \tag{1.24}$$

The canonical quantization of a singular system is similar but presents special problems. First of all, one must translate the constraints into quantum operators, $\phi_i \rightarrow \widehat{\phi}_i$, and there may be problems with operator ordering ambiguities. The sometimes difficult problem of operator ordering will not be addressed in this work. Second, the algebra of first-class constraints should be preserved. This means that one must ensure that the relations (1.20) hold as operator equations with the constraint on the right

$$[\widehat{\psi}_i, \widehat{\psi}_j] = \widehat{f}_{ij}^k \widehat{\psi}_k. \tag{1.25}$$

The prescription for dealing with the second-class constraints is to modify (1.24) by taking the Dirac bracket over to the commutator

$$i\hbar\{q, p\}_{DB} \rightarrow [\widehat{q}, \widehat{p}].$$

Analogously to the reduction of the phase space, $\Gamma_\xi \subset \Gamma_1 \subset \Gamma_2 \subset \Gamma$, a physical subspace of the Hilbert space must be chosen. The oldest scheme for choosing a physical subspace is that employed by Gupta and Bleuler for quantizing electrodynamics. The essence of this procedure is the requirement that the matrix element of any constraint operator vanish between physical states.

$$\langle phys|\widehat{\psi}_i|phys'\rangle = 0. \tag{1.26}$$

The interpretation of this condition is that the Hilbert space splits into the space of physical states and the space orthogonal to the physical Hilbert space. The constraint operators map the physical states onto unphysical states orthogonal to the physical states. Since the unphysical states may have non-zero norm in the canonical Hilbert space inner product, they may not be set to zero in general. We call the condition (1.26) the *weak Dirac condition.* A much stronger condition was originally considered by Dirac. This condition is that the constraint operator should annihilate the physical states,

$$\widehat{\psi}_i|phys\rangle = 0. \tag{1.27}$$

It is clear that the strong Dirac condition (1.27) implies the weak Dirac condition (1.26). To employ the strong Dirac condition the second-class constraints must be eliminated through the use of the Dirac bracket, otherwise contradictions may result as in equation (1.16). No such stricture is needed to employ the weak Dirac condition (1.26).

An example of the strong Dirac quantization condition is electrodynamics in the wave-functional form. The dynamical variables are $A_\mu(x)$ and its conjugate momentum $\Pi^\mu = F^{\mu 0}$. The system has two constraints, $\Pi^0 \approx 0$ and $\nabla \cdot \mathbf{\Pi} \approx \mathbf{0}$. Imposing them on a wave-functional Ψ

$$\begin{aligned}
\Psi &= \Psi[\mathbf{A}] \\
0 &= -\int \chi \left[\nabla \cdot \frac{\delta}{\delta \mathbf{A}}\right] \Psi d^3x = \int (\nabla\chi) \cdot \frac{\delta}{\delta \mathbf{A}} \Psi d^3x \\
&= \Psi[\mathbf{A} + \nabla\chi] - \Psi[\mathbf{A}],
\end{aligned} \tag{1.28}$$

leaves a functional which is independent of the non-dynamical field A_0 and is invariant under a time independent gauge transformation.

1.5 The Path Integral Quantization of Constrained Systems

Classical mechanics has as its goal the description of the motion of a system as a solution, $q^i = q^i(t)$, of the equations of motion with specified boundary conditions $q^i(t_0) = q^i_0$, $\dot{q}^i(t_0) = \dot{q}^i_0$. Quantum mechanics only allows one to calculate the amplitude, $\langle q_f, t_f | q_0, t_0 \rangle$, for finding the system at q_f at time t_f given that it was at q_0 at time t_0. This amplitude may be expressed in the following standard fashion:

$$\langle q_f, t_f | q_0, t_0 \rangle = \langle q_f | e^{-\frac{i}{\hbar}(t_f - t_0)\widehat{H}} | q_0 \rangle \tag{1.29}$$

where $\widehat{H}$ is the Hamiltonian of the system. One may split up $t_f - t_0$ into N intervals (t_i, t_{i+1}) with $t_{N+1} = t_f$ and write

$$e^{-\frac{i}{\hbar}(t_f - t_0)\widehat{H}} = \prod_{i=0}^{N} e^{-\frac{i}{\hbar}(t_{i+1} - t_i)\widehat{H}}. \tag{1.30}$$

Through the repeated insertion of

$$1 = \int dp_i |p_i\rangle\langle p_i| = \int dq_i |q_i\rangle\langle q_i|, \tag{1.31}$$

one may bring the amplitude to the form

$$\begin{aligned}\langle q_f, t_f | q_0, t_0 \rangle &= \lim_{N\to\infty} \int \frac{dp_{N+1}}{2\pi\hbar} \prod_{i=1}^{N} \frac{dp_i\, dq_i}{2\pi\hbar}\, e^{\frac{i}{\hbar}(p_i(q_i - q_{i-1}) - (t_i - t_{i-1})H(p_i, q_{i-1}))} \\ &=: \int \mathcal{D}p\mathcal{D}q\, e^{\frac{i}{\hbar}\int dt(p\dot{q} - H(p,q))} \\ &= \int \mathcal{D}q\mathcal{M}(q) e^{\frac{i}{\hbar}S}.\end{aligned} \tag{1.32}$$

The measure factor, $\mathcal{M}(q)$, in the last integral results from doing the momentum integrations.

Constrained systems may be quantized in the same way except that the path integral must be modified. Just as the classical evolution of the system stays within the submanifold of phase space satisfying the constraints, the "quantum evolution" should also be in the constrained submanifold in order that physical states propagate into physical states. This can be achieved by modifying the measure of the phase space path integral.

First let us consider systems without second-class constraints. The physical subspace of a constrained system with n first-class constraints is chosen by n "gauge conditions" which intersect all of the flows generated by the first-class constraints. Thus for each constraint $\psi_a \approx 0$ a gauge condition $\xi_b \approx 0$ must be chosen so that

$$\det \{\psi_a, \xi_b\} \not\approx 0. \tag{1.33}$$

One procedure is to integrate only over those field configurations which satisfy $\psi_a \approx 0$ and $\xi_b \approx 0$. Clearly, the measure of the path integral should be independent of the functional form of the constraints so long as the same submanifold of phase space is specified by the set of constraints.

In particular, the measure

$$\mathcal{D}p\mathcal{D}q\,\delta[\psi_a]\delta[\xi_b]\mathcal{M}(\psi, \xi) \tag{1.34}$$

should be invariant under a change of constraints

$$\begin{aligned} \psi_{a'} &= A_{a'a}\psi_a, \\ \xi_{b'} &= B_{b'b}\xi_b, \end{aligned} \tag{1.35}$$

with $\det A \neq 0$ and $\det B \neq 0$. This implies that the factor $\mathcal{M}$ must be proportional to $\det\{\psi_a, \xi_b\}$, which is the phase space transcription of the measure found by Faddeev and Popov [24] for the path integral in configuration space.

Theories which have second-class constraints are really no harder to handle in principle but there may be computational difficulties if the second-class constraints have awkward Poisson brackets.

The first step is to eliminate the second-class constraints through the use of Dirac brackets. Once this is done, the path integral must be restricted to those paths which lie in the constrained submanifold Γ_2. As before, one restricts to this space by the insertion of a delta function $\delta[\chi_a]$ into the measure. Invariance under redefinitions of the second-class constraints requires the insertion of a factor $(\mathrm{sdet}\{\chi_a, \chi_b\}_{PB})^{\frac{1}{2}}$ into the measure in analogy with the first-class case. The general S-matrix is then of the form [50,27]

$$Z = \int \mathcal{D}p\mathcal{D}q\, \mathrm{sdet}\,\{\xi_a, \psi_b\}_{DB}\, (\mathrm{sdet}\,\{\chi_c, \chi_d\}_{PB})^{\frac{1}{2}} \delta[\psi_a]\delta[\xi_b]\delta[\chi_c] e^{i\int dt(p\dot{q}-H_0)}. \quad (1.36)$$

An alternative to choosing a gauge, $\xi_a(p,q) = 0$, which depends upon the original dynamical variables, is first to impose only the constraints and write the path integral as

$$Z = \int \mathcal{D}q\, \mathcal{D}p\, \mathcal{D}\lambda\, e^{i\int dt(p\dot{q}-H_0-\lambda_a\psi_a)} \quad (1.37)$$

and to notice that the symmetries of the “action” in the exponential,

$$\begin{aligned} \delta q &= \{q, \psi_a\}\epsilon_a, \\ \delta p &= \{p, \psi_a\}\epsilon_a, \\ \delta\lambda_a &= (\delta_a^b \frac{d}{dt} - V_a^b - f_{ac}^b\lambda_c)\epsilon_b, \end{aligned} \quad (1.38)$$

include a change in the Lagrange multiplier λ_a. A gauge choice which includes the multipliers λ_a would be just as suitable for defining the functional integral (1.37). A possible gauge condition is one of the so-called *relativistic* gauges [7,26,27]

$$\dot{\lambda}_a = \Lambda_a(p, q, \lambda). \quad (1.39)$$

In a relativistic gauge the action takes the form

$$S(p,q,\lambda,\pi) = \int dt\ \left(p\dot{q} + \pi\dot{\lambda} - H_0 - \lambda_a\psi_a - \pi_a\Lambda_a\right). \tag{1.40}$$

The variable π is a Lagrange multiplier whose equation of motion just enforces the relativistic gauge condition (1.39). A functional integral using the action (1.40) includes integration over both of the multipliers λ and π. The only subtlety is to determine the correct measure for the functional integral, analogous to the measure (1.34). One of the properties of the correct measure is that the S-matrix be independent of the gauge fixing scheme used to define it. For this purpose the phase space BRS transformation is most useful.

1.6 BRS Symmetry and the BFV Path Integral

The existence of a global fermionic symmetry in gauge-fixed Yang-Mills theories was discovered by Becchi, Rouet, and Stora [8] and independently by Tyutin [55]. There is a more general symmetry which is possessed by any constrained Hamiltonian system. This symmetry was found by Batalin and Vilkovisky [7]. The first step in the construction of the BRS symmetry is the enlargement of the phase space of the dynamical system. Let us assume that the second-class constraints, if there are any, have been eliminated through the use of the Dirac bracket. Thus it is without loss of generality that we may assume that the system has first-class constraints only.

The first variables we wish to add to the system are the Lagrange multipliers λ and π, a pair for each first-class constraint ψ_a. The variables λ and π are canonically conjugate and have the same statistics (Grassmann parity) as the constraint ψ_a. Along with the Lagrange multipliers, one introduces the ghost variables c^a, b_a, $\bar{c}_a$ and $\bar{b}^a$. The ghosts all have statistics opposite to ψ_a. The unbarred ghosts are canonical conjugates as are the barred ghosts and the Lagrange multipliers λ and π.

These extra variables can be loosely justified in the following way. Suppose for a moment that the constraint ψ_a is bosonic. In this case we add to the system two bosonic and four fermionic degrees of freedom. We do not impose the constraint.

Instead, the fermionic degrees of freedom act [1,2,47] as "negative" dimensions. The phase space thus has two fewer degrees of freedom by virtue of the added variables. Of course, we will justify these variables by showing how to construct the path integral and demonstrate that the path integral is correct by reducing it to the expression (1.36).

The key to doing this is existence of a global fermionic symmetry generated by the quantity, Ω, which satisfies $\{\Omega, \Omega\} = 0$. This generator, called the BRS charge, can be expressed as

$$\Omega = c^\alpha \psi_\alpha - (-1)^{|c^\beta|} \tfrac{1}{2} f^\gamma_{\alpha\beta} c^\beta c^\alpha b_\gamma + \sum_{n\geq 2} b_{\alpha_n} \cdots b_{\alpha_1} \Omega^{\alpha_1 \cdots \alpha_n} + \bar{b}^\alpha \pi_\alpha. \tag{1.41}$$

The BRS charge Ω, discovered by Batalin and Vilkovisky [7], generates the canonical version of the global fermionic symmetry discovered in Yang-Mills theories by Becchi, Rouet and Stora [8] and Tyutin [55]. The BRS charge is, in general, not unique. All the possible BRS charges are unitarily equivalent [34].

It is also necessary to have a Hamiltonian for the dynamical system in the extended phase space which is BRS invariant.

$$\{H_{BRS}, \Omega\} = 0. \tag{1.42}$$

This Hamiltonian may be constructed as follows

$$H_{BRS} = H_0 + b_\alpha V^\alpha_\beta c^\beta + \sum_{n\geq 2} b_{\alpha_n} \cdots b_{\alpha_1} H^{\alpha_1 \ldots \alpha_n}, \tag{1.43}$$

where V^β_α is the coefficient in the relation

$$\{\psi_\alpha, H_0\} = V^\beta_\alpha \psi_\beta. \tag{1.44}$$

If the structure functions $f^\gamma_{\alpha\beta}$ and V^β_α are not constants, then the terms $\Omega^{\alpha_1 \ldots \alpha_n}$ and $H^{\alpha_1 \ldots \alpha_n}$ in the expansions (1.41) and (1.43) may not be zero. In most cases it is fairly easy to obtain these extra pieces from the requirement that Ω and H_{BRS} satisfy $\{H_{BRS}, \Omega\} = \{\Omega, \Omega\} = 0$. The existence of the extra pieces for a general dynamical system was demonstrated in reference [34] where a general procedure for calculating them is given also.

To each variable a quantity called ghost number may be assigned. The original degrees of freedom, q^i and p_j, are assigned ghost number zero. The extra degrees of freedom are assigned the following ghost numbers

$$\begin{aligned} \mathrm{gh}(\lambda^\alpha) &= \mathrm{gh}(\pi_\alpha) = 0, \\ \mathrm{gh}(c^\alpha) &= -\mathrm{gh}(b_\alpha) = 1, \\ \mathrm{gh}(\bar{c}_\alpha) &= -\mathrm{gh}(\bar{b}^\alpha) = -1. \end{aligned} \tag{1.45}$$

Time evolution under the Hamiltonian H_{BRS} preserves the ghost numbers, while the generator Ω increases the ghost number by one because ghost number is additive and the ghost numbers of H_{BRS} and Ω are 0 and 1 respectively.

In this enlarged phase space the path integral for the S-matrix may be given the very general form [26]

$$Z_\Psi = \int \mathcal{D}P\mathcal{D}Q\, \delta[\chi_a](\mathrm{sdet}\{\chi_a, \chi_b\})^{\frac{1}{2}} e^{i\int dt(P\dot{Q} - H_{BRS} + \{\Psi, \Omega\}_{DB})}. \tag{1.46}$$

Here Ψ is an arbitrary function of all of the variables of the system, and must have ghost number -1 if the path integral (1.46) is to conserve ghost number, and the variables Q and P stand for all of the coordinates and momenta of the system, including the ghost and Lagrange multiplier degrees of freedom.

In an ordinary path integral such as the one given by (1.32), the boundary conditions are chosen to correspond to the initial and final states of the amplitude we are evaluating. Here, however, we have more variables in our system and we must decide what boundary conditions must be put on the ghosts and Lagrange multipliers. We shall choose those boundary conditions which ensure the existence of a global symmetry generated by the BRS charge Ω. From the properties of the BRS charge

$$\begin{aligned} \{\Omega, \Omega\}_{DB} &= 0, \\ \{H_{BRS}, \Omega\}_{DB} &= 0, \end{aligned} \tag{1.47}$$

one finds the variation of the action of (1.46),

$$S_{BRS} = \int dt(P\dot{Q} - H_{BRS} + \{\Psi, \Omega\}_{DB}), \tag{1.48}$$

under the infinitesimal canonical transformation

$$\delta z = \{z, \Omega\epsilon\}_{DB}$$

to be

$$\delta S_{BRS} = \left[{\sum}' z^i \frac{\partial \Omega}{\partial z^i} - \Omega\right]_1^2 \epsilon, \tag{1.49}$$

where the prime on the sum denotes that the sum is carried out only over those variables which are not fixed at the boundaries. The proper "BRS invariant" boundary conditions will be those which cause the expression (1.49) to vanish.

One possible way to do this is to choose some set of variables, ζ_i, to set to zero at the boundary so that the following conditions are met.

$$\begin{aligned} \Omega|_{\zeta_i=0} &= 0, \\ \{\zeta_j, \Omega\}_{DB}|_{\zeta_i=0} &= 0. \end{aligned} \tag{1.50}$$

There are three sets of variables which will satisfy (1.50). From the structure of the BRS charge,

$$\Omega = c\psi + \sum_{n\geq 1} b^n c^{n+1} \Omega_n + \bar{b}\pi, \tag{1.51}$$

it is immediately obvious that each of the following sets of boundary conditions makes

Ω vanish [34].

$$\begin{aligned}(i)\quad & \bar{c} = b = \pi = \psi(p) = 0,\\ (ii)\quad & \bar{b} = c = 0,\\ (iii)\quad & c = \bar{c} = \pi = 0.\end{aligned} \tag{1.52}$$

One may also check that the second condition of (1.50) is satisfied by each of the conditions (1.52). Thus the three sets of conditions are sufficient to guarantee the vanishing of the BRS variation of the action (1.49). In fact, slightly weaker conditions may be imposed at the boundary which will also ensure the existence of the global BRS symmetry. These can be obtained directly from the expression (1.49):

$$\begin{aligned}(i)\quad & \psi(p) = 0,\ b = 0,\ \bar{c} = \bar{c}_o,\ \pi = \pi_o,\\ (iia)\quad & c = 0,\ \bar{b} = \bar{b}_o,\ \lambda = \lambda_o,\\ (iib)\quad & c = 0,\ \bar{b} = 0,\ \pi = \pi_o,\\ (iic)\quad & c = 0,\ \bar{b} = \bar{b}_o,\ \pi = 0,\\ (iii)\quad & c = 0,\ \bar{c} = \bar{c}_o,\ \pi = \pi_o.\end{aligned} \tag{1.53}$$

The first conditions appear to be more restrictive than the others but, in fact, are not. They just specify the boundary conditions on the original variables (in this case on p.) The boundary conditions on the original variables are left unspecified in the other cases but, nonetheless, they are assumed implicitly to exist. The boundary conditions ($1.53i$) are only useful for constraints, ψ, which satisfy an abelian algebra so that they may be treated as momenta of the system, otherwise the boundary conditions on the rest of the original variables q^i and p_i are difficult to set. The other conditions of (1.53) are easier to use in practice.

Having in hand BRS invariant boundary conditions, we now turn to the Fradkin-Vilkovisky theorem which asserts that the path integral (1.46) is independent of the gauge fixing function Ψ. Following Henneaux [34], we prove the independence of Z_Ψ under infinitesimal changes of Ψ.

Let z denote *all* canonical variables in the extended phase space. Let us perform the infinitesimal canonical transformation

$$
\begin{aligned}
z_i'(t) &= z_i(t) + i\{z_i, \Omega\}(t)\,\Delta\Psi, \\
&= z_i(t) + i\omega_{ij}\frac{\partial\Omega}{\partial z_j}(t)\,\Delta\Psi,
\end{aligned}
\tag{1.55}
$$

where

$$
\Delta\Psi = \int_{t_1}^{t_2} dt\,\delta\Psi \tag{1.56}
$$

and

$$
\delta\Psi = \Psi' - \Psi \tag{1.57}
$$

are infinitesimal and where

$$
\omega_{ij} = \{z_i, z_j\} \tag{1.58}
$$

is the canonical form. With any of the BRS invariant boundary conditions, the action is invariant under the infinitesimal canonical transformation (1.55). The path integral measure is not invariant, however. Let us compute the Jacobian of the change of variables (1.55).

$$
\begin{aligned}
J &= \mathrm{sdet}^{-1}\Big(\frac{\delta z_i'(t')}{\delta z_j(t)}\Big) \\
&= \mathrm{sdet}^{-1}\Big[\delta_{ij}\delta(t'-t) + i\omega_{ik}\frac{\partial^2\Omega}{\partial_L z_j \partial_L z_k}\Delta\Psi\delta(t'-t) \\
&\qquad - i(-1)^{|z_k|}\frac{\partial\,\delta\Psi}{\partial_R z_j}(t')\omega_{ik}\frac{\partial\Omega}{\partial_L z_k}(t)\Big].
\end{aligned}
\tag{1.59}
$$

Since $\delta\Psi$ is infinitesimal, the Jacobian is just

$$\begin{aligned}
\mathrm{sdet}^{-1}(\frac{\delta z_i'(t')}{\delta z_j(t)}) &= 1 - \mathrm{str}(\frac{\delta z_i'(t')}{\delta z_j(t)}) \\
&= 1 - i\sum_{i,j}\big[(-1)^{|z_i|}\omega_{ij}\int dt\,\frac{\partial^2\Omega}{\partial_L z_i \partial_L z_j}\Delta\Psi\big] \\
&\quad + i\sum_{i,j}\omega_{ij}\int dt\,(\frac{\partial\,\delta\Psi}{\partial_R z_i}\frac{\partial\Omega}{\partial_L z_j}) \\
&= 1 + i\int dt\{\delta\Psi,\Omega\} \\
&= \exp(i\int dt\{\Psi' - \Psi,\Omega\}),
\end{aligned} \tag{1.60}$$

to first order. From this it follows that the measure transforms as

$$\mathcal{D}P\,\mathcal{D}Q = \mathcal{D}P'\,\mathcal{D}Q'\,e^{i\int\{\Psi'-\Psi,\Omega\}dt}, \tag{1.61}$$

which, when substituted into the path integral (1.46), yields the desired result

$$Z_\Psi = Z_{\Psi'}. \tag{1.62}$$

The Fradkin-Vilkovisky theorem may be used to reduce the BFV path integral to the form (1.36). To do so, we first choose our gauge fixing function Ψ to be

$$\Psi = \beta^{-1}\chi(p,q)\bar{c} + b\lambda, \tag{1.63}$$

where β is arbitrary for now. Using the shorthand

$$\Omega = c\psi + \sum_n c^{n+1}b^n\Omega_n + \bar{b}\pi, \tag{1.64}$$

we may write the bracket

$$\{\Psi,\Omega\} = \beta^{-1}\chi\pi + \lambda\psi + b\bar{b} + \beta^{-1}\bar{c}\{\chi,\psi\}c + \sum(cb)^n\lambda\Omega_n + \sum\beta^{-1}\bar{c}c(cb)^n\{\chi,\Omega_n\}. \tag{1.65}$$

The full action is

$$
\begin{aligned}
S_{BRS} = \int dt\, \big[& p\dot{q} + \pi\dot{\lambda} + \bar{b}\dot{\bar{c}} + b\dot{c} - H_0 \\
& - \sum_{n\geq 1} b^n c^n H_n + \beta^{-1}\chi\pi + \lambda\psi + b\bar{b} \\
& + \beta^{-1}\bar{c}\{\chi, \psi\}c + \bar{c}\sum_{n\geq 1} b^n F_n(p, q, c)\big].
\end{aligned}
\tag{1.66}
$$

The factors of β^{-1} may be eliminated from the above expression at the expense of introducing β's on the kinetic terms $\bar{b}\dot{\bar{c}}$ and $\pi\dot{\lambda}$ through the change of variables

$$
\begin{aligned}
\pi &\rightarrow \beta\pi, \\
\bar{c} &\rightarrow \beta\bar{c}.
\end{aligned}
\tag{1.67}
$$

In fact, this change of variables has unit Jacobian because the ghost $\bar{c}$ and the Lagrange multiplier π have opposite Grassmann parity. The Fradkin-Vilkovisky theorem assures us that the path integral is independent of the gauge fixing function Ψ, and hence is independent of β. We thus may take the limit of vanishing β, whose only effect is to excise the kinetic terms mentioned above from the exponential of the path integral.

In the limit of vanishing β, the path integral is easily done. The integral over $\bar{b}$ yields a factor $\delta[b]$ which is set to one by the integral over b. The integrals over the Lagrange multipliers π and λ bring down factors of $\delta[\chi]$ and $\delta[\psi]$ respectively. Finally, the ghost integrals on c and $\bar{c}$ produce the superdeterminant $\mathrm{sdet}\{\chi, \psi\}$. We thus obtain the desired result

$$
Z_\Psi|_{\beta=0} = \int \mathcal{D}p\mathcal{D}q\, \delta[\chi]\delta[\psi]\mathrm{sdet}\{\chi, \psi\} e^{i\int(p\dot{q}-H_0)dt}. \tag{1.68}
$$

1.7 Operatorial BRS Quantization

One may use the BFV formalism to construct states in the enlarged Hilbert space and to study their evolution. The first task is to define the physical states. This can be done analogously to the Dirac condition (1.27). First, one must transcribe the canonical variables to operators on this Hilbert space and one must construct an operatorial BRS charge $\widehat{\Omega}$ which is Hermitian and satisfies the condition

$$\widehat{\Omega}^2 = 0. \tag{1.69}$$

Physical states are defined to be those which are annihilated by the BRS charge.

$$\widehat{\Omega}|phys\rangle = 0. \tag{1.70}$$

The states may also be chosen to have a definite ghost number.

$$\mathrm{gh}(|phys\rangle) = n. \tag{1.71}$$

From the condition (1.69) it follows that any state of the form $\widehat{\Omega}|\phi\rangle$ is physical for any $|\phi\rangle$. These states are orthogonal to all the physical states of the theory and are irrelevant. That the physical state condition (1.70) for zero ghost number states is equivalent to the strong Dirac physical state condition is easy to see when either the constraints form a true Lie algebra or when they have zero brackets with each other. When the constraints have zero brackets with each other, the BRS charge is simply

$$\widehat{\Omega} = c^\alpha \widehat{\psi}_\alpha + \bar{b}^\alpha \widehat{\pi}_\alpha. \tag{1.72}$$

A zero ghost number state is not a function of the ghost coordinates c^α, $\bar{b}^\alpha$ but is a function of the original phase space variables and the Lagrange multipliers λ^α.

The physical state condition (1.70) then implies that

$$\begin{aligned} \widehat{\psi}_\alpha|phys\rangle &= 0, \\ \frac{\partial}{\partial\lambda^\alpha}|phys\rangle &= 0. \end{aligned} \tag{1.73}$$

When the constraints form a Lie algebra, then the BRS charge has the form

$$\widehat{\Omega} = c^\alpha \widehat{\psi}_\alpha + \tfrac{1}{2} f^\gamma_{\alpha\beta} c^\beta c^\alpha \widehat{b_\gamma} + \bar{b}^\alpha \widehat{\pi}_\alpha. \tag{1.74}$$

The second term in the BRS charge (1.74) gives zero when operating upon a zero ghost number state. The physical state condition again yields the conditions (1.73).

Time evolution of a state is governed by the BRS invariant Hamiltonian $\widehat{H}_{BRS}$ which must commute with the fermionic BRS charge $\widehat{\Omega}$.

$$\widehat{H}_{BRS}\widehat{\Omega} = \widehat{\Omega}\widehat{H}_{BRS}. \tag{1.75}$$

The condition (1.75) ensures that the physical state condition (1.70) is preserved by time evolution under the Hamiltonian $\widehat{H}_{BRS}$.

II. Point Particle Theories

2.1 The Scalar Particle

The simplest point particle theory one could imagine is that of the scalar particle. This particle is completely defined by its position in spacetime x^μ. Such a particle has no internal structure or intrinsic angular momentum. The action for this particle is just the length of its world line. A massive particle moves along a classical path which extremizes the action

$$S = -m \int ds = -m \int d\tau \sqrt{-\dot{x}^2}. \tag{2.1}$$

In defining the conjugate momentum to the position,

$$P_\mu = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}}, \tag{2.2}$$

one is naturally led to the constraint

$$P_\mu P^\mu + m^2 \approx 0, \tag{2.3}$$

and to the interesting result that the Hamiltonian vanishes identically

$$H_0 \equiv 0. \tag{2.4}$$

The vanishing of the Hamiltonian is a consequence of the reparametrization invariance of the action (2.1). The demonstration of this fact is quite simple [57]. A Lagrangian which is reparametrization invariant transforms as

$$\delta L = \frac{d}{dt}(\xi L) = \dot{\xi} L + \xi \ddot{q} \frac{\partial L}{\partial \dot{q}} + \xi \dot{q} \frac{\partial L}{\partial q} \tag{2.5}$$

when the time is changed by $\tau \rightarrow \tau + \xi(\tau)$. On the other hand, the variation of the Lagrangian can be computed directly from $\delta q = \xi \dot{q}$ and the chain rule:

$$\delta L = \dot{\xi} \dot{q} \frac{\partial L}{\partial \dot{q}} + \xi \ddot{q} \frac{\partial L}{\partial \dot{q}} + \xi \dot{q} \frac{\partial L}{\partial q}. \tag{2.6}$$

A direct comparison of the two expressions demonstrates that

$$\dot{q} \frac{\partial L}{\partial \dot{q}} = L. \tag{2.7}$$

If a Lagrangian has reparametrization invariance which is not manifest, the Hamiltonian will not vanish identically but will be a combination of the first-class constraints. This is the case for both point particle theories and for string theories. In both the point particle and string theories the dynamics is completely determined by the constraints.

Because the Hamiltonian vanishes identically, there are no secondary constraints. Physical states may be chosen to be those which satisfy the constraints as operator equations,

$$(P^2 + m^2)|phys\rangle = 0, \tag{2.8}$$

which is the Klein-Gordon equation

$$(\Box - m^2)\phi = 0. \tag{2.9}$$

The free propagator may be calculated from the phase space path integral via the Batalin-Fradkin-Vilkovisky formalism.

$$\langle x^\mu_f, \tau_f | x^\mu_i, \tau_i \rangle = \int \mathcal{D}P \, \mathcal{D}Q \, e^{i \int (P\dot{Q} + \{\Psi, \Omega\}) d\tau} \tag{2.10}$$

where Q and P are generic labels for $(x^\mu, c, \bar{b}, \lambda)$ and $(p_\mu, b, \bar{c}, \pi)$ respectively. The coordinates are fixed at the endpoints while the momenta at the endpoints are integrated. The full BRS generator is

$$\Omega = c(p^2 + m^2) + \bar{b}\pi \tag{2.11}$$

and the gauge fixing function may be chosen to be

$$\Psi = -\frac{b\lambda}{\Delta\tau.} \tag{2.12}$$

Evaluation of the expression (2.10) is not difficult and yields the result

$$\langle x_f | x_i \rangle = \int d^4p\, d\lambda\, e^{i[p\cdot(x_f - x_i) - \lambda(p^2+m^2)]}. \tag{2.13}$$

If one makes the reasonable requirement that particles with positive energy travel forward in time while particles with negative energy travel backward in time, one is forced to restrict the integration over λ to be positive because of the equation of motion

$$\frac{dx^\mu}{d\tau} = \{x^\mu, H_{BFV}\} = 2\lambda p^\mu. \tag{2.14}$$

When this is done, the Feynman propagator is obtained. (The same result could be obtained by inserting the factor $\theta(p^0)\theta(x_f^0 - x_i^0) + \theta(-p^0)\theta(x_i^0 - x_f^0)$ into the integral to enforce these boundary conditions.)

Of course, a free theory is not very interesting. Ultimately, one would like to use this "first quantization" to construct appropriate "second quantized" field theories. In this case the free second quantized theory should be constructed so that its classical solutions yield precisely the physical states specified by the physical state condition (2.6). Following Siegel [51] we postulate that the action is

$$S = \int d^4x\, dc\, \phi\widehat{\mathcal{K}}\phi, \tag{2.15}$$

where $\widehat{\mathcal{K}}$ is a differential operator chosen so that the action is invariant under unphysical changes state $\phi \to \phi + \widehat{\Omega}\chi$. This can be accomplished by choosing $\widehat{\mathcal{K}}$ so that

$[\widehat{\mathcal{K}}, \widehat{\Omega}]_+ = 0$. Let us choose $\widehat{\mathcal{K}}$ to be $\widehat{\Omega}$, then

$$\begin{aligned} S &= \int d^4x\, dc\, \phi(x,c)\, c(\Box - m^2)\phi(x,c) \\ &= \int d^4x \phi(x)(\Box - m^2)\phi(x), \end{aligned} \tag{2.16}$$

is the ordinary Klein-Gordon action.

The Klein-Gordon particle is a trivial example of a constrained system. More insight can be obtained from a consideration of the classical Dirac electron and its quantization, which not only contains classically anticommuting variables but also has second-class constraints.

2.2 The Dirac Electron

The first classical action for the Dirac electron using anticommuting, or Grassmann variables, was written by Berezin and Marinov [11].

$$S = \int d\tau[-m\sqrt{-\dot{x}^2} + \frac{i}{2}(\xi^\mu \dot{\xi}_\mu + \xi_5 \dot{\xi}_5) - (u_\mu \xi^\mu + \xi_5)\lambda]. \tag{2.17}$$

In this action the variables ξ_μ, ξ_5 and λ are anticommuting while x^μ is commuting. The notation u_μ is a shorthand for $\dot{x}_\mu/\sqrt{-\dot{x}^2}$. The canonical momenta

$$\begin{aligned} P_\mu &= m u_\mu + i\lambda(-\dot{x}^2)^{-\frac{1}{2}}(\xi_\mu + u_\nu \xi^\nu u_\mu), \\ P_\xi &= \frac{i}{2}\xi, \\ P_5 &= \frac{i}{2}\xi_5, \\ P_\lambda &= 0, \end{aligned} \tag{2.18}$$

lead to the constraints

$$\begin{aligned} P_\lambda &\approx 0, \\ P^2 + m^2 &\approx 0, \\ P_\mu \xi^\mu + m\xi_5 &\approx 0, \\ P_{\xi_\mu} - \frac{i}{2}\xi^\nu \eta_{\mu\nu} &\approx 0, \\ P_5 - \frac{i}{2}\xi_5 &\approx 0. \end{aligned} \tag{2.19}$$

Of these, the first three are first-class while the last two are second-class. The two second-class constraints may be removed by introducing the Dirac brackets

$$\begin{aligned} \{\xi_\mu, \xi_\nu\}_{DB} &= i\eta_{\mu\nu}, \\ \{\xi_5, \xi_5\}_{DB} &= i. \end{aligned} \tag{2.20}$$

The Hamiltonian

$$H = \lambda_1(P^2 + m^2) + \lambda_2(P_\mu \xi^\mu + m\xi_5)$$

generates the equations of motion

$$\begin{aligned} \dot{x}_\mu &= 2\lambda_1 P_\mu + \xi_\mu \lambda_2, \\ \dot{P}_\mu &= 0, \\ \dot{\xi}_\mu &= iP_\mu \lambda_2, \\ \dot{\xi}_5 &= im\lambda_2. \end{aligned} \tag{2.21}$$

One may fix a gauge by choosing the multipliers to be

$$\begin{aligned} \lambda_1 &= \frac{1}{2m}, \\ \lambda_2 &= 0. \end{aligned}$$

This Hamiltonian yields a simple set of equations of motion.

The constraint $(P^2 + m^2)$ generates τ reparametrizations just as it does for the scalar particle, while the constraint $(P \cdot \xi + m\xi_5)$ generates translations in an anticommuting time. We might call this anticommuting time ϑ. This system, in fact, is an example of a one dimensional supersymmetric theory. This can be demonstrated by writing the theory in a manifestly supersymmetric form. We accomplish this by pairing up the fields which are "superpartners" into single "superfields" which are functions of both τ and ϑ. We define the following superfields

$$\begin{aligned}
\mathbf{X}^\mu(\tau,\vartheta) &= x^\mu(\tau) + i\vartheta\xi^\mu(\tau), \\
\mathbf{X}_5(\tau,\vartheta) &= \xi_5(\tau) + \vartheta\phi_5(\tau), \\
\mathbf{E}(\tau,\vartheta) &= e_0(\tau) + \vartheta e_1(\tau).
\end{aligned} \tag{2.22}$$

The five superfields $\mathbf{X}^\mu$ and $\mathbf{E}$ are commuting superfields while $\mathbf{X}_5$ is an anticommuting superfield. The supersymmetric covariant derivative, D, and the supersymmetry generator, Q, defined by

$$\begin{aligned}
D &:= \frac{d}{d\vartheta} - i\vartheta\frac{d}{d\tau}, \\
Q &:= \frac{d}{d\vartheta} + i\vartheta\frac{d}{d\tau},
\end{aligned} \tag{2.23a}$$

satisfy the algebra

$$\begin{aligned}
Q^2 &= i\frac{d}{d\tau}, \\
D^2 &= -i\frac{d}{d\tau}, \\
QD + DQ &= 0.
\end{aligned} \tag{2.23b}$$

Using the properties (2.23b) and the superfields (2.22), we may construct an action which is manifestly invariant under the global supersymmetry transformations

$$\delta\mathbf{\Phi} = \epsilon Q\mathbf{\Phi}.$$

The action

$$S = \int d\tau d\vartheta[-\frac{1}{2\mathbf{E}}D\mathbf{X}^\mu D^2\mathbf{X}_\mu + \frac{1}{2\mathbf{E}}\mathbf{X}_5 D\mathbf{X}_5 + m\mathbf{X}_5] \tag{2.24}$$

is also invariant under local τ-reparametrizations

$$\begin{aligned}
\delta\mathbf{X}^\mu &= \epsilon\dot{\mathbf{X}}^\mu, \\
\delta\mathbf{X}_5 &= iD(\epsilon D\mathbf{X}_5), \\
\delta\mathbf{E} &= \frac{d}{d\tau}(\epsilon\mathbf{E}) + i(D\epsilon)(D\mathbf{E}),
\end{aligned} \tag{2.25}$$

with ϵ a commuting function of τ and ϑ.

When written out in component form,

$$S = \int d\tau \left[\frac{1}{2e_0}(\dot{x}^2 + i\xi \cdot \dot{\xi} + \frac{e_1}{e_0}\xi \cdot \dot{x} + i\xi_5\dot{\xi}_5 + \phi_5^2 - \frac{e_1}{e_0}\xi_5\phi_5) + m\phi_5 \right], \tag{2.26}$$

the action (2.24) can easily be seen to be equivalent to the action written by Berezin and Marinov (2.17). A locally supersymmetric superfield formulation has been written by Brink, Deser, Zumino, Di Vecchia, and Howe [13] for the massless case.

Upon quantization, the anticommuting variables ξ^μ and ξ_5 will become operators with anticommutators given by the Dirac brackets (2.20). Various authors [11,12,29,35,42] have identified the quantum operators corresponding to ξ_μ and ξ_5 as the elements of the Dirac gamma algebra $i\sqrt{\frac{\hbar}{2}}\gamma_5\gamma_\mu$ and $i\sqrt{\frac{\hbar}{2}}\gamma_5$. In fact, as we shall see, the situation is a bit more subtle than this. The second-class constraints must be handled carefully in order to obtain a consistent quantization. First let us observe that there are an even number of Grassmann variables and an odd number of second-class constraints on those variables. The reduced phase space is thus odd dimensional. The variables ξ_μ are easily separated into pairs conjugate under the Dirac brackets (2.20):

$$\begin{aligned} \eta_1 &= \frac{\xi_0 + \xi_3}{\sqrt{2}}, \\ \eta_1^* &= \frac{-\xi_0 + \xi_3}{\sqrt{2}}, \\ \eta_2 &= \frac{\xi_1 + i\xi_2}{\sqrt{2}}, \\ \eta_2^* &= \frac{\xi_1 - i\xi_2}{\sqrt{2}}, \end{aligned} \tag{2.27}$$

satisfying the relations

$$\{\eta_i, \eta_j^*\}_{DB} = i\delta_{ij}. \tag{2.28}$$

The constraint on ξ_5 is not as easily solved. In fact, because it is second-class, it cannot be imposed on a state directly. The easiest way to take it into account [12] is to impose the condition (1.26):

$$\int d\xi_5\, \chi^*(\xi_5)[\widehat{P}_5 - i\xi_5/2]\chi(\xi_5) = 0. \tag{2.29}$$

This implies that the physical states are all proportional to $\chi_\alpha = (\sqrt{2} + e^{i\alpha}\xi_5)$, all with the same value of α. All of the wavefunctions are of the form

$$\Psi = \chi_\alpha(\xi_5)\phi(\eta_1^*, \eta_2^*, x), \tag{2.30}$$

with ϕ, as well as χ_α, of mixed Grassmann parity. The wavefunction ϕ is specified by four complex functions $\phi_{ij}(x)$:

$$\phi(\eta_1^*, \eta_2^*, x) = \phi_{00}(x) + \phi_{10}(x)\eta_1^* + \phi_{01}(x)\eta_2^* + \phi_{11}(x)\eta_1^*\eta_2^*. \tag{2.31}$$

The ϕ carry exactly the same information as a Dirac spinor wavefunction.

We may find an operator for ξ_5 which realizes the relation (2.20) as an anticommutation relation. This operator is

$$\widehat{\xi}_5 = \frac{\partial}{\partial \xi_5} - \frac{\xi_5}{2}, \tag{2.32}$$

which, strangely enough, is the direct transcription of the last constraint in (2.19) into operator form. To realize the relations (2.28) we assign the operators

$$\begin{aligned} \eta_1 &\to i\frac{\partial}{\partial \eta_1^*}, \\ \eta_2 &\to i\frac{\partial}{\partial \eta_2^*}. \end{aligned} \tag{2.33}$$

The constraints then become the conditions

$$\begin{aligned} 0 &= (\Box - m^2)\phi_{ij}(x), \\ 0 &= me^{i\alpha}\chi_{-\alpha}(-\xi_5)\phi(\eta_1^*, \eta_2^*, x) + \chi_\alpha(-\xi_5)(i\sqrt{2}\widehat{\xi}^\mu\partial_\mu)\phi(\eta_1^*, \eta_2^*, x). \end{aligned} \tag{2.34}$$

This last condition cannot be satisfied unless $\alpha = 0, \pi$ or $\phi(\eta_1^*, \eta_2^*, x) \equiv 0$. Thus one is forced to choose $\alpha = 0, \pi$ in order to obtain any quantum theory at all. For these values of α the fermionic constraint is equivalent to the condition

$$(\pm m + \sqrt{2} i \hat{\xi} \cdot \partial)\phi(\eta_1^*, \eta_2^*, x) = 0. \tag{2.35}$$

If we require the norm of the state functions

$$\langle \chi_\alpha | \chi_\alpha \rangle = \int d\xi_5 \, \chi_\alpha^*(\xi_5)\chi_\alpha(\xi_5) = \sqrt{8} \cos \alpha \tag{2.36}$$

to be positive, then α must be zero.

The condition (2.35) is obviously the Dirac equation, and does not contain the spurious factor of γ_5. The correct propagator may be obtained by the BFV prescription. Other authors [12,29,42] have obtained an incorrect propagator by not treating the boundary conditions in a careful manner.

To begin the BFV quantization, one first computes the BRS charge. Because the constraints do not form an abelian algebra, there is a term containing more than one ghost. The ghosts c_1 and b_1 are anticommuting while c_2 and b_2 are commuting. The BRS charge is

$$\Omega = c_1(P^2 + m^2) + c_2(P \cdot \xi + m\xi_5) - \frac{i}{2} c_2 c_2 b_1 + \bar{b}_1 \pi_1 + \bar{b}_2 \pi_2. \tag{2.37}$$

It is simplest to choose the gauge-fixing function to be

$$\Psi = -(b_1 \lambda_1 + b_2 \lambda_2)/\Delta\tau. \tag{2.38}$$

The propagator is computed from the BFV path integral

$$Z_{BFV} = \int \mathcal{D}P \, \mathcal{D}Q \, e^{i \int d\tau \, (P\dot{Q} - H_0 + \{\Psi, \Omega\}_{DB})}. \tag{2.39}$$

Here P and Q stand for all of the phase space degrees of freedom, including the ghosts and the Lagrange multiplier variables. Since the canonical Hamiltonian, H_0, of any reparametrization invariant action (such as the Berezin-Marinov action (2.17)) vanishes, the "Hamiltonian" governing τ evolution is just the "gauge fixing" piece

$$\begin{aligned} H &= -\{\Psi, \Omega\}_{DB} \\ &= \lambda_1(P^2 + m^2) - \lambda_2(\xi \cdot P + m\xi_5) + i\lambda_2 c_2 b_1 + b_i \bar{b}_i. \end{aligned} \tag{2.40}$$

Imposing the Feynman boundary conditions that the positive (negative) energy particles move forward (backward) in time on the motion of the particle, one finds from the x^μ equations of motion that λ_1 must be restricted to nonnegative values:

$$\dot{x}^\mu = \frac{\partial H}{\partial P_\mu} = 2\lambda_1 P^\mu - \lambda_2 \xi^\mu. \tag{2.41}$$

Unlike the simple case of the bosonic particle, there is a nontrivial ghost integral to evaluate. With the ghost boundary conditions $c_i = \pi_i = 0$, the ghost integrations lead to a factor

$$\delta^2(\bar{b}_f)\delta^2(\bar{b}_i)\,\mathrm{sdet}^{-1}\begin{pmatrix} \frac{\partial^2}{\partial\tau^2} & i\lambda_2\frac{\partial}{\partial\tau} \\ 0 & \frac{\partial^2}{\partial\tau^2} \end{pmatrix} = \delta^2(\bar{b}_f)\delta^2(\bar{b}_i), \tag{2.42}$$

which will be implicit in the following.

The path integral over the variable ξ_5 can be done as in Bordi and Casalbuoni [12], or directly by discretization of the integral

$$K(\xi_f|\xi_i) := \int\limits_{\xi(0)=\xi_i}^{\xi(\tau)=\xi_f} \mathcal{D}\xi\, e^{i\int_0^\tau dt(\frac{i}{2}\xi\dot{\xi}+\eta\xi)} = (\xi_f - \xi_i + i\eta\tau)e^{-\xi_f\xi_i/2}. \tag{2.43}$$

This expression is not the propagator, since it does not preserve the physical states under time evolution. The propagator defined from the expression (2.43) as [25]

$$\widetilde{K}(\xi_f|\xi_i) = K(\xi_f|\xi_i)e^{\xi_f\xi_i} \tag{2.44}$$

preserves the orthogonality of the physical states χ_o and the unphysical states χ_π

under τ evolution.

$$\int d\xi_f\, d\xi_i\, \chi_\pi^*(\xi_f)\widetilde{K}(\xi_f|-\xi_i)\chi_o(\xi_i) = 0. \tag{2.45}$$

The rest of the transition amplitude is easy to evaluate. The full propagator is

$$\begin{aligned} K(x_f, \eta_f, \xi_{5f}|x_i, \eta_i^*, \xi_{5i}) = \int d\lambda_2\, d^4p\, & \frac{e^{ip\cdot(x_f-x_i)}}{p^2+m^2-i\epsilon} \\ & \cdot (\xi_{5f} - \xi_{5i} - im\lambda_2)\, e^{\xi_{5f}\xi_{5i}/2} e^{\eta_f\cdot\eta_i^*} e^{i\lambda_2 p\cdot\xi}. \end{aligned} \tag{2.46}$$

The boundary conditions must be handled carefully. The transition element (2.46) must be evaluated between physical states. In particular, the ξ_5 dependence is crucial. To obtain the correct propagator, we take the matrix element of (2.46) between states whose dependence on ξ_5 is physical. Thus

$$\begin{aligned} \langle x_f, \eta_f|x_i, \eta_i^*\rangle &= \int d\xi_{5i}\, d\xi_{5f}\, \chi_o^*(\xi_{5f}) K(x_f, \eta_f, \xi_{5f}|x_i, \eta_i^*, -\xi_{5i})\chi_o(\xi_{5i}) \\ &\propto \int d^4p \frac{e^{ip\cdot(x_f-x_i)}}{p^2+m^2-i\epsilon}(m - \sqrt{2}p\cdot\xi), \end{aligned} \tag{2.47}$$

with ξ_μ identified as the expressions in (2.27), is the the Dirac propagator when the identification

$$\sqrt{2}\xi_\mu \rightarrow \gamma_\mu \tag{2.48}$$

is made. There is no factor of γ_5 to spoil unitarity, which is as it must be since the Hamiltonian path integral is manifestly unitary.

2.3 The Brink-Schwarz Superparticle

As an analog to the Klein-Gordon and Dirac particles, one may write an action for a particle moving in a space whose coordinates are a commuting position in space-time and at least one anticommuting spinor. These variables are the coordinates of *superspace*, analogous to the one dimensional superspace variables τ and ϑ which were used to write the action for the Dirac particle in a manifestly supersymmetric fashion. Let us simplify the discussion by taking a superspace consisting of a commuting vector variable x^μ and a single anticommuting Majorana-Weyl spinor θ. It is useful to write theories as theories in superspace because the coordinates (x^μ, θ) have simple transformation laws under the Poincaré superalgebra. Under the action of the usual Poincaré generators, $a_\mu P^\mu + \frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}$, with $J^{\mu\nu} = 2x^{[\mu}P^{\nu]} + \frac{1}{2}\zeta\gamma^{\mu\nu}\theta$, the coordinates of superspace transform as

$$
\begin{aligned}
x^\mu &\rightarrow x^\mu + a^\mu + \omega^\mu_\nu x^\nu, \\
\theta &\rightarrow \theta + \tfrac{1}{4}\omega_{\mu\nu}\gamma^{\mu\nu}\theta.
\end{aligned}
\tag{2.49}
$$

Supersymmetry transformations, generated by $Q\epsilon$, with $Q = \zeta - i\bar{\theta}\not{P}$ and $\not{P} = P_\mu\gamma^\mu$, transform the coordinates as

$$
\begin{aligned}
x^\mu &\rightarrow x^\mu - i\bar{\theta}\gamma^\mu\epsilon, \\
\theta &\rightarrow \theta + \epsilon.
\end{aligned}
\tag{2.50}
$$

By building a field theory whose fields are functions of position in superspace, one is able to construct actions which are manifestly supersymmetric.

The objects one is allowed to use to construct manifestly supersymmetric actions are the superfields themselves, the supersymmetric covariant derivative, and integration over superspace. The supersymmetric covariant derivative, $D = \frac{\partial}{\partial\theta} - i\bar{\theta}\gamma^\mu\frac{\partial}{\partial x^\mu}$, has the property that it anticommutes with the supersymmetry generator $Q = \frac{\partial}{\partial\theta} + i\bar{\theta}\gamma^\mu\frac{\partial}{\partial x^\mu}$.

In this section we investigate actions describing the motion of particles in superspace. These actions have as their dynamical variables the positions x^μ and θ in superspace and should be invariant under changes

$$
\begin{aligned}
\delta x^\mu &= -i\bar{\theta}\gamma^\mu\delta\theta, \\
\delta\theta &= \epsilon.
\end{aligned}
\tag{2.51}
$$

where ϵ is a constant anticommuting spinor. The first such action is the one written by Casalbuoni [17] without a one-dimensional metric on the world-line and by Brink and Schwarz [16]:

$$S_{BS} = \tfrac{1}{2}\int d\tau[e^{-1}(\dot{x}^\mu - i\bar{\theta}\gamma^\mu\dot{\theta})^2 + em^2]. \tag{2.52}$$

Here e is the "einbein" which takes the role of the metric tensor on the world-line. The case which is most relevant to the goal of understanding the dynamics of the manifestly supersymmetric string is the case of a massless ten-dimensional superparticle with a single spinor which is both Majorana and Weyl. If we had chosen a spinor which was Weyl but not Majorana, then the expression $\bar{\theta}\gamma^\mu\dot{\theta}$ would have to have been replaced by $\bar{\theta}\gamma^\mu\dot{\theta} - \dot{\bar{\theta}}\gamma^\mu\theta$. From now on we shall consider the massless case exclusively since it is the case which is most analogous to the superstring in any dimension.

The momenta for the variables e, x^μ and θ are defined as

$$\begin{aligned} P_e &:= \frac{\partial L}{\partial \dot{e}} = 0, \\ P_\mu &:= \frac{\partial L}{\partial \dot{x}^\mu} = e^{-1}(\dot{x}^\mu - i\bar{\theta}\gamma_\mu\dot{\theta}), \\ \zeta &:= \frac{\partial_R L}{\partial_R \dot{\theta}} = -i\bar{\theta}\gamma_\mu e^{-1}(\dot{x}^\mu - i\bar{\theta}\gamma^\mu\dot{\theta}), \end{aligned} \tag{2.53}$$

where the subscript R denotes the right-handed derivative. The canonical Hamiltonian is

$$H_0 = \tfrac{1}{2}eP^2, \tag{2.54}$$

and there are two primary constraints.

$$\begin{aligned} \phi_1 &:= P_e \approx 0, \\ \phi_2 &:= \zeta + i\bar{\theta}\not{P} \approx 0. \end{aligned} \tag{2.55}$$

Conservation of the constraints (2.55) requires the imposition of a third constraint,

$$\phi_3 := \tfrac{1}{2}P^2 \approx 0. \tag{2.56}$$

One may check that the most general allowed Hamiltonian is

$$H = \tfrac{1}{2}(e + \lambda_3)P^2 + \zeta\not{P}\lambda_2 + P_e\lambda_1, \tag{2.57}$$

with $\lambda_{1,2,3}$ arbitrary, and that there are no further constraints. The constraints $P_e \approx 0$

and $\frac{1}{2}P^2 \approx 0$ are purely first-class, generating τ-reparametrizations, while ϕ_2 contains both first- and second-class pieces as one may see from the canonical Poisson brackets

$$\{\phi_2, \phi_2\}_{PB} = 2i\gamma^0 \not{P} \not\approx 0. \tag{2.58}$$

Because P_μ is null, the first-class piece of ϕ_2 may be identified as

$$\phi_2^{(1st)} = \phi_2 \not{P} \approx \zeta \not{P} \approx 0. \tag{2.59}$$

This constraint generates the analog of the superstring's κ-symmetry in the superparticle system:

$$\begin{aligned} \delta\theta &= \{\theta, \phi_2^{(1st)}\epsilon\}_{PB} \\ &= e^{-1}(\dot{x}^\mu - i\bar{\theta}\gamma^\mu\dot{\theta})\gamma_\mu\epsilon, \\ \delta x^\mu &= \{x^\mu, \phi_2^{(1st)}\epsilon\}_{PB} \\ &= i\bar{\theta}\gamma^\mu\delta\theta, \\ \delta e &= 4i\bar{\epsilon}\dot{\theta}. \end{aligned} \tag{2.60}$$

The second-class piece of the constraint may not be identified covariantly unless the system is further enlarged and constrained. If one wishes not to add any more variables to the system, the second-class piece of ϕ_2 may be identified only non-covariantly, for instance, as

$$\phi_2^{(2nd)} = \phi_2\gamma^0 \not{P} \gamma^0 \approx 0 \qquad (2.61).$$

Discounting the einbein and its momentum, the canonical phase space for the Brink-Schwarz superparticle has $2D$ bosonic and $2S$ fermionic dimensions, where D is the dimensionality of spacetime and S is the number of real components of whatever spinor we are considering. For a ten-dimensional Majorana-Weyl spinor, S is sixteen. There is a single first-class bosonic constraint, namely ϕ_3, $S/2$ fermionic first-class constraints and $S/2$ fermionic second-class constraints. Thus the gauge fixed phase space has $2D - 2$ bosonic and $S/2$ fermionic physical degrees of freedom.

2.4 The Siegel Superparticle

Obviously, the fact that the second-class constraints of the Brink-Schwarz superparticle cannot be identified covariantly in a simple manner means that the theory cannot be quantized in a simple, covariant fashion. It would be ideal, then, if one could do away with the second-class problems and also have a manifestly covariant gauge choice. This was the idea of Siegel when he proposed the action [52]

$$S_{Siegel} = \int d\tau[(\dot{x} - i\bar{\theta}\gamma\dot{\theta}) \cdot P - \tfrac{1}{2}gP^2 + \dot{\bar{\theta}}\pi + \psi \not{P}\pi]. \tag{2.62}$$

This action, excepting the last two terms, is the first-order form of the massless Brink-Schwarz action (2.52).

Just as in the first-order form of electrodynamics (the Palatini formalism), one may treat both x and P as "positions" and give each a momentum. The Dirac procedure will identify P with the conjugate momentum to x through second-class constraints. The analysis of the constraints is as straightforward as before, but is somewhat messier.

For the Siegel action (2.62) take the coordinates to be $x^\mu, P_\mu, \theta, \pi, \psi,$ and g. Their canonically conjugate momenta are then $P_x, P_P, \zeta, P_\pi, P_\psi,$ and P_g respectively. A conserved set of constraints is obtained after two iterations of the Dirac procedure. These constraints are

$$\begin{aligned}
P_x - P &\approx 0,\\
P_P &\approx 0,\\
P_\pi &\approx 0,\\
\zeta + \bar{\pi} + i\bar{\theta}\not{P} &\approx 0,\\
\not{P}\pi &\approx 0,\\
P_g &\approx 0,\\
P_\psi &\approx 0,\\
P^2 &\approx 0.
\end{aligned} \tag{2.63}$$

The first pair are second-class and may be ignored if P is identified as conjugate to

x. The second pair are similarly ignorable if $\bar{\pi}$ is identified everywhere with $-\zeta - i\bar{\theta}\not{P}$. This leaves a consistent set of constraints

$$\begin{aligned} P^2 &\approx 0, \\ \zeta\not{P} &\approx 0, \\ P_g &\approx 0, \\ P_\psi &\approx 0, \end{aligned} \tag{2.64}$$

which are *all* first-class. The corresponding Hamiltonian is

$$H = \tfrac{1}{2}\lambda_1 P^2 + \lambda_2 \not{P}\bar{\zeta} + \lambda_3 P_g + \lambda_4 P_\psi, \tag{2.65}$$

with $\lambda_{1,2,3,4}$ arbitrary.

Again discounting the einbein g, the gravitino ψ, and their momenta, the canonical phase space has, as before, $2D$ bosonic and $2S$ fermionic dimensions. Here again, there is a single bosonic and $S/2$ fermionic first-class constraints; but there are no second-class constraints. Thus the gauge-fixed phase space has $2D-2$ bosonic and S fermionic physical degrees of freedom. The action of Siegel thus describes a different system from the one described by the Brink-Schwarz action (2.52). The Siegel superparticle, because it lacks second-class constraints, is easily quantizable in a covariant fashion.

2.5 Light-Cone Quantization

Let us examine the Brink-Schwarz superparticle in ten dimensions with a single Majorana-Weyl spinor. It was noted before that this is the most important case to consider because it shares the main difficulty of the Green-Schwarz string, namely second-class constraints. The first-class constraints of (2.55) and (2.56) are

$$P^2 \approx 0, \quad D\not{P} \approx 0, \tag{2.66}$$

while the second-class constraints are not so easily identified in a covariant fashion. In this section we will identify them in a particular non-covariant fashion. If we define the light-cone gamma matrices

$$\gamma^{\pm} := \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma^9) \tag{2.67}$$

we may choose the second-class constraints to be

$$\phi_2^{(2nd)} = D\gamma^+, \tag{2.68}$$

and calculate the matrix of Poisson brackets

$$\{\phi_2^{(2nd)}, \phi_2^{(2nd)}\} = -4i\sqrt{2}P_-(\frac{\gamma^-\gamma^+}{2}). \tag{2.69}$$

This matrix can be easily inverted so that the Dirac bracket may be defined. This inverse is

$$\Delta_{ab}^{-1} = \frac{i}{4\sqrt{2}P_-}(\frac{\gamma^-\gamma^+}{2})_{ab} \tag{2.70}$$

on the subspace defined by the projection operator $\gamma^-\gamma^+/2$. The Dirac brackets which are computed from the second-class constraints are

$$\begin{aligned}
\{x^\mu, P_\nu\}_{DB} &= \delta^\mu_\nu, \\
\{x^\mu, x^\nu\}_{DB} &= \frac{i}{4P_-}\overline{\theta}\gamma^{\mu+\nu}\theta, \\
\{x^\mu, \theta_a\}_{DB} &= -\frac{1}{4P_-}(\gamma^+\gamma^\mu\theta)_a, \\
\{\theta_a, \theta_b\}_{DB} &= \frac{i}{4\sqrt{2}P_-}(\gamma^+\gamma^-)_{ab}.
\end{aligned} \tag{2.71}$$

The fact that the coordinates x^μ do not have vanishing brackets with each other is a general feature of theories with second-class constraints. Through much tedious algebra one finds that the x^μ may be replaced by either of two new coordinates

$$\begin{aligned} q^\mu &= x^\mu + \frac{iP_\sigma}{2P_-}\overline{\theta}\gamma^{\mu+\sigma}\theta, \\ \tilde{q}^\mu &= x^\mu - \frac{i}{2}\overline{\theta}\gamma^{\mu+-}\theta, \end{aligned} \tag{2.72}$$

whose Dirac brackets vanish.

$$\{q^\mu, q^\nu\}_{DB} = \{\tilde{q}^\mu, \tilde{q}^\nu\}_{DB} = 0. \tag{2.73}$$

The tilded set of variables is completed by the fermionic coordinates

$$\begin{aligned} \tilde{\vartheta}_+ &:= \gamma^-\gamma^+\theta, \\ \tilde{\vartheta}_- &:= \frac{\gamma^+\gamma^-}{\sqrt{P_-}}\theta. \end{aligned} \tag{2.74}$$

The untilded variables are completed by the fermionic coordinates

$$\begin{aligned} \vartheta_+ &:= \gamma^-\gamma^+\theta, \\ \vartheta_- &:= \frac{\gamma^+\not{P}}{\sqrt{P_-}}\theta. \end{aligned} \tag{2.75}$$

These new sets of coordinates satisfy

$$\begin{aligned} \{q^\mu, \vartheta_\pm\}_{DB} &= 0, \\ \{\tilde{q}^\mu, \tilde{\vartheta}_\pm\}_{DB} &= 0, \\ \{\vartheta_+, \vartheta_+\}_{DB} &= \{\tilde{\vartheta}_+, \tilde{\vartheta}_+\}_{DB} = 0, \\ \{\vartheta_+, \vartheta_-\}_{DB} &= \{\tilde{\vartheta}_+, \tilde{\vartheta}_-\}_{DB} = 0, \\ \{\vartheta_-, \vartheta_-\}_{DB} &= \frac{i}{\sqrt{2}(P_-)^2}\gamma^+\gamma^-, \\ \{\tilde{\vartheta}_-, \tilde{\vartheta}_-\}_{DB} &= \frac{i}{\sqrt{2}}\gamma^+\gamma^-. \end{aligned} \tag{2.76}$$

We define the (non-canonical) momenta for the $\vartheta_\pm$ and $\tilde{\vartheta}_\pm$ as

$$\begin{aligned} \zeta_+ &:= \tfrac{1}{4}\zeta \not{P}\gamma^+, \\ \tilde{\zeta}_+ &:= \tfrac{1}{4}\zeta\gamma^-\gamma^+, \\ \zeta_- = \tilde{\zeta}_- &:= \tfrac{1}{4}\zeta\gamma^+\gamma^-. \end{aligned} \tag{2.77}$$

The momenta $\zeta_\pm$, in particular, satisfy

$$\begin{aligned} \{\vartheta_+, \zeta_+\}_{DB} &= P_-(\frac{\gamma^-\gamma^+}{2}), \\ \{\vartheta_-, \zeta_+\}_{DB} &= 0, \\ \{\vartheta_+, \zeta_-\}_{DB} &\approx 0, \\ \{\vartheta_-, \zeta_-\}_{DB} &\approx \tfrac{3}{2}\sqrt{P_-}(\frac{\gamma^+\gamma^-}{2}). \end{aligned} \tag{2.78}$$

A quantum theory might be built upon either of the sets of variables $P_\mu, q^\mu, \vartheta_\pm, \zeta_\pm$, or $P_\mu, \tilde{q}^\mu, \tilde{\vartheta}_\pm, \tilde{\zeta}_\pm$. Perhaps the simplest way to build a quantum theory is to work in momentum space for the x^μ and position space for the ϑ. One solves the constraint $P^2 = P_iP_i - 2P_+P_- \approx 0$ for $P_+ = P_iP_i/2P_-$. The state is taken to be a function of the momentum variables P_i, P_- only. On such a state, the fermionic momenta may be represented as the differential operators

$$\begin{aligned} \widehat{\zeta}_+ &= iP_-\frac{\partial}{\partial\vartheta_+}(\frac{\gamma^-\gamma^+}{2}), \\ \widehat{\zeta}_- &= \frac{3i}{2}\sqrt{P_-}\frac{\partial}{\partial\vartheta_-}(\frac{\gamma^+\gamma^-}{2}). \end{aligned} \tag{2.79}$$

The first-class constraint $\bar{D}\not{P} \approx 0$ may be written

$$\bar{D}\not{P} \approx \zeta\not{P} \approx \frac{1}{2P_-}\zeta_+\not{P} = \frac{i}{4}\frac{\partial}{\partial\vartheta}\gamma^-\gamma^+\not{P}. \tag{2.80}$$

The physical state Φ, then, is a function of the aforementioned momentum variables, the spinor ϑ_+ and half of the spinor ϑ_- which anticommutes with itself (here denoted by η) satisfying the condition

$$\frac{\partial}{\partial\vartheta_+}\gamma^-\gamma^+\not{P}\Phi(P_i, P_-, \vartheta_+, \eta) = 0. \tag{2.81}$$

A position space dependent state may the be obtained by Fourier transforming the solution to (2.81) with P_+ identified as $P_iP_i/2P_-$.

Clearly this is not a useful characterization of a physical state and it is not even covariant at that. Before going on to give a covariant version of the ten-dimensional Majorana-Weyl Brink-Schwarz particle, let us examine the four-dimensional version which can be made quite simple through the elimination of the second-class constraints without changing the physical content of the theory.

2.6 The Four Dimensional Brink-Schwarz Superparticle

In four dimensions a spinor may not be both Majorana and Weyl at the same time. For simplicity let us choose to examine the case of a single Weyl spinor which is analogous to the case of two Majorana spinors in ten dimensions. We will follow the conventions* of Wess and Bagger [56]. The free superparticle action

$$S = \int d\tau \frac{1}{2e}(\dot{x}^\mu - i\overline{\theta}\sigma^\mu\dot{\theta} + i\dot{\overline{\theta}}\sigma^\mu\theta)^2 \tag{2.82}$$

leads in the usual way to the constraints $P^2 \approx D_\alpha \approx \bar{D}_{\dot{\alpha}} \approx 0$. (There are two fermionic constraints because there are two real components to the spinor θ.) They satisfy the usual algebra

$$\begin{aligned}
&\{D_\alpha, D_\beta\} = 0,\\
&\{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0,\\
&\{D_\alpha, \bar{D}_{\dot{\beta}}\} = 2i\sigma^\mu_{\alpha\dot{\beta}}P_\mu.
\end{aligned} \tag{2.83}$$

Again, since P_μ is null, there are both first- and second-class constraints present. Essentially, D_α and $\bar{D}_{\dot{\beta}}$ are each half first- and half second-class. If we could eliminate

---

* Spinor indices are either undotted or dotted according to their handedness. Undotted indices are summed upper left to lower right, while dotted indices are summed lower left to upper right. Both are raised and lowered with the antisymmetric tensors $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot{\alpha}\dot{\beta}}$, $\epsilon^{12} = -\epsilon_{12} = 1$. The matrices $\sigma^\mu_{\alpha\dot{\alpha}}$ and $\bar{\sigma}^{\mu\dot{\beta}\beta}$ are $\sigma^0 = \bar{\sigma}^0 = -1$ and the usual Pauli matrices $\sigma^{1,2,3} = -\bar{\sigma}^{1,2,3}$.

half of the second-class constraints, then the remaining half would become first-class and *the dynamical content of the theory would be unchanged.* In fact, it is quite easy to eliminate half of the second-class constraints. All we need to do is to eliminate the pieces of D_α (say) which are annihilated by the matrix $\not{P}$. A first-class reduction of the system which is dynamically equivalent to the original system is

$$\begin{aligned} D\not{P} &\approx 0, \\ \bar{D} &\approx 0, \\ P^2 &\approx 0. \end{aligned} \tag{2.84}$$

All of the brackets of the system of constraints vanish identically except for

$$\{D\not{P}, \bar{D}\} = -2iP^2 \approx 0, \tag{2.85}$$

which vanishes only weakly. The absence of any second-class constraints allows us the easy identifications

$$\begin{aligned} &\bar{D}_{\dot\alpha} \to i\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - \theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu, \\ &D_\beta \to i\frac{\partial}{\partial\theta^\beta} - \sigma^\mu_{\beta\dot\beta}\bar\theta^{\dot\beta}\partial_\mu, \\ &P^2 \to -\Box. \end{aligned} \tag{2.86}$$

A state's wavefunction, Φ, is a commuting function of the coordinates

$$\Phi = \Phi(x^\mu, \theta, \overline{\theta}) \tag{2.87}$$

which satisfies the constraints (2.84) as operatorial equations. If Φ satisfies the condition $\bar{D}\Phi = 0$ it is known as a scalar superfield and has the component expansion

$$\begin{aligned} \Phi(x,\theta,\overline{\theta}) = \phi(x) &+ \theta\psi(x) + i\theta\sigma^\mu\overline{\theta}\partial_\mu\phi(x) + \theta\theta F(x) \\ &- \frac{i}{2}\theta\theta\partial_\mu\psi(x)\sigma^\mu\overline{\theta} + \frac{1}{4}\theta\theta\overline{\theta\theta}\Box\phi(x), \end{aligned} \tag{2.88}$$

where $\psi(x)$ is an anticommuting Weyl-spinor function and $\phi(x)$ and $F(x)$ are commuting scalar functions. The other fermionic constraint,

$$\partial_\mu \bar{\sigma}^{\mu\dot{\alpha}\alpha} D_\alpha \Phi = 0, \tag{2.89}$$

yields the field equations of the Wess-Zumino scalar multiplet.

$$\begin{aligned} \partial_\mu F(x) &= 0, \\ \sigma^\mu \partial_\mu \psi(x) &= 0, \\ \Box \phi(x) &= 0. \end{aligned} \tag{2.90}$$

This is in contradiction to the findings of [22,43] that the four-dimensional Brink-Schwarz superparticle describes a vector multiplet.

The simplicity of the system without second-class constraints points the way to a method for handling the ten dimensional Brink-Schwarz superparticle in a covariant fashion. It is important that any second-class constraints have simple Poisson brackets, or better yet, that there be no second-class constraints at all.

2.7 Null Systems

A null system is a set of dynamical variables together with a set of constraints which exactly eliminate the dynamics of the variables. The simplest example one can think of is the single pair of conjugate variables p and q and the single first-class constraint $p \approx 0$. The value of q at any time is a purely gauge dependent quantity determined by the Lagrange multiplier λ in the Hamiltonian of the system.

$$\begin{aligned} H &= \lambda(t) p, \\ \dot{q} &= \{q, H\} = \lambda. \end{aligned} \tag{2.91}$$

A less trivial example is that of a set of D^2 vielbeine e_μ^m and their D^2 momenta p_n^ν. A set of D^2 first-class constraints will eliminate their dynamics. One such set of constraints is the set of $gl(D)$ generators

$$D_{mn} = p_n^\mu e_\mu^m \approx 0. \tag{2.92}$$

Alternatively, one could trade the symmetric generators for constraints which are more useful. For instance, the symmetric constraints

$$\delta_{\mu\nu} e_m^\mu e_n^\nu - \delta_{mn} \approx 0, \tag{2.93}$$

have weakly vanishing Poisson brackets with the antisymmetric $SO(D)$ generators of (2.92) and are therefore possible alternatives.

The quantization of the null system of vielbeine is straightforward. The constraints $D_{[mn]} \approx 0$ become differential operator generators of the $SO(D)$ group. One assumes that the wavefunction $\phi(e_m^\mu)$ has a power series expansion in the e_m^μ. Any such function ϕ which satisfies the operatorial equation

$$\widehat{D}_{[mn]}\phi = 0, \tag{2.94}$$

must be constructed from $SO(D)$ singlet functions. The solution to the constraint equation (2.94) is, in fact, a constant independent of the e_m^μ when the constraint (2.93) is imposed.

$$\begin{aligned}
\phi(e_m^\mu) = &\sum_{k=0}^{\infty} (e_n^{\mu_1} e_n^{\nu_1}) \cdots (e_m^{\mu_k} e_m^{\nu_k}) \phi_{\mu_1\nu_1 \ldots \mu_k\nu_k} \\
&\quad + (e_{a_1}^{\alpha_1} \cdots e_{z_1}^{\zeta_1} \epsilon^{a_1 \cdots z_1}) \cdots (e_{a_k}^{\alpha_k} \cdots e_{z_k}^{\zeta_k} \epsilon^{a_k \cdots z_k}) \psi_{\alpha_1 \cdots \zeta_k}, \\
= &\sum_{k=0}^{\infty} \delta^{\mu_1\nu_1} \cdots \delta^{\mu_k\nu_k} \phi_{\mu_1\nu_1 \ldots \mu_k\nu_k} \\
&\quad + (\epsilon^{\alpha_1 \cdots \zeta_1} \cdots \epsilon^{\alpha_k \cdots \zeta_k}) \psi_{\alpha_1 \cdots \zeta_k}.
\end{aligned} \tag{2.95}$$

2.8 Covariant Quantization in Ten Dimensions

It is possible to mimic closely the non-covariant separation of the second-class constraints while still maintaining manifest covariance. To separate the constraints we must define an embedding of the light-cone spinors in the Lorentz group. That is, we will need to parametrize the embedding $SO(8) \subset SO(9,1)$. To this end, let us add a null system to the original phase space. We add coordinates Υ^a_A and $\Upsilon^a_{\dot{A}}$ which are commuting ten-dimensional Majorana-Weyl spinors of the same handedness as the spinors θ of the original theory. The indices A and $\dot{A}$ each take the values $1, \ldots, 8$. The conjugate momenta to the Υ^a_A and $\Upsilon^a_{\dot{A}}$ are denoted by P^A_a and $P^{\dot{A}}_a$ respectively, and they satisfy the canonical Poisson bracket relations

$$\begin{aligned} \{\Upsilon^a_A, P^B_b\} &= \delta^B_A \delta^a_b, \\ \{\Upsilon^a_{\dot{A}}, P^{\dot{B}}_b\} &= \delta^{\dot{B}}_{\dot{A}} \delta^a_b. \end{aligned} \tag{2.96}$$

The phase space now has 512 extra degrees of freedom which must be eliminated by the imposition of constraints if the dynamics are to remain unchanged. The simplest constraints to impose would be the vanishing of the momenta P^B_b and $P^{\dot{B}}_b$. These constraints would carry Lorentz indices, so it would be more convenient to introduce instead the constraints

$$\begin{aligned} D_{AB} &:= \Upsilon^a_A P_{aB} \approx 0, \\ D_{A\dot{B}} &:= \Upsilon^a_A P_{a\dot{B}} \approx 0, \\ D_{\dot{A}B} &:= \Upsilon^a_{\dot{A}} P_{aB} \approx 0, \\ D_{\dot{A}\dot{B}} &:= \Upsilon^a_{\dot{A}} P_{a\dot{B}} \approx 0. \end{aligned} \tag{2.97}$$

which are the generators of the group $GL(16)$. In order to separate the fermionic constraints $D \approx 0$, we must replace some of the constraints of (2.97) with other, more useful, constraints.

The constraints (2.97) satisfy Poisson bracket relations showing them to be the elements of $gl(16)$

$$\begin{aligned}
&\{D_{AB}, D_{CD}\} = \delta_{BC} D_{AD} - \delta_{AD} D_{CB}, \\
&\{D_{A\dot{B}}, D_{CD}\} = -\delta_{AD} D_{C\dot{B}}, \\
&\qquad\qquad \text{etc},
\end{aligned} \tag{2.98}$$

and are thus first-class. Because these constraints are all first-class, imposing them would eliminate all of the extra 512 degrees of freedom added to the system. The Υ are quite similar to the vielbeine null system discussed in the last section. It will be shown later that to separate the constraints it will be necessary to break the symmetry to $GL(8) \times SO(8)$. It is useful to replace some of the constraints of the $GL(16)$ algebra by constraints which express the null space of the matrix $\not{P}$. Because the matrix $\gamma^0 \not{P}$ satisfies the projection-operator-like relation

$$(\gamma^0 \not{P})^2 = -2P^0 (\gamma^0 \not{P}), \tag{2.99}$$

we may postulate the following alternative constraints

$$\begin{aligned}
\Phi_{\dot{A}\dot{B}} &:= \overline{\Upsilon}_{\dot{A}} \not{P} \Upsilon_{\dot{B}} \approx 0, \\
\Phi_{A\dot{B}} &:= \overline{\Upsilon}_{A} \not{P} \Upsilon_{\dot{B}} \approx 0, \\
\Phi_{AB} &:= \overline{\Upsilon}_{A} \not{P} \Upsilon_{B} - \delta_{AB} \approx 0,
\end{aligned} \tag{2.100}$$

which are symmetric in their indices. Here we have included the first constraint $\Phi_{\dot{A}\dot{B}}$, even though it is not independent. In fact, it is implied by the second constraint:

$$\Phi_{\dot{A}\dot{B}} \approx \Phi_{\dot{A}C} \Phi_{C\dot{B}}. \tag{2.101}$$

The constraints (2.100) are to replace the generators of (2.97) which are symmetric in their indices. We will keep those constraints which generate $GL(8) \times SO(8) \times$

$[SO(16)/(SO(8)\times SO(8))]$.

$$\begin{aligned} D_{\dot{A}\dot{B}} &\approx 0,\\ D_{[AB]} &\approx 0, \\ D_{\dot{A}B} &\approx 0. \end{aligned} \tag{2.102}$$

The new constraints (2.100) satisfy the Poisson brackets with (2.102)

$$\begin{aligned} \{D_{\dot{A}\dot{B}}, \Phi_{\dot{C}\dot{D}}\} &= -2\delta_{\dot{B}(\dot{C}}\Phi_{\dot{D})\dot{A}},\\ \{D_{\dot{A}\dot{B}}, \Phi_{\dot{C}D}\} &= -\delta_{\dot{B}\dot{C}}\Phi_{\dot{A}D},\\ \{D_{[AB]}, \Phi_{CD}\} &= -\delta_{B(C}\Phi_{D)A} + \delta_{A(C}\Phi_{D)B},\\ \{D_{[AB]}, \Phi_{\dot{C}D}\} &= \delta_{D[A}\Phi_{B]\dot{C}},\\ \{D_{\dot{A}B}, \Phi_{\dot{C}D}\} &= -\delta_{BD}\Phi_{\dot{A}\dot{C}},\\ \{D_{\dot{A}B}, \Phi_{CD}\} &= -2\delta_{B(C}\Phi_{D)\dot{A}},\\ \text{all others} &\equiv 0. \end{aligned} \tag{2.103}$$

These brackets reveal that all the constraints (2.100) and (2.102) are first-class. Taking the symmetries and reducibility of the constraints into account, one may count that there are 100 independent constraints of (2.100) and 156 constraints of type (2.102). Together they fix out 512 degrees of freedom, exactly the number of variables added to the original system. The constraint $\bar{D}_a = \zeta_a + i\bar{\theta}\not{P}_a = 0$ may now be separated covariantly. The constraints

$$\begin{aligned} \phi_{\dot{A}} &:= \bar{D}\Upsilon_{\dot{A}} \approx 0,\\ \Gamma_A &:= \bar{D}\Upsilon_A \approx 0, \end{aligned} \tag{2.104}$$

neatly separate the constraint $\bar{D} \approx 0$ into first- and second-class pieces since they obey

$$\begin{aligned} \{\phi_{\dot{A}}, \phi_{\dot{B}}\} &= 2i\Phi_{\dot{A}\dot{B}} \approx 0,\\ \{\phi_{\dot{A}}, \Gamma_B\} &= 2i\Phi_{\dot{A}B} \approx 0,\\ \{\Gamma_A, \Gamma_B\} &= 2i(\Phi_{AB} + \delta_{AB}) \approx 2i\delta_{AB}. \end{aligned} \tag{2.105}$$

The new first-class constraints are the $\phi_{\dot{A}}$ and the new second-class constraints are the Γ_B. Further, the second-class constraints have the simplest brackets one could hope to obtain.

It is possible to perform a first-class reduction of the system (2.100), (2.102), and (2.104). It need not have been possible to do so except that the Γ_A will form the Clifford algebra of $SO(8)$ upon quantization and gamma matrices are covariantly constant if their spinor indices are rotated along with their vector index.

When using the constraints $D_{[AB]} \approx 0$, one may not ignore half of the Γ_A's because under the Poisson bracket algebra

$$\begin{aligned} \{D_{[AB]}, D_{[CD]}\} &= \tfrac{1}{2}\big(\delta_{AC} D_{[BD]} - \delta_{AD} D_{[BC]} \\ &\qquad + \delta_{BD} D_{[AC]} - \delta_{BC} D_{[AD]}\big), \\ \{D_{[AB]}, \Gamma_C\} &= \tfrac{1}{2}(\delta_{AC}\Gamma_B - \delta_{BC}\Gamma_A), \end{aligned} \tag{2.106}$$

the Γ_A transform as an **8** of $SO(8)$. The redefinition of constraints

$$\mathcal{D}_{[AB]} = D_{[AB]} - \tfrac{i}{4}\Gamma_A\Gamma_B \approx 0, \tag{2.107}$$

yields the same $SO(8)$ algebra (2.106) for the $\mathcal{D}_{[AB]}$'s, but now the Γ_A are $SO(8)$ singlets:

$$\{\mathcal{D}_{[AB]}, \Gamma_C\} = \tfrac{1}{2}(\Gamma_A\Phi_{BC} - \Gamma_B\Phi_{AC}) \approx 0. \tag{2.108}$$

The weakly equals sign in (2.108) means we have used the constraint $\Phi_{AB} \approx 0$.

The simplest way of choosing a first-class reduction of the second-class constraints Γ_A is to choose four linear combinations of them which have vanishing Poisson brackets among themselves. The following combinations will work.

$$\Gamma_{2n-1} + i\Gamma_{2n} \approx 0, \quad n = 1, \ldots, 4. \tag{2.109}$$

These are equivalent to the quantum "chirality" conditions

$$i\widehat{\Gamma}_{2n-1}\widehat{\Gamma}_{2n} = 1, \quad n = 1, \ldots, 4. \tag{2.110}$$

As yet there does not seem to be a straightforward implementation of the system of constraints (2.100), (2.102) and (2.104) or their first-class replacements (2.107) and

(2.109). The correct implementation of these constraints should yield a superfield formulation of the linearized ten-dimensional supersymmetric Yang-Mills theory. Other groups [15,23,45,46,54] have considered similar null systems for use in quantizing the Brink-Schwarz action but have not achieved any useful formalism.

III. String Theories

The bosonic string is a simple model in which to test ideas which may prove useful for the more relevant superstring models; either the NSR "covariant" superstring or the manifestly supersymmetric string of Green and Schwarz. In this chapter we will discuss the bosonic and the Green-Schwarz strings.

The full power of the BFV quantization is not needed for either the bosonic string or the NSR string. These strings have constraint algebras with constant structure functions and they do not have any second-class constraints. Thus the method of Faddeev and Popov is sufficient for them. This chapter first explores the quantization of the bosonic string. In particular, it is shown that the Nambu action and the Polyakov action both lead to the same quantum theory.

Next, the two most promising actions for a manifestly supersymmetric string are examined. These actions, the one proposed by Siegel and the other proposed by Green and Schwarz, are shown to be essentially equivalent. The problems of quantization of these actions are briefly discussed, and a method of quantization is proposed.

3.1 The Bosonic String

The first classical action that we discuss is the one proposed by Nambu to generalize the action (2.1) for the point particle. The action (2.1) for the point particle is the length of its world line. The natural generalization is that the action for a string moving through spacetime should be the area of the sheet it sweeps out during its evolution. Thus, the bosonic string is described classically by its position $X^{\mu}(\sigma)$ in spacetime and its evolution is governed by the action [44]

$$
\begin{aligned}
S_{Nambu} &= -\int d^2\sigma \sqrt{-\det(\partial_\alpha X^\mu \partial_\beta X_\mu)} \\
&= -\int d^2\sigma \sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2}.
\end{aligned}
\tag{3.1}
$$

The momentum conjugate to $X^\mu(\sigma)$,

$$
\begin{aligned}
P_\mu(\sigma) &:= \frac{\delta L}{\delta \dot{X}^\mu(\sigma)} \\
&= \frac{X'_\mu(\dot{X}\cdot X') - \dot{X}_\mu(X'^2)}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2}},
\end{aligned}
\tag{3.2}
$$

must satisfy the Virasoro constraints

$$
\begin{aligned}
P\cdot X' &\approx 0, \\
P^2 + X'^2 &\approx 0.
\end{aligned}
\tag{3.3}
$$

Because the action is reparametrization invariant, the Hamiltonian vanishes identically. It is convenient to combine the constraints (3.3) into the form

$$
T_\pm = \tfrac{1}{2}(P \pm X')^2 \approx 0,
\tag{3.4}
$$

in order to express their Poisson brackets simply:

$$
\begin{aligned}
&\{T_\pm(\sigma), T_\pm(\varrho)\} = \pm 2(T_\pm(\sigma) + T_\pm(\varrho))\delta'(\sigma-\varrho), \\
&\{T_\pm(\sigma), T_\mp(\varrho)\} = 0.
\end{aligned}
\tag{3.5}
$$

The non-zero structure functions are “constants” given by $\pm 2\delta'(\sigma-\varrho)$ and the constraints are obviously first-class.

Another action for the bosonic string is that given by Brink, Di Vecchia and Howe [14] and Deser and Zumino [19] and promoted by Polyakov [48]. The "Polyakov" action is quadratic in the fields X^μ and would therefore seem to be the more appropriate action to describe the system quantum mechanically. To write the action, we introduce the metric, $g_{\alpha\beta}$, on the world sheet of the string. The action used in the Polyakov path integral is

$$S_P = -\tfrac{1}{2}\int d^2\sigma \sqrt{-g}\, g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu. \tag{3.6}$$

The world-sheet metric has no dynamics and its momentum consequently vanishes. The Virasoro constraints ensure the conservation of the metric's momentum,

$$\begin{aligned} &P_{g\alpha\beta} \approx 0, \\ &\dot{P}_{g\alpha\beta} \approx 0 \Longleftrightarrow \tfrac{1}{2}(P \pm X')^2 \approx 0, \end{aligned} \tag{3.7}$$

which follows from the Hamiltonian

$$H = \frac{1}{2g^{00}\sqrt{g}}(P^2 + X'^2) - \frac{g^{01}}{g^{00}} P \cdot X' \tag{3.8}$$

with the momentum defined by

$$P_\mu(\sigma) = -\sqrt{-g}\, g^{0\beta}\partial_\beta X_\mu(\sigma). \tag{3.9}$$

The $g^{\alpha\beta}$ are obviously irrelevant variables from the classical point of view. The $X^\mu(\sigma)$ obey the same dynamics as they did in the Nambu theory. Quantum mechanics will resurrect the action (3.6) in a manner which could only be described as circuitous. In this fashion the bosonic string is quite analogous to electrodynamics where the Lagrange multiplier enforcing the Gauss law constraint in the Hamiltonian formalism becomes the time component of the quantum vector potential.

Had we started from the path integral in the form of equation (1.37), we would have found the integral

$$Z = \int \mathcal{D}X^\mu \mathcal{D}g_{\alpha\beta}(-g)^{-2}(g^{00})^{-\frac{7}{4}}\delta[g-g_0]e^{iS_P} \tag{3.10}$$

once the momentum integrations were done. This integral is still not gauge fixed, and is thus not well defined perturbatively.

The BFV method for constructing the path integral starts from the BRS charge [37]

$$\Omega = \int d\sigma(c_+T_+ + c_-T_- + 2b_+c'_+c_+ - 2b_-c'_-c_- + \bar{b}_+\pi_+ + \bar{b}_-\pi_-). \tag{3.11}$$

The correct gauge fixed action may be obtained from the gauge fixing fermionic function

$$\Psi = -\tfrac{1}{2}\int d\varrho(b_+ + b_-). \tag{3.12}$$

The gauge fixed action

$$S_{BFV} = \int d^2\sigma(P\dot{Q} + \{\Psi, \Omega\}) \tag{3.13}$$

becomes

$$\begin{aligned} S_{BFV} = \int d^2\sigma(&P\cdot\dot{X} + \pi_i\dot{\lambda}_i + b_i\dot{c}_i + \bar{b}_i\dot{\bar{c}}_i \\ &-\tfrac{1}{2}(T_+ + T_-) - c_+b'_+ + c_-b'_-). \end{aligned} \tag{3.14}$$

After evaluating the P, λ, π, $\bar{b}$, and $\bar{c}$ integrals, the gauge fixed path integral is obtained.

$$Z = \int \mathcal{D}X^\mu \mathcal{D}b\,\mathcal{D}c\, e^{i\int d^2\sigma(-\frac{1}{2}(-\dot{X}^2+X'^2)+c_+\partial_-b_+ + c_-\partial_+b_-)}. \tag{3.15}$$

3.2 The Manifestly Supersymmetric String

Just as the bosonic string describes the propagation of a string through ordinary space, the manifestly supersymmetric string describes the motion of a string through superspace. The formalism of superspace allows one to describe a field theory in which the global spacetime supersymmetry of that theory is manifest.

The Neveu-Schwarz-Ramond (NSR) version of the superstring is known to have spacetime supersymmetry for many solutions, such as ten-dimensional Minkowski space, in addition to the manifest superconformal symmetry on the worldsheet. Consequently, any string field theory built from the NSR form will be (spacetime) supersymmetric, but its supersymmetry will not be manifest. To construct a string field theory with manifest spacetime supersymmetry, one needs a first-quantized theory which has superspace coordinates as the fundamental fields on the two-dimensional worldsheet. A classical Lagrangian having global supersymmetry and a local fermionic worldsheet symmetry has been constructed by Green and Schwarz [30,31]. This action can be quantized in light-cone gauge and is the same as the NSR string in light-cone gauge.

While the Green-Schwarz action is free in light-cone gauge, in general gauges it is an interacting two-dimensional theory. This, in part, is why the covariant quantization of the theory is difficult. Because the action is free in light-cone gauge, one might expect that the action could be quantized straightforwardly in a covariant gauge, and that the covariantly quantized action would have a simple form. A formal quantization of the Green-Schwarz action shows that, on the contrary, there are difficulties even in passing from the Hamiltonian to a Lagrangian description.

One should keep in mind, therefore, the possibility that the Green-Schwarz action is not the appropriate action for quantization and may need to be amended in whole or in part. One such possible action has been proposed by Siegel [53]. The motivation for the Siegel modification is to incorporate the smallest algebra containing the generators of reparametrizations and of local fermionic transformations as the invariance algebra of the string. Unfortunately, the phase space constraints of the Siegel modification are equivalent to those of the Green-Schwarz string in generic regions of phase space, as I will show in this section. It seems then that one is forced to use the Green-Schwarz action, or some other action as yet unknown, in order to obtain a covariantly first-quantized string with manifest spacetime supersymmetry.

Even though there are difficulties in constructing a covariantly quantized string with manifest spacetime supersymmetry, such a construction is a necessary ingredient of the corresponding superstring field theory. The dynamical variables, including ghosts and auxiliary fields, of the first-quantized string become the coordinates on which the string field depends. And, as Siegel has explained [51], the constraints of the first-quantized theory will determine the free dynamics of the string field. The free field Lagrangian is $\Psi\hat{H}\Psi$, with $\hat{H}$ the first-quantized Hamiltonian (operator) and Ψ the string field.

A manifestly supersymmetric string field theory may make the possibilities for, or necessity of, supersymmetry breaking more apparent and may put some constraints on the allowable vacua.

The ghost structure of the theory is likely to yield as much insight into string physics as it has for the bosonic and NSR strings [28,41].

The first step on the road to string field theory is the construction of a first-quantized theory. Again, because the classical string actions are singular systems, their quantization begins with the analysis of their constraints. A canonical analysis of the manifestly supersymmetric string was first begun by Hori and Kamimura [36] who recognized most of the relevant features of the system. It is more useful to use a path integral quantization of the system because it eliminates some of the problems of ordering in the canonical formalism and also yields the ghost parts of the action. Through the BFV formalism one may formally quantize the Green-Schwarz action. The difficulties of, and remaining technical steps in this quantization are discussed in section 3.5. The Siegel modification will be shown to be essentially equivalent to the Green-Schwarz string, and thus is shown to have no particular advantage over the Green-Schwarz string in the canonical formalism.

The quantities which appear in the Green-Schwarz action are the two-dimensional metric $g_{\alpha\beta}$, a ten-dimensional position X^μ, and two anticommuting Majorana-Weyl spinors $\theta^A, A = 1,2$. Both X^μ and θ^A transform as scalars under worldsheet reparametrizations. When light-cone gauge is fixed, the remaining pieces of θ^A together transform as a spinor on the world-sheet, with the label A becoming a two-dimensional spinor index. The covariant classical Green-Schwarz action is [30]

$$\begin{aligned} I_{GS} = \frac{1}{\pi}\int d^2\sigma\sqrt{-g}\Big\{&-\tfrac{1}{2}g^{\alpha\beta}\,\Pi_\alpha\cdot\Pi_\beta \\ &- i\epsilon^{\alpha\beta}\Pi_\alpha\cdot[\bar\theta^1\gamma\,\partial_\beta\,\theta^1 - \bar\theta^2\gamma\,\partial_\beta\,\theta^2] \\ &- \epsilon^{\alpha\beta}\bar\theta^1\gamma\,\partial_\alpha\theta^1\cdot\bar\theta^2\gamma\,\partial_\beta\theta^2\Big\} \end{aligned} \tag{3.16}$$

where

$$\Pi^\mu_\alpha := \partial_\alpha X^\mu - i\sum_A \bar\theta^A\gamma^\mu\,\partial_\alpha\theta^A. \tag{3.17}$$

Just as the Brink-Schwarz superparticle has primary constraints (2.55) there are similar primary constraints for the Green-Schwarz action [36,38]:

$$\begin{aligned} &P_g \approx 0, \\ &\bar D^A := \zeta^A + i\bar\theta^A\,\gamma_\mu(P^\mu + (-)^A\,X^{\mu\prime} - (-)^A i\bar\theta^A\,\gamma^\mu\theta^{A\prime}) \approx 0. \end{aligned} \tag{3.18}$$

Here P_g is canonically conjugate to g, prime denotes derivative with respect to σ, and ζ^A is the conjugate momentum to θ^A satisfying the (symmetric) canonical Poisson bracket

$$\Big\{\zeta^A(\sigma)\,,\,\theta^B(\varrho)\Big\} = h^A\,\delta^{AB}\,\delta(\sigma-\varrho) \tag{3.19}$$

where h^A is the chirality projector for the spinor $\theta^A : h^A\theta^A = \theta^A$. The second relation in (3.18) defines the momentum, $\zeta^A := \partial_R L/\partial_R\dot\theta^A$, which is the right derivative of the Lagrangian with respect to the velocity of θ^A. Already one can see that there is something about the θ^A which is peculiar for scalar fields. The momentum conjugate to the field θ^A is constrained to be a function of fields other than the velocity of θ^A. This is more the behavior of a two-dimensional spinor field.

Two more constraints need to be imposed in order to conserve the first constraint of (3.18). These two constraints are the vanishing of the (traceless) stress tensor $T_{\alpha\beta}$:

$$0 \approx T_{\alpha\beta} := \dot{P}_{g\alpha\beta} = \sqrt{-g}\Big(\tfrac{1}{2}g_{\alpha\beta}g^{\gamma\delta} - \delta_\alpha^\gamma\delta_\beta^\delta\Big)\Pi_\gamma \cdot \Pi_\delta. \tag{3.20}$$

The Π_α^μ in eq. (3.20) are the expressions (3.17) expressed in canonical variables. In conformal coordinates the constraints (3.20) are particularly simple. Writing

$$\Pi_A^\mu := \Pi_0^\mu + (-)^A\Pi_1^\mu = (P^\mu + (-)^A X^{\mu'}) - 2i(-)^A\bar{\theta}^A\,\gamma^\mu\theta^{A'}, \tag{3.21}$$

one obtains $\Pi_A^2 = 0$ for the constraints (3.20).

One may check that there are no more constraints which need to be imposed in addition to (3.18) and (3.20). Upon examining the constraints one finds that some of them are second-class. Specifically, one computes the Poisson bracket of the fermionic constraints with themselves and finds

$$\Big\{\bar{D}^{Aa}(\sigma),\, \bar{D}^{Bb}(\varrho)\Big\} = 2i\delta^{AB}\,\delta(\sigma-\varrho)(\gamma^0 h_A\gamma^\mu)^{ab}\Pi_{A\mu}(\sigma). \tag{3.22}$$

(Roman minuscules are ten-dimensional spinor indices.) Because Π_A^μ is null (from (3.20)), exactly half of the components of $\bar{D}^A \approx 0$ are second-class and half are first-class. The null vectors Π_A^μ are useful for separating these constraints covariantly [36,38]. Contrary to the claims of reference [9], one may check that the first-class constraints are separated covariantly from the second-class constraints by

$$F^A := \bar{D}^A\,\gamma_\mu\Pi_A^\mu \approx 0\,, \quad A = 1, 2, \tag{3.23a}$$

$$G^A := \bar{D}^A\,\gamma_\mu\Pi_{\bar{A}}^\mu \approx 0\,, \quad A = 1, 2. \tag{3.23b}$$

The bar over the label A in (3.23b) is to denote the other value A may take; that is $\bar{1} = 2, \bar{2} = 1$. The $F^A(\sigma)$ are first-class and the $G^A(\sigma)$ are second-class. The choice of second-class constraints is not unique. One could choose *any* null vector V_A with

$V_A \cdot \Pi_A \not\equiv 0$ and define a new second-class constraint $\tilde{G}^A = \bar{D}^A \gamma_\mu V_A^\mu \approx 0$. One is forced into choosing the null vector V_A to be $\Pi^\mu_{\bar{A}}$ because the choice of any null vector not given by the theory itself would break manifest covariance. It turns out [36] that the generator of reparametrizations is not purely Π_A^2 but is $\frac{1}{2}\Pi_A^2 + 2(-)^A \bar{D}^A \theta^{A\prime}$. The full set of constraints for the Green-Schwarz string is

$$\begin{aligned} &(P_g)_{\alpha\beta} \approx 0, \\ &T_A := \tfrac{1}{2}\Pi_A^2 + 2(-)^A \bar{D}^A \theta^{A\prime} \approx 0 \,, \quad A = 1,2, \\ &F_A := \bar{D}^A \gamma_\mu \Pi_A^\mu \approx 0 \qquad\qquad\quad , \quad A = 1,2, \\ &G_A := \bar{D}^A \gamma_\mu \Pi_{\bar{A}}^\mu \approx 0 \qquad\qquad\quad , \quad A = 1,2. \end{aligned} \tag{3.24}$$

The first two of these constraints generate Weyl rescalings of the metric and two-dimensional reparametrizations of the world-sheet. The constraints $F^A \approx 0$ generate the local fermionic κ-symmetry and together with the first two constraints are the first-class constraints of the theory. The conditions $G^A \approx 0$ are second-class and must be treated differently from the rest of the constraints in (3.24). Before analyzing the Siegel modification, let us count the degrees of freedom of the Green-Schwarz string. Ignoring the metric degrees of freedom one has twenty bosonic and sixty-four fermionic phase space variables on which there are two bosonic first-class, sixteen fermionic first-class and sixteen fermionic second-class constraints. The first-class constraints each fix out two degrees of freedom while the second-class constraints each fix out a single degree of freedom. (This counting works irrespective of the choice of gauge fixing conditions.) Thus there are sixteen bosonic and sixteen fermionic physical phase space degrees of freedom at each point along the string.

3.3 The Siegel String Action

The Siegel string is motivated by a desire not to have the whole of $\bar{D}^A$ fixed to zero, but to have the smallest symmetry algebra containing the generator of the "κ-symmetry." This is reasonable because the troublesome second-class constraints are contained in $\bar{D}^A$. If only three quarters of the components of $\bar{D}^A$ were constrained to vanish, then there would be the correct number of degrees of freedom, and the constraints would all be first-class. The Siegel string action does have different constraints from the Green-Schwarz string but they are equivalent to the Green-Schwarz string constraints in generic regions of phase space.

The two-dimensional fields used to construct the Siegel string are, in addition to the fields of the Green-Schwarz string, a ten-dimensional and worldsheet vector P^μ_α, two ten-dimensional Majorana-Weyl spinors D^A_α which are also vectors on the worldsheet, and three auxiliary fields $\psi^{A\ \beta}_\alpha$, $\chi^{A\alpha}_{\ \ \beta\mu\nu\rho}$, and $\phi^{A\alpha\beta}_{\ \ \gamma\mu}$. The ψ^A are two Majorana-Weyl spinors in ten dimensions while χ^A and ϕ^A are an antisymmetric tensor and vector respectively. The full Weyl invariant classical action is [53,49]

$$\begin{aligned}
I_S = \int d^2\sigma\sqrt{-g}\Big\{ & g^{\alpha\beta}\Big(\tfrac{1}{2}P_\alpha\cdot P_\beta + P_\alpha\cdot(\partial_\beta X - i\sum_A \bar\theta^A\gamma\partial_\beta\theta^A)\Big) \\
& + i\epsilon^{\alpha\beta}\,\partial_\alpha X\cdot(\bar\theta^2\gamma\partial_\beta\theta^2 - \bar\theta^1\gamma\,\partial_\beta\theta^1) + \epsilon^{\alpha\beta}\bar\theta^1\gamma\partial_\alpha\theta^1\cdot\bar\theta^2\gamma\partial_\beta\theta^2 \\
& + i\sum_A \bar D^A_\alpha\partial_\beta\theta^A\Pi^{A\alpha\beta} + \sum_A \Pi^{A\eta\alpha}\Pi^{A\delta}_{\ \ \beta}\,\psi^{A\ \beta}_{\alpha}\not{P}_\eta D^A_\delta \\
& + \sum_A \chi^{A\alpha}_{\ \ \beta\,\mu\nu\rho}\,\Pi^{A\delta}_{\ \ \alpha}\Pi^{A\gamma\beta}\bar D^A_\delta\gamma^{\mu\nu\rho}D^A_\gamma \\
& + \sum_A \phi^{A\alpha\beta}_{\ \ \gamma\mu}\,\Pi^{A\delta}_{\ \ \alpha}\Pi^{A\epsilon}_{\ \ \beta}\Pi^{A\gamma\eta}\bar D^A_\delta\gamma^\mu\partial_\eta D^A_\epsilon\Big\}.
\end{aligned} \tag{3.25}$$

The quantities $\Pi^{A\alpha\beta}$ are projection operators,

$$\Pi^{A\alpha\beta} := \tfrac{1}{2}(g^{\alpha\beta} + (-)^A\epsilon^{\alpha\beta}), \tag{3.26}$$

and are not related to the expression (3.17) even though they are, unfortunately, denoted by the same symbol. The Dirac analysis proceeds analogously to the Siegel superparticle in chapter 2. The canonical phase space has the conjugate pairs $(X, P_X), (P, P_P)$, (θ, ζ), $(D, B), (g, P_g), (\psi, P_\psi)$, (χ, P_χ) and (ϕ, P_ϕ) as canonical variables.

The definition of momenta leads directly to the primary constraints.

$$
\begin{aligned}
&\phi_1 := \zeta^A + i\bar{\theta}^A\gamma_\mu(P_\alpha^{\ \mu} g^{\alpha 0} + (-)^A X^{\mu\prime} - i(-)^A\bar{\theta}^{\bar{A}}\gamma^\mu\theta^{\bar{A}\prime}) - i\bar{D}^A_\beta \Pi^{A\beta 0} \approx 0, \\
&\phi_2 := P_X^{\ \mu} - P_\alpha^{\ \mu} g^{\alpha 0} - i\sum_B (-)^B\bar{\theta}^B\gamma^\mu\theta^{B\prime} \approx 0, \\
&\phi_3 := B^{A\eta} - \phi^{A\alpha\beta}_{\ \gamma\mu}\bar{D}^A_\delta\gamma^\mu\Pi^{A\delta}_{\ \ \alpha}\Pi^{A\eta}_{\ \ \ \beta}\Pi^{A\gamma 0} \approx 0, \\
&\phi_4 := P_P^{\ \alpha\mu} \approx 0, \\
&\phi_5 := P_{\psi\ \beta}^{\ A\alpha} \approx 0, \\
&\phi_6 := P^{A\ \beta\,\mu\nu\rho}_{\chi\ \alpha} \approx 0, \\
&\phi_7 := P_{g\alpha\beta} \approx 0, \\
&\phi_8 := P_{\phi\,\alpha\beta}^{\ A\,\gamma\mu} \approx 0.
\end{aligned}
\tag{3.27}
$$

In these primary constraints the variables are all mixed up in a complicated fashion, but it is still possible to see that ϕ_1, ϕ_2, ϕ_4 and pieces of ϕ_3 are second-class. Whether or not the rest are first-class is less clear, but one must suspect that $\phi_{5,6,7,8}$ are first-class as they shift the Lagrange multiplier fields. In order to simplify the analysis one may fix these suspected gauge invariances with further constraints, and then must check that there are no inconsistencies that follow from the imposition and conservation of the extra constraints. With this caveat, set

$$
\begin{aligned}
\omega_1 &:= g^{\alpha\beta} - \eta^{\alpha\beta} \approx 0, \\
\omega_2 &:= \chi^A \approx 0, \\
\omega_3 &:= \phi^A \approx 0, \\
\omega_4 &:= \psi^A \approx 0, \\
\omega_5 &:= D^A_\alpha \Pi^{\bar{A}\alpha 1} \approx 0,
\end{aligned}
\tag{3.28}
$$

and require their conservation.

Conservation of the constraints (3.27) and (3.28) requires the additional constraints

$$
\begin{aligned}
&\phi_9 := P_\gamma^{\ \mu}\gamma_\mu\, D^A_\epsilon\, \Pi^{A\gamma}_{\ \ \alpha}\, \Pi^{A\epsilon}_{\ \ \beta} \approx 0,\\
&\phi_{10} := \tfrac{1}{2}\bar{D}^A_\gamma \gamma^{\mu\nu\rho}\, D^A_\delta \Pi^{A\gamma}_{\ \ \alpha}\Pi^{A\delta\beta} \approx 0,\\
&\phi_{11} := \bar{D}^A_\delta \gamma^\mu\, D^{A'}{}_\epsilon\, \Pi^{A\delta}_{\ \ \alpha}\, \Pi^{A\epsilon}_{\ \ \beta} \approx 0,\\
&\phi_{12} := T_{\alpha\beta} \approx 0,\\
&\phi_{13} := P_1{}^\mu + X^{\mu'} - i\sum_A \bar{\theta}^A\gamma^\mu\theta^{A'} \approx 0,
\end{aligned}
\tag{3.29}
$$

where

$$
\begin{aligned}
&T_{00} = T_{11} = \tfrac{1}{2}(P_0^2 + P_1^2) - i\sum_A \bar{D}^A_1\theta^{A'},\\
&T_{01} = P_0\cdot P_1 - i\sum_A \bar{D}^A_0\theta^{A'}
\end{aligned}
\tag{3.30}
$$

is the stress tensor and is the same as the modified stress tensor in (3.24). The set of constraints (3.27), (3.28), and (3.29) are conserved without the imposition of any further constraints. The set of fixing conditions (3.28) is consistent and leaves an algebra [53] of constraints generated by $\phi_{9,10,11,12}$. The Hamiltonian which preserves the constraints,

$$
\begin{aligned}
\mathcal{H} = &-\tfrac{1}{2}\eta^{\alpha\beta}P_\alpha\cdot P_\beta - P_1\cdot(X' - i\sum_B \bar{\theta}^B\gamma\theta^{B'})\\
&- i\sum_B \bar{\theta}^B\gamma_\mu\theta^{B'}((-)^B P_0{}^\mu - X^{\mu'} + i\bar{\theta}^{\bar{B}}\gamma^\mu\theta^{\bar{B}'})\\
&+ P_X^2 + P_X\cdot(P_0 - i\sum_B(-)^B\bar{\theta}^B\gamma\theta^{B'})\\
&+ \sum_A(-)^A(\zeta^A\theta^{A'} + B^{A\eta}D^{A'}{}_\eta) - \epsilon_{\alpha\beta}P_P^\alpha\cdot P^{\beta'},
\end{aligned}
\tag{3.31}
$$

is unique up to the addition of first-class constraints and is equal to T_{00} upon setting the second-class constraints to zero strongly. (That is, taking the second-class constraints to vanish *identically* and replacing the Poisson bracket by the Dirac bracket so that no contradictions will result from taking the constraints to vanish identically.) Since the constraints $\phi_{1,\ldots,8,13}$ and $\omega_{1,\ldots,5}$ together are second-class, one must think of $\bar{D}^A_\alpha$ and $P_\alpha{}^\mu$ as derived quantities given by their expressions in (3.27, 3.28, 3.29). When this is done, the only independent variables left are X, P_X, θ and ζ, which must still satisfy $\phi_{9,10,11,12} \approx 0$. Two of these constraints have *identical* counterparts in the Green-Schwarz theory. The constraint ϕ_9 is F^A and ϕ_{12} is T_A. The Green-Schwarz string has eight additional second-class constraints $G^A \approx 0$ while the Siegel string has instead twenty-nine additional independent constraints which have vanishing Poisson brackets with all other constraints on the constraint surface. One might be tempted to call these twenty-nine constraints first-class, but if they were first-class then there would be a mismatch in the number of physical degrees of freedom between the Green-Schwarz and Siegel string. In fact, both theories have the same number of physical degrees of freedom and have the same second-class constraints in generic regions of phase space. To see this, one must analyze the constraints $\phi_{9,10,11,12}$ carefully. First, it is useful to have a simple notation. Set A = 1 because the case A = 2 is analogous. Let D^A_α become D because A = 1 and α has only one non-zero component by (3.28). Similarly, let $P = P_\alpha \Pi^{A\alpha\beta}$. The constraints (3.29) are now easily written as

$$\tfrac{1}{2}P^2 + 2\bar{D}\theta' \approx 0, \qquad \not{P} D \approx 0, \qquad \bar{D}\gamma^{\mu\nu\rho} D \approx 0, \qquad \bar{D}\gamma^\mu D' \approx 0. \tag{3.32}$$

Here we must appeal to the arguments in appendix B in order to solve these constraints. Because the variable P_μ is a global supersymmetry invariant, we may restrict it to be a real number, and not just a commuting supernumber, without ruining the global supersymmetry. As is demonstrated in appendix B, a path integral over commuting supernumbers may be restricted to be over the real numbers only.

Let us restrict the variable P_μ to be real. Because $\not{P}$ is invertible for $P^2 \neq 0$, the first two constraints of (3.32) together imply that

$$P^2 \approx 0 \qquad \text{and} \qquad \bar{D}\theta' \approx 0 \tag{3.33}$$

separately. The argument proceeds by multiplying the second constraint by $\not{P}$ and then dividing by P^2 if it is non-zero.

The relevant constraints on the derived quantity D are

$$\not{P} D \approx 0, \qquad \bar{D}\theta' \approx 0, \qquad \bar{D}\gamma^{\mu\nu\rho} D \approx 0, \qquad \bar{D}\gamma^{\mu} D' \approx 0. \tag{3.34}$$

The solution of these constraints, $D = f(\theta', \not{P})$ will be the constraints analogous to $\bar{D} = 0$ in the Green-Schwarz theory. For any f, except $f = (\not{P} - i\bar{\theta}\gamma\theta'\gamma)\theta +$ constant, the constraints $D = f(\theta', \not{P})$ are obviously second-class. One may dispose of the possibility $f = (\not{P} - i\bar{\theta}\gamma\theta'\gamma)\theta$ by showing that it is not a solution.

One may show that the only solution for generic θ' is $f = 0$ or that (3.34) are equivalent to $\bar{D} \approx 0$. The third constraint is most easily solved. It implies that

$$D^a(\sigma) = \lambda^a(\sigma)\epsilon(\sigma) \tag{3.35}$$

where $\lambda^a(\sigma)$ is a commuting spinor function and $\epsilon(\sigma)$ is an anticommuting scalar. The second equation implies

$$\epsilon(\sigma) \propto (\bar{\lambda}\theta'), \tag{3.36}$$

while the last requires that $\epsilon\epsilon' = 0$ or, equivalently,

$$\epsilon(\sigma) \propto \epsilon'(\sigma). \tag{3.37}$$

Generically, all of the components of θ' are independent Grassmann numbers and have zeros as functions of σ. Eq. (3.37) requires $\lambda(\sigma)$ to have poles of the same order as the zeros of these generic θ configurations unless ϵ is identically zero. The expression D, which is tacitly assumed to be a differentiable function of σ, is expressed through (3.35, 3.36) as

$$D^a = \beta\lambda^a(\bar{\lambda}\theta'). \tag{3.38}$$

Therefore D^a has poles as a function of σ for generic $\theta(\sigma)$ field configurations unless it vanishes.

There is one loophole in the above argument. There is a way to solve the constraints (3.34) which is not of the form (3.35). The last two constraints can be solved by setting D proportional to a constant nilpotent commuting number (such as $\epsilon_1\epsilon_2, \epsilon_1$ and ϵ_2 both Grassmann). That is, the expression (3.38) satisfies the last three constraints if $\beta^2 = 0$. It also satisfies the first constraint if the commuting spinor λ is annihilated by $\not{P}$. These solutions must be considered "pathological."

These pathological solutions are assumed to be unimportant for the quantum theory. To illuminate the peculiar nature of solutions involving nilpotent commuting numbers, consider a constraint $P_\mu P^\mu = 0$. Any P_μ of the form $P_\mu = \beta M_\mu$ with M arbitrary and $\beta^2 = 0$ satisfies the constraints. The finite dimensional analog of the path integral measure over such a constrained surface is $d^n P\, \delta(P^2)$ which becomes $\beta^n \delta(\beta^2) d^n M/M^2$ upon replacement of P_μ by βM_μ. From the rules in appendix B, we would define $\beta^n \delta(\beta^2)$ as $\beta^{n+2}\delta'(0)$ which is ambiguous but should be set to zero because β is nilpotent. These pathological solutions can be eliminated if we define the integrals over these subspaces to vanish.

The existence of these pathological solutions is of secondary importance to the fact that they, like $\bar{D} = 0$, are also second-class constraints.

3.4 Quadratic Constraint Algebras

It is peculiar that the algebra of constraints (3.34) hides second-class constraints. Usually one believes that constraints which form an algebra are first-class and generate symmetries. An analogous, though simpler, model of this situation can be made. Suppose there is a system with constraints $p \approx 0$ and $q \approx 0$. These constraints cannot be imposed simultaneously on the system because their Poisson brackets do not vanish; $\{q, p\} = 1$. These are second-class constraints. An *equivalent* set of constraints may be imposed on the system. Consider the set of constraints

$$p^2 \approx 0, \qquad q^2 \approx 0, \qquad pq \approx 0. \tag{3.39}$$

The constraints (3.39) are equivalent to $p \approx 0, q \approx 0$ in that the hypersurfaces defined by both sets of constraints are identical. The difference is that (3.39) form an algebra:

$$\{q^2, p^2\} = 4pq, \qquad \{p^2, pq\} = -2p^2, \qquad \{q^2, pq\} = 2q^2. \tag{3.40}$$

Thus the quadratic constraints (3.39) appear to be first-class. When they are solved (written in a form linear in the dynamical variables) one can see they are actually second-class. This simple example shows how algebras of non-linear quantities may hide second-class constraints.

The quantization of theories with quadratic constraint algebras is not straightforward. If we insist on dealing with the constraints in their non-linear form, we will be unable to obtain any states at all, despite the fact that the constraints form an algebra. This can be demonstrated with the simple example (3.39) above. First, we transcribe the constraints into operators. In order to preserve the algebra (3.40), we must order the constraints as follows.

$$\begin{aligned} &p^2 \rightarrow \hat{p}^2, \\ &q^2 \rightarrow \hat{q}^2, \\ &pq \rightarrow \tfrac{1}{2}(\hat{p}\hat{q} + \hat{q}\hat{p}). \end{aligned} \tag{3.41}$$

Imposing these operators on a wavefunction leads to the conclusion that the wavefunction must vanish. It has been verified explicitly [40] that the same conclusion holds for the set of constraints (3.32) of the Siegel string modification. The BFV formalism may not be applied to the system (3.39) directly because the constraints are reducible. When using the reducibility conditions

$$\begin{aligned} 0 &= q(p^2) + p(pq), \\ 0 &= q(pq) + p(q^2). \end{aligned} \tag{3.42}$$

in the BFV formalism for reducible theories, one finds that one cannot obtain a consistent BRS charge Ω.

Because the solutions of the constraints (3.34) are second-class, the "symmetries" of the Siegel string system generated by the last two constraints of (3.34) are not true symmetries. The Siegel string, because it has the same constraints as the Green-Schwarz string, also has sixteen bosonic and sixteen fermionic physical phase space degrees of freedom. As classical theories the two formulations of the string are equivalent. For quantization the (linear) Green-Schwarz constraints are more suitable.

3.5 Quantization of the Manifestly Supersymmetric String

The bosonic string (and the NSR superstring) may be covariantly quantized through the Lagrangian Faddeev-Popov procedure, similar to (1.36), in which the canonical structure of the theories need never enter. Instead, the integration over metric degrees of freedom is rewritten to factor out the diffeomorphisms explicitly through a change of variables. The resulting Jacobian becomes the ghost action once the ghosts are introduced.

Theories which have complicated phase space structure, such as nontrivial second-class constraints, or algebras of first-class constraints which have phase-space dependent structure constants, cannot be quantized using the Faddeev-Popov method. Theories with phase-space dependent structure constants in the first-class constraint algebra have a more complicated BRS charge which leads to a Lagrangian containing ghost-ghost interactions. Complicated second-class constraints require a modification of the path integral measure and a modification of the Poisson brackets. It is unfortunate that the manifestly supersymmetric string has both complications.

In order to quantize covariantly one of these complicated theories, the constraints must first be separated into first- and second-classes. The Poisson bracket is redefined so that the second-class constraints have vanishing brackets with any function on phase space. The measure of the path integral is modified by the introduction of delta functions of the second-class constraints multiplied by the square root of the superdeterminant of the matrix of Poisson brackets of all second-class constraints.

$$\text{Measure factor} = \delta^N(\chi_i)\sqrt{\text{sdet}\{\chi_i,\chi_j\}_{PB}} \tag{3.43}$$

Next, the first-class constraints must be considered. The first-class constraints may be used to construct a BRS symmetry generator which will later be used to fix out the first-class symmetries. One starts by enlarging the phase space. For each first-class constraint $\phi \approx 0$ a Lagrange multiplier λ and its conjugate momentum π are introduced. A ghost c, antighost $\bar{c}$ and their conjugate momenta b and $\bar{b}$ round out the set additional phase space variables needed for each first-class constraint ϕ.

From the constraints one must construct the BRS generator (1.41)

$$\Omega = \bar{b}\pi + c\phi + \text{more} \tag{3.44}$$

to satisfy

$$\{\Omega, \Omega\}_{\text{DB}} = 0. \tag{3.45}$$

This condition is problematical because the Dirac bracket is extremely complicated, attributable to the nontrivial brackets the second-class constraints have with each other.

The correct generating functional for the system is

$$Z_\Psi = \int \mathcal{D}P\,\mathcal{D}Q\,\delta[\chi_i]\sqrt{\text{sdet}\{\chi_i, \chi_j\}_{PB}}\exp i \int dt(P\dot{Q} - H + \{\Psi, \Omega\}_{DB}). \tag{3.46}$$

with Ψ any imaginary fermionic function on the extended phase space of original variables and ghost, antighost and Lagrange multiplier degrees of freedom, with ghost number -1. The Fradkin-Vilkovisky theorem (1.62) states that the generating functional is independent of the gauge fixing function Ψ.

The Fradkin method cannot be straightforwardly applied to systems with quadratic constraints which are second-class. As was stated in section 3.4, the constraints (3.39) do not yield a consistent BRS charge when the reducibility is taken into account. Further, if one treats them as being irreducible, then one may show that the correct measure factor, $\delta(p)\delta(q)$, cannot be obtained. Similarly, the treatment of the constraints (3.32) as first-class constraints in the Fradkin formalism will not yield the correct result (i.e., the result one gets when the second-class constraints are separated out explicitly as in (2.9)). Perhaps there is a modification of the Fradkin formalism which allows more flexibility in the treatment of second-class constraints, but the replacement of second-class constraints by quadratic first-class constraints does not work.

Without possessing a more flexible formalism one is forced to treat the system (3.24) according to the rules of the Fradkin formalism. Thus one can write down, at least formally, the most general quantum version of the manifestly supersymmetric string.

The measure factor is

$$\delta[G^{Aa}(\sigma)]\Big(\det(\{G^{Aa}(\sigma), G^{Bb}(\varrho)\}_{PB})\Big)^{-\frac{1}{2}} \tag{3.47}$$

where

$$\{G^{Aa}(\sigma), G^{Bb}(\varrho)\}_{PB} \approx -4i\delta(\sigma-\varrho)\delta^{AB}\Pi_A \cdot \Pi_{\bar{A}}(\gamma^0 h_A \gamma_\mu \Pi^\mu_{\bar{A}})^{ab}. \tag{3.48}$$

Next, one must construct the BRS charge Ω to satisfy $\{\Omega,\Omega\}_{DB} = 0$ and show that the quantum BRS charge only squares to zero for ten spacetime dimensions. This still has yet to be done, but there is no reason to doubt that it can be done.

There is, perhaps, little calculational power to be gained in continuing the quantization in this fashion because the Poisson bracket (3.48) is cubic in the momentum P^μ. The elevation of this bracket from the measure to the action with appropriate "second-class ghosts" will yield an action cubic in momenta. The momentum integrals cannot be done explicitly to yield a conventional Lagrangian, but one could consider this momentum space path integral as the configuration space path integral of a first-order Lagrangian, which, unfortunately, is not free. All that is needed to use this formal quantization is the explicit form of the BRS operator Ω whose quantum analog is nilpotent. This quantization could be used for any (worldsheet) perturbative calculation.

To conclude this section, I resolve the puzzle of why the above remarks do not apply to the light-cone gauge

$$X^+ + P^+\tau \approx 0, \qquad P^{+'} \approx 0, \qquad \gamma^+\theta^A \approx 0, \qquad g_{\alpha\beta} \approx \eta_{\alpha\beta}. \tag{3.49}$$

In other words, why can the light-cone gauge fixed string be quantized so easily and why is it free? The answer crucially depends upon the observation that the constraints (3.49) may be treated on the same footing with the constraints (3.24). The constraints (3.49) must also be conserved and so fix the Hamiltonian to be [36]

$$\mathcal{H} = \tfrac{1}{2}(P^2 + X'^2) + \zeta^2\theta^{2'} - \zeta^1\theta^{1'}. \tag{3.50}$$

The whole set of constraints (3.49) and (3.24) are all together second- class constraints as they must be since (3.49) fix the gauge completely. The generating functional (3.46) may be used to quantize the theory. Because there are no first-class constraints there is no BRS charge. The superdeterminant in the measure factor is not field dependent, and the delta functions may be solved easily. When the momentum integral is done, one is left with a free theory for the transverse modes. The main point of this is that the theory is much simpler if one does not have to separate the constraints into first- and second-classes. It also helps that the gauge conditions (3.49) are simple and linear.

3.6 Discussion and Prospects

The main result of this chapter is the demonstration that the Siegel string action is essentially equivalent to the Green-Schwarz string action.

The extra symmetries of the Siegel string are not actually symmetries at all but hide second-class constraints. Because they are not symmetries, they do not need to be fixed out through a gauge choice and do not properly belong in the BRS generator of the theory. The existence of a formalism for quantizing a theory with a quadratic algebra of constraints which are actually second-class is an open question.

The existing formalism for quantization requires the explicit identification of the second-class constraints and therefore the Green-Schwarz form is the most appropriate for quantization. A formal quantization of the Green-Schwarz system has been given. The construction of the BRS charge which has zero Dirac brackets with itself and the demonstration that the quantum mechanical BRS charge is nilpotent, are the remaining steps necessary to complete the quantization. This formal quantization does not possess the most useful attribute of the NSR and bosonic strings; the freedom of the world sheet σ-model. Nevertheless, one knows how to begin constructing the associated string field theory. The set of fields on which the wave functional depends is the original set X^μ, θ^A, the ghosts for the diffeomorphisms and local supersymmetry, the "second-class ghosts," and the fields used to elevate the delta functional in the measure factor (3.46) to the action. Much less clear is the explicit form for the dynamics of the free string field theory.

I would like to end with two suggestions for future research. The first suggestion is to look at systems which are analogous to the auxiliary superspace variables Υ for the superparticle. It is obvious that the direct transcription of the constraints (2.87) used to separate the constraints of the superparticle in ten dimensions to the superstring cannot work because the Π^μ_A, defined in equation (3.21) obey the brackets

$$\{\Pi^\mu_A(\sigma), \Pi^\nu_B(\varrho)\} = 2(-1)^A \delta_{AB}\eta^{\mu\nu}\delta'(\sigma - \varrho), \tag{3.51}$$

and therefore the constraints Φ_{AB} would not commute among themselves.

The idea which is interesting to consider is to diagonalize the projection operator

$$\Pi^\mu_1(\sigma)\Pi^\nu_2(\sigma)\gamma_\mu\gamma_\nu. \tag{3.52}$$

This will make the constraints $\Upsilon^\dagger_A \Pi^\mu_1 \Pi^\nu_2 \gamma_\mu\gamma_\nu \Upsilon_B$ commute to give something proportional to the stress tensor. The only possible problem is that the constraint $D^A \approx 0$ may not commute with the new constraints to give another constraint. This has yet to be checked.

To explain the second idea, we must digress to a discussion of the equations of motion for the Green-Schwarz string. In reference [31] the equations of motion were found to be

$$\begin{aligned}
&\Pi_\alpha \cdot \Pi_\beta = \tfrac{1}{2} g_{\alpha\beta} g^{\gamma\delta} \Pi_\gamma \cdot \Pi_\delta, \\
&\gamma \cdot \Pi_\alpha P_-^{\alpha\beta} \partial_\beta \theta^1 = 0, \\
&\gamma \cdot \Pi_\alpha P_+^{\alpha\beta} \partial_\beta \theta^2 = 0, \\
&\partial_\alpha \Big\{ \sqrt{-g} \Big[g^{\alpha\beta} \partial_\beta X^\mu - 2i P_-^{\alpha\beta} \overline{\theta}^1 \gamma^\mu \partial_\beta \theta^1 - 2i P_+^{\alpha\beta} \overline{\theta}^2 \gamma^\mu \partial_\beta \theta^2 \Big] \Big\} = 0.
\end{aligned} \tag{3.53}$$

Here the symbol $P_A^{\alpha\beta}$ is just another name for the projection operator $\Pi^{A\alpha\beta}$ defined in (3.26). A crucial observation is made in reference [31] that the gauge conditions (3.49) lead to free equations of motion on the worldsheet. It turns out that there is a gauge choice one may make which also implies that the equations of motion are free. This gauge condition is

$$P_A^{\alpha\beta} \partial_\beta \theta^A = 0. \tag{3.54}$$

It would appear that the conditions (3.54) set sixteen equations for each θ, but *on shell* the they in fact set exactly eight equations for each θ. This is because on shell the Π_A^μ are null and the equations of motion already have the vanishing of half of the gauge conditions (3.54). It is problematical to translate the gauge condition (3.54) into canonical form (an attempt is made in reference [18]), but it is perhaps the most natural covariant gauge choice. The gauge condition (3.54) has been treated in the Lagrangian formalism by Kallosh [39], who has to introduce two arbitrary null vectors in order to fix the gauge. These arbitrary null vectors complicate the theory and also break the manifest Lorentz invariance of the theory.

This thesis has not considered the mechanics of particles or strings moving in background fields. Introduction of background fields into the action complicates the Hamiltonian analysis. If it is difficult to understand the flat background theory, it is even more difficult to understand the theory in a nontrivial background. Once a theory has been analyzed in a flat background, though, it becomes important to understand that theory in nontrivial backgrounds because the consistency of the first-quantized theory could put restrictions on those background fields.

APPENDIX A

Spinor Conventions

Metric

The spacetime metric is taken to be of signature $D-2$,

$$\eta_{\mu\nu} = (-1, 1, 1, \ldots, 1). \tag{A.1}$$

With this convention the gamma matrices obey the anticommutators

$$\{\gamma_\mu, \gamma_\nu\} = -2\eta_{\mu\nu}. \tag{A.2}$$

Symmetrization and antisymmetrization of indices are defined with unit weight. For example, the product of n gamma matrices which is antisymmetric in its indices is

$$\begin{aligned}\gamma^{\mu_1\mu_2\ldots\mu_n} &= \gamma^{[\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_n]} \\ &= \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^{|\sigma|}\gamma^{\mu_{\sigma(1)}}\gamma^{\mu_{\sigma(2)}}\ldots\gamma^{\mu_{\sigma(n)}}.\end{aligned} \tag{A.3}$$

Majorana spinors

If the matrices γ^μ may taken as purely real or purely imaginary, then the spinors may be taken as purely real objects. In ten dimensions the gamma matrices may be taken to be purely imaginary. They thus satisfy

$$\gamma^{\mu T} = -\gamma^0\gamma^\mu\gamma^0. \tag{A.4}$$

Spinors may be taken to be real so that they satisfy

$$\overline{\psi} = \psi^T\gamma^0. \tag{A.5}$$

Any two anticommuting Majorana spinors, ψ and ϕ, satisfy

$$\overline{\psi}\gamma^{\mu_1\ldots\mu_n}\phi = (-1)^{n+[\frac{n}{2}]}\overline{\phi}\gamma^{\mu_1\ldots\mu_n}\psi. \tag{A.6}$$

Majorana spinors may be defined only in spacetimes of dimension $D = 8n+2$, $8n+3$, or $8n+4$.

Weyl spinors

In even dimensional spacetimes one may construct the matrix

$$\gamma_{D+1} = e^{i\pi(D-2)/4}\gamma^0\gamma^1\dots\gamma^{D-1} \tag{A.7}$$

which can be used to define Weyl spinors. Weyl spinors are spinors satisfying

$$\gamma_{D+1}\psi = \pm\psi. \tag{A.8}$$

If γ_{D+1} is real, then one may define spinors which are both Majorana and Weyl. Obviously these spinors may be defined only when the dimension of spacetime is $D = 8n+2$.

Dirac Algebra

One may construct all $2^{[\frac{D}{2}]} \times 2^{[\frac{D}{2}]}$ matrices from antisymmetric products of the gamma matrices γ^μ. These matrices are denoted

$$\gamma^{\mu_1\dots\mu_n} = \gamma^{[\mu_1}\dots\gamma^{\mu_n]} \tag{A.9}$$

and are simply

$$\gamma^{\mu_1}\dots\gamma^{\mu_n}$$

when all of the indices μ_1 through μ_n are different and are zero otherwise. These matrices have the Hermitian conjugates

$$\gamma^{\mu_1\dots\mu_n\dagger} = \gamma^0\gamma^{\mu_n\dots\mu_1}\gamma^0. \tag{A.10}$$

In a Majorana representation where the matrices may be taken as purely imaginary, (as is true in ten dimensions), we have

$$\gamma^{\mu_1\dots\mu_n T} = (-1)^{n+[\frac{n}{2}]}\gamma^0\gamma^{\mu_1\dot{s}\mu_n}\gamma^0. \tag{A.11}$$

For the special case of the ten-dimensional Majorana-Weyl representation the matrices $\gamma^0\gamma^\mu$ and $\gamma^0\gamma^{\mu_1\dots\mu_5}$ are symmetric while $\gamma^0\gamma^{\mu_1\mu_2\mu_3}$ is antisymmetric.

Fierz Identity

It is useful to be able to rearrange spinors in expressions involving four or more spinors contracted into expressions involving gamma matrices. The Fierz identity allows one to do just this. For ζ and η commuting spinors in ten dimesions one has

$$\begin{aligned}16\zeta\overline{\eta} = \overline{\eta}\zeta + \gamma_\mu\overline{\eta}\gamma^\mu\zeta - \frac{1}{2!}\gamma_{\mu\nu}\overline{\eta}\gamma^{\mu\nu}\zeta - \frac{1}{3!}\gamma_{\mu\nu\rho}\overline{\eta}\gamma^{\mu\nu\rho}\zeta \\ + \frac{1}{4!}\gamma_{\mu\nu\rho\sigma}\overline{\eta}\gamma^{\mu\nu\rho\sigma}\zeta + \frac{1}{2\cdot 5!}\gamma_{\mu\nu\rho\sigma\tau}\overline{\eta}\gamma^{\mu\nu\rho\sigma\tau}\zeta.\end{aligned} \tag{A.12}$$

If η and ζ are Weyl with the same chirality, then we keep those terms with gamma matrices having an odd number of indices only. If η and ζ are Weyl of opposite chirality, we keep those terms with gamma matrices having an even number of indices only. Finally, if ψ_i are Majorana-Weyl spinors and are all either commuting or anticommuting then the following important identity holds.

$$\gamma_\mu\psi_i\overline{\psi}_j\gamma^\mu\psi_k + \gamma_\mu\psi_j\overline{\psi}_k\gamma^\mu\psi_i + \gamma_\mu\psi_k\overline{\psi}_i\gamma^\mu\psi_j = 0. \tag{A.13}$$

APPENDIX B

Mathematics of Supernumbers

Superclassical Dynamical Systems

We suppose that our dynamical variables may be either commuting or anticommuting numbers. The evolution of the system is governed by a commuting function of the dynamical variables, the Lagrangian. To obtain a Hamiltonian description, one defines momenta. If a dynamical variable is Grassmann odd, care must be taken in defining its momentum. We take the definition that the momentum is the right derivative of the Lagrangian with respect to the velocity.

$$p := \frac{\partial_R L}{\partial_R \dot{q}}. \tag{B.1}$$

The expression for the Hamiltonian is then the usual expression with the momentum to the left of the velocity.

$$H(p,q) = p_i q^i - L(q^i, \dot{q}^i). \tag{B.2}$$

The Hamiltonian generates time evolution of the canonical variables

$$\dot{z} = \{z, H\}_{PB}, \tag{B.3}$$

through the use of the generalized Poisson brackets

$$\{A,B\}_{PB} = \sum_i \left\{ \frac{\partial_R A}{\partial_R q^i} \frac{\partial_L B}{\partial_L p_i} - (-1)^{|i|} \frac{\partial_R A}{\partial_R p_i} \frac{\partial_L B}{\partial_L q^i} \right\}. \tag{B.4}$$

The symbol $|i|$ denotes the Grassmann parity of the variables q^i and p_i. If q^i is a commuting number then $|i|$ is 0, and $|i|$ is 1 if q^i is anticommuting.

The generalized Poisson brackets satisfy the "antisymmetry" and "Jacobi" relations. The antisymmetry relation is

$$\{A, B\}_{PB} = -(-1)^{|A||B|}\{B, A\}_{PB}, \tag{B.5}$$

where $|A|$ and $|B|$ denote the Grassmann parities of A and B respectively. The Jacobi identity is generalized to

$$\sum_{cyclic\,perms} (-1)^{|A||C|}\{A, \{B, C\}_{PB}\}_{PB} = 0. \tag{B.6}$$

Calculus of Commuting and Anticommuting Grassmann Numbers [20,10]

Following DeWitt [20], we define *supernumbers* by starting with an infinite dimensional Grassmann algebra with basis ζ^a $a = 1, 2, \ldots$ satisfying only the relations

$$\begin{aligned} \zeta^a\zeta^b &= -\zeta^b\zeta^a, \\ (\zeta^a)^2 &= 0. \end{aligned} \tag{B.7}$$

We denote this algebra over a base field $\mathbf{F}$ by $\Lambda_\infty(\mathbf{F})$. We shall be concerned mostly with $\Lambda_\infty(\mathbf{R})$. Any supernumber in $\Lambda_\infty(\mathbf{R})$ may be split into its *body* and *soul*

$$\begin{aligned} &x \in \Lambda_\infty(\mathbf{R}), \quad x = x_B + x_S \\ &x_B \in \mathbf{R}, \quad x_S = \sum_{n=1}^{\infty} \frac{1}{n!} c_{\alpha_1\ldots\alpha_n}\zeta^{\alpha_n}\cdots\zeta^{\alpha_1} \\ &c_{\alpha_1\ldots\alpha_n} \in \mathbf{R} \end{aligned} \tag{B.8}$$

Functions on $\Lambda_\infty(\mathbf{R})$ may be defined by extending any infinitely differentiable real function by the *formal* series

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_B) x_S^n. \tag{B.9}$$

Because the series (B.9) is purely formal, there is no problem with convergence.

More important for physics is the distinction between *even* and *odd* (that is, commuting and anticommuting) supernumbers. Any supernumber x can be split into two pieces, $x_c \in \mathbf{R}_c$ and $x_a \in \mathbf{R}_a$.

$$\begin{aligned} x &= x_c + x_a; \\ x_c &= x_B + \sum_{n=1}^{\infty} \frac{1}{(2n)!} c_{\alpha_1 \ldots \alpha_{2n}} \zeta^{\alpha_{2n}} \cdots \zeta^{\alpha_1}, \\ x_a &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} c_{\alpha_1 \ldots \alpha_{2n+1}} \zeta^{\alpha_{2n+1}} \cdots \zeta^{\alpha_1}. \end{aligned} \tag{B.10}$$

Analytic functions of a single anti-commuting variable are precisely the linear functions

$$f(x_a) = a + b x_a, \tag{B.11}$$

because x_a is nilpotent, $x_a^2 = 0$.

Functions of a real commuting variable, obtained from infinitely differentiable real functions, are defined by the formal series (B.9).

A definite integral along a path in $\mathbf{R}_c$ of a function of a function of a commuting variable is given by

$$\int_a^b f(x)\, dx = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{a_B}^{b_B} f^{(n)}(x_B(t)) x_S^n(t) \left[\frac{dx_B}{dt} + \frac{dx_S}{dt} \right] dt \tag{B.12}$$

where t is a real number. Here we take the parametrization to be $x_B(t) = t$. The striking thing about the integral (B.12) is that it is independent of the path $(x_B(t), x_S(t))$ used to define it. This fact is easily demonstrated. First we rewrite the integral (B.12) as

$$\int_a^b f(x)\, dx = \sum_{n=0}^{\infty} \int_{a_B}^{b_B} f^{(n)}(t) \left[\frac{1}{n!} x_S^n(t) + \frac{1}{(n+1)!} \frac{d}{dt} x_S^{n+1}(t) \right] dt \tag{B.13}$$

Next we split the sums apart and integrate by parts.

$$
\begin{aligned}
\int\limits_a^b f(x)\,dx = \int\limits_{a_B}^{b_B} & f(t)\left[1 + x_S'(t)\right]\,dt \\
& + \sum_{n=1}^{\infty} \frac{1}{n!}\left[f^{(n-1)}(b_B)b_S^n - f^{(n-1)}(a_B)a_S^n\right] \\
& + \sum_{n=1}^{\infty} \int\limits_{a_B}^{b_B} \left[-\frac{1}{n!} f^{(n-1)} \frac{dx_S^n(t)}{dt} + \frac{1}{(n+1)!} f^{(n)}(t) \frac{dx_S^{n+1}}{dt}\right]\,dt.
\end{aligned} \tag{B.14}
$$

We obtain the desired result

$$
\begin{aligned}
\int\limits_a^b f(x)\,dx &= \int\limits_{a_B}^{b_B} f(t)\,dt + \sum_{n=1}^{\infty} \frac{1}{n!}\left[f^{(n-1)}(b_B)b_S^n - f^{(n-1)}(a_B)a_S^n\right] \\
&= F(b) - F(a),
\end{aligned} \tag{B.15}
$$

as long as f and all of its derivatives are finite at the bodies a_B and b_B. A corollary of this is that the improper integral over $\mathbf{R}_c$ is the same as that over $\mathbf{R}$. This follows from the fact that

$$
\lim_{t \to \infty} F(t + x_S) = \lim_{t \to \infty} F(t) \tag{B.16}
$$

holds for all smooth functions F and all finite $x_S \in \mathbf{R}_c$.

By contrast, the integral over anticommuting numbers is a simpler operation. This integral, the *Berezin* integral [10], is motivated not by measure theoretical ideas but by simple consideration of the properties one would like such an integral to have. The properties one desires are that the integrals are translation invariant

$$
\int\limits_{\mathbf{R}_a} f(x_a)\,dx_a = \int\limits_{\mathbf{R}_a} f(x_a + y_a)\,dx_a \quad \forall y_a \in \mathbf{R}_a, \tag{B.17}
$$

and that integration may be done by parts

$$
\int\limits_{\mathbf{R}_a} \frac{d}{dx_a} f(x_a)\,dx_a = 0. \tag{B.18}
$$

By these requirements, the integral over anticommuting numbers is specified by

$$\begin{aligned}\int_{\mathbf{R}_a} 1\, dx_a &= 0,\\ \int_{\mathbf{R}_a} x_a\, dx_a &= Z.\end{aligned} \tag{B.19}$$

Here Z is a constant supernumber, usually taken to be 1. The rules (B.19) specify the integral because the functions one works with are taken to be analytic and hence linear.

Supertrace and Superdeterminant

Suppose that we have a vectorspace over $\mathbf{R}_c$, whose vectors have components which are either commuting or anticommuting supernumbers. Let us suppose that the first n components are commuting while the last m components are anticommuting,

$$\mathbf{V} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \\ u_1 \\ \vdots \\ u_m \end{pmatrix}. \tag{B.20}$$

where $v_i \in \mathbf{R}_c$ and $u_j \in \mathbf{R}_a$. Any matrix which preserves this characterization of the vectors must be of the form

$$M = \begin{pmatrix} A & C \\ D & B \end{pmatrix}, \tag{B.21}$$

where A is an $n \times n$ matrix whose elements are commuting, B is $m \times m$ and also commuting, C and D are anticommuting $n \times m$ and $m \times n$ matrices respectively.

The *supertrace* of a matrix M given in (B.21) is defined as

$$\mathrm{str} M = \mathrm{tr} A - \mathrm{tr} B. \tag{B.22}$$

The *superdeterminant* is

$$\operatorname{sdet}\begin{pmatrix} A & C \\ D & B \end{pmatrix} = \det(A - CB^{-1}D)(\det B)^{-1} \\ = (\det(B - DA^{-1}C))^{-1}\det A \tag{B.23}$$

With these definitions we have the properties

$$\begin{aligned} \operatorname{str}(M_1 M_2 \cdots M_n) &= \operatorname{str}(M_2 \cdots M_n M_1), \\ \operatorname{sdet}(M_1 M_2) &= \operatorname{sdet}(M_1)\operatorname{sdet}(M_2), \\ \delta \ln \operatorname{sdet}(M) &= \operatorname{str}(M^{-1}\delta M), \\ \operatorname{sdet}(\mathbf{1} + \epsilon M) &= 1 + \epsilon \operatorname{str}(M) + O(\epsilon^2). \end{aligned} \tag{B.24}$$

Under a change of variables in a region of $\mathbf{R}_c^n \times \mathbf{R}_a^m$, the "measure" of an integral transforms as we expect. The general rule is

$$d^n x_c\, d^m x_a = \operatorname{sdet}\left(\frac{\partial(x_c, x_a)}{\partial(\bar{x}_c, \bar{x}_a)}\right) d^n \bar{x}_c d^m \bar{x}_a. \tag{B.25}$$

Delta functions

In integrals over $\mathbf{R}_c$, we may define the delta function through its formal series (B.9) because it is infinitely differentiable, as all distributions are.

$$\delta_c(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\delta^{(n)}(x_B)x_S^n. \tag{B.26}$$

The delta function has the property

$$\delta_c(f(x)) = \sum_{\text{zeros}\, x_i} \frac{1}{f'(x_i)}\delta_c(x_i). \tag{B.27}$$

The anticommuting delta function is defined by

$$\delta_a(x_a) = x_a, \tag{B.28}$$

and satisfies

$$\delta_a(f(x_a)) = \sum_{\text{zeros}\, x_i} f'(x_i)\delta_a(x_i). \tag{B.29}$$

APPENDIX C

Miscellaneous Poisson Brackets

In computing Poisson brackets for the manifestly supersymmetric string, one needs to use the formula

$$\begin{aligned} f(\sigma)g(\varrho)\delta'(\sigma-\varrho) &= \tfrac{1}{2}(f(\sigma)g(\sigma)+f(\varrho)g(\varrho))\delta'(\sigma-\varrho) \\ &\quad + \tfrac{1}{2}(f(\sigma)g'(\sigma)-f'(\sigma)g(\sigma))\delta(\sigma-\varrho). \end{aligned} \tag{C.1}$$

The Poisson brackets between the string variables

$$\begin{aligned} \Pi^{A\mu} &= P^\mu + (-)^A X^{\mu\prime} - 2i(-)^A \overline{\theta}^A \gamma^\mu \theta^{A\prime}, \\ D^A &= \zeta^A + i\overline{\theta}^A \gamma_\mu (P^\mu + (-)^A X^{\mu\prime} - (-)^A i\overline{\theta}^A \gamma^\mu \theta^{A\prime}), \end{aligned}$$

$$\begin{aligned} \{\Pi^\mu_A(\sigma), \Pi^\nu_B(\varrho)\} &= 2(-)^A \delta_{AB}\, \eta^{\mu\nu} \delta'(\sigma-\varrho), \\ \{D^{Aa}(\sigma), D^{Bb}(\varrho)\} &= 2i\delta^{AB}\delta(\sigma-\varrho)(h_A\gamma^0\gamma^\mu)\Pi_{A\mu}(\sigma), \\ \{D^{Aa}(\sigma), \Pi^{B\mu}(\varrho)\} &= -4i(-)^A\delta^{AB}\delta(\sigma-\varrho)(\overline{\theta}^{A\prime}\gamma^\mu)^a, \end{aligned} \tag{C.2}$$

yield the Poisson brackets of the first-class fermionic constraints

$$\begin{aligned} \{D^A \not{\Pi}_A(\sigma), D^B \not{\Pi}_B(\varrho)\} &= -4i\delta^{AB}\delta(\sigma-\varrho)\Big[(\gamma^0 h_A \not{\Pi}_A)_{(ab)}\left(\tfrac{1}{2}\Pi_A^2 + 2(-)^A D^A \theta'_A\right) \\ &\quad + 4(-)^A \overline{\theta}^{A\prime}_{(a} D^A \not{\Pi}^A_{b)} + (\gamma^0 h_A \gamma^\mu)_{(ab)} D^A \not{\Pi}^A \gamma_\mu \theta^{A\prime} \\ &\quad - \frac{i}{4}(\gamma^0 h_A \gamma^\mu)_{(ab)} D^A \gamma_\mu \overline{D}^{A\prime}\Big] \\ &\quad + (-)^A \delta^{AB}\delta'(\sigma-\varrho)\Big[(D^A\gamma_\mu)_{[a}(D^A\gamma^\mu)_{b]}(\sigma) + (\sigma \leftrightarrow \varrho)\Big] \end{aligned} \tag{C.3}$$

Here $\not{\Pi}$ is a shorthand for $\Pi \cdot \gamma$.

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