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Inequivalence of the Brink-Schwarz and Siegel Superparticles*

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Abstract

Dirac's procedure is applied to the Brink-Schwarz and Siegel superparticle systems. Both systems are exhibited to have the same first-class constraints. The difference between the systems is the lack of second-class fermionic constraints in the Siegel superparticle. Thus the Siegel superparticle has a phase space with twice as many fermionic degrees of freedom as the Brink-Schwarz superparticle.

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1. Introduction

Since its discovery in 1984 the manifestly supersymmetric string action [1] has resisted quantization in a covariant gauge [2, 3, 4]. The supersymmetric particle has served as a laboratory for those wishing to practice tier methods before attacking the more complicated mechanics of the superstring.

The phase space structures of the superstring and, to a lesser extent, of the superparticle, are complicated and so the methods of Dirac [6] seem particularly appropriate for their quantization. In this letter I wish to discuss the classical mechanics of the Brink-Schwarz and Siegel superparticles. The Dirac analysis is also useful in this context as it allows the true dimension of the phase space to be examined easily.

In a forthcoming paper I shall describe and compare the two actions proposed as manifestly supersymmetric descriptions of the superstring [1, 9].

2. Review of the Dirac Procedure

Whenever a Lagrangian is first order in derivatives, or has a variable whose velocity does not appear, the system is said to be singular. In most cases one must be careful when quantizing a singular system. Singular systems have constraints which first appear in the course of defining the momenta. For such systems there are velocities which are not expressible in terms of positions and momenta.

The definition of the momenta become the primary constraints of the system,

$$\phi_i(p, q) \approx 0. \tag{2.1}$$

Here the curly equals sign reminds one that, while the system's evolution occurs on the surface in phase space satisfying eq. (2.1) as an equality, the equality should not be taken before evaluating a Poisson bracket.

One obtains the Hamiltonian in the usual way except that there is the possibility of adding arbitrary combinations of the constraints to the Hamiltonian because any two such Hamiltonians would be numerically equal on the constraint surface.

In the course of time evolution, the system should remain on the constraint surface specified by eq. (2.1). This is equivalent to the conservation of the constraints:

$$\dot{\phi}_i = \{\phi_i, H\}_P \approx 0. \tag{2.2}$$

The preservation of the constraints in time may require the imposition of further constraints or a restriction on the form of the Hamiltonian. Any further constraints must also be conserved.

Eventually one is left with a set of consistent conserved constraints and a Hamiltonian which may have some arbitrariness in its from.

The maximal algebra of constraints with the property that the Poisson bracket of any element of the algebra and any other constraint vanishes on the constraint surface is the algebra of first-class constraints. Any remaining constraints are called second-class.

The first-class constraints generate "gauge transformations" while the second-=class constraints are superfluous variables which must be excised from the system. The consistent removal of second-class constraints is accomplished using a redefined bracket, the Dirac bracket [6], which has the property that the bracket of any function on phase space with any second-class constraint vanishes identically. Usually one can deal with the second-class constraints trivially. Exceptions appear to be the superparticle and the corresponding superstring in which the second-class constraints serve to frustrate quantization.

The existence of gauge symmetry and the corresponding first-class constraints allows one to impose as many gauge conditions as one has symmetries. The corresponding statement in the Hamiltonian language is that a single first-class constraint reduces the dimension of the phase space by two, while the second-class constraints, which do not generate symmetries, each reduce the dimension of the phase space by one.

3. The Brink-Schwarz Superparticle

The massless superparticle considered in ref. [8] has the simple action

$$I = \tfrac{1}{2}\int d\tau [e^{-1}(\dot{x}^{\mu} - i\bar{\theta}\gamma^{\mu}\dot{\theta})^2]. \tag{3.1}$$

Here e is an "einbein" playing the role of the metric tensor along the world-line, θ is the fermionic position of the particle in superspace which is the superpartner of the particle's bosonic position X^{μ}. The spinor θ is taken to be a Majorana spinor. In ten dimensions θ can be taken to be both Majorana and Weyl and has sixteen real components.

The momenta for the fields e, X^{μ} and θ are defined as

$$\begin{aligned} P_e &:= \frac{\partial L}{\partial \dot{e}} = 0, \\ P_{\mu} &:= \frac{\partial L}{\partial \dot{x}^{\mu}} = e^{-1}(\dot{x}^{\mu} - i\bar{\theta}\gamma_{\mu}\dot{\theta}), \\ \zeta &:= \frac{\partial_R L}{\partial_R \dot{\theta}} = -i\bar{\theta}\gamma_{\mu}e^{-1}(\dot{x}^{\mu} - i\bar{\theta}\gamma^{\mu}\dot{\theta}) = -i\bar{\theta}\not{P}, \end{aligned} \tag{3.2}$$

where R denotes the right-handed derivative. The canonical Hamiltonian is

$$H_0 = \tfrac{1}{2}eP^2, \tag{3.3}$$

and there are two primary constraints.

$$\begin{aligned} \phi_1 &:= P_e \approx 0, \\ \phi_2 &:= \zeta + i\bar{\theta}\not{P} \approx 0. \end{aligned} \tag{3.4}$$

Conservation of the constraints (2.55) requires the imposition of a third constraint,

$$\phi_3 := \tfrac{1}{2}P^2 \approx 0. \tag{3.5}$$

One may check that the most general allowed Hamiltonian is

$$H = \tfrac{1}{2}(e + \lambda_3)P^2 + \zeta\not{P}\lambda_2 + P_e\lambda_1, \tag{2.57}$$

with $\lambda_{1,2,3}$ arbitrary, and that there are no further constraints. The constraints ϕ_1 and ϕ_3 are purely first-class, generating τ-reparametrizations, while ϕ_2 is half first-class and half second-class. The first-class piece in ϕ_2 is just

$$\phi_2^{1st} = \zeta \not{P} \approx 0. \tag{3.7}$$

This constraint generates a superparticle analog of the superstring's κ-symmetry. The identification of the second-class piece of ϕ_2 cannot be made covariantly for the massless superparticle [5], but it may be identified as

$$\phi_2^{2nd} = \phi_2 \gamma^0 \not{P} \gamma^0 \approx 0. \tag{3.8}$$

Discounting the einbein and its momentum, the canonical phase space has $2D$ bosonic dimensions and $2S$ fermionic dimensions. Here S is the number of real independent degrees of freedom of whatever type of spinor we are considering. For a ten-dimensional Majorana-Weyl spinor S is sixteen. There is a single bosonic first-class constraint, namely ϕ_3, $\frac{1}{2}S$ fermionic first-class constraints and $\frac{1}{2}S$ fermionic second-class constraints. Thus the completely gauge fixed phase space has $2D-2$ bosonic and $\frac{1}{2}S$ fermionic dimensions.

4. The Siegel Superparticle

The action considered in ref. [7] is

$$I = \int d\tau [(\dot{x} - i\bar{\theta}\gamma\dot{\theta}) \cdot P - \tfrac{1}{2} g P^2 + \dot{\bar{\theta}}\pi + \psi \not{P} \pi]. \tag{4.1}$$

This action, excepting the last two terms, is just (3.1) written in first-order form. Just as in first-order electrodynamics (Palatini formalism) one may treat both X and P as positions and give each a momentum. The Dirac procedure will identify P with the momentum conjugate to X through second-class constraints. Thus the analysis is as straightforward as before but somewhat messier.

For the action (4.1) take the coordinates to be X^μ, P_μ, θ, π, and g. Their canonically conjugate momenta are then $P_x, P_P, \zeta, P_\pi, P_\psi$, and P_g respectively. The Dirac procedure yields a conserved set of constraints after two iterations. The constraints are

$$
\begin{aligned}
P_X - P &\approx 0,\\
P_P &\approx 0,\\
P_\pi &\approx 0,\\
\zeta + \bar{\pi} + i\bar{\theta}\not{P} &\approx 0,\\
\not{P}\pi &\approx 0,\\
P_g &\approx 0,\\
P_\psi &\approx 0,\\
\tfrac{1}{2}P^2 &\approx 0.
\end{aligned}
\tag{4.2}
$$

The first pair are second-class and may be ignored if P is identified as conjugate to X. The second pair are also second-class and may similarly be ignored if $\bar{\pi}$ is everywhere identified with $-\zeta - i\bar{\theta}\not{P}$. This leaves a consistent set of constraints

$$
\begin{aligned}
\tfrac{1}{2}P^2 &\approx 0,\\
\zeta\not{P} &\approx 0,\\
P_g &\approx 0,\\
P_\psi &\approx 0,
\end{aligned}
\tag{4.3}
$$

which are *all* first-class. The corresponding Hamiltonian is

$$
H = \tfrac{1}{2}(g + \lambda_1)P^2 + (\psi + \lambda_2)\not{P}\bar{\zeta} + \lambda_3 P_g + \lambda_4 P_\psi,
\tag{4.4}
$$

with $\lambda_{1,2,3,4}$ arbitrary.

Again discounting the einbein g, the gravitino ψ, and their momenta, the canonical phase space has, as before, $2D$ bosonic and $2S$ fermionic dimensions. Here again one has a single bosonic and $S/2$ fermionic first-class constraints; but no second-class constraints. Thus the gauge-fixed phase space has $2D - 2$ bosonic and S fermionic dimensions.

5. Conclusions

Because the two superparticle actions considered here have phase spaces of different dimensions, they are not equivalent. It seems that a necessary condition for having $\frac{1}{4}S$, rather than $\frac{1}{2}S$, unconstrained fermionic coordinates is the existence of second-=class constraints. Since the second-class constraints of the massless Brink-Schwarz superparticle cannot be identified covariantly it is unlikely that the Brink-Schwarz superparticle can be quantized in a formalism which requires the separation of first- from second-class constraints.

This problem is less severe for the string in that, contrary to the claims of ref. [5], the corresponding second-class constrains can be identified in a ten-dimensionally covariant manner [3,4].

Note Added

After completion of this work, it was pointed out to me that an observation tantamount to the inequivalence of the two superparticle actions has been made elsewhere [10].

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