MAD/TH-94-10
September, 1994

Reality Conditions, Reducibility and Ghosts

Theodore J. Allen *and* Dennis B. Crossley*

Department of Physics, University of Wisconsin

1150 University Avenue, Madison, WI 53706 USA

Abstract

We examine two methods for incorporating complex first-class constraints into the BRST formalism when the complex conjugates of the constraints are linearly dependent upon the constraints, as is the case for general relativity in the Ashtekar variables.

PACS: 03.70.+k, 04.60.Ds, 11.15.−q

⋆ Present Address: Dept. of Physics, U. Toronto, 60 St. George St., Toronto, ON M5S 1A7 Canada

1. Introduction

The gravitational force, the first fundamental force to be understood classically, has yet to reconciled with quantum mechanics. Attempts at constructing a quantum theory have ranged from Dirac's work on the canonical quantization of general relativity, through perturbative quantum field theory to supersymmetric string theory. The perturbative approach failed because general relativity, long known to be non-renormalizable, is not finite at two loops.[1] The canonical approach pioneered by Dirac did not fail as much as it simply came to an impasse because of the computational complexity of the quantum constraints. Supersymmetric string theory remains an exciting and viable candidate but it too has become a very involved edifice.

In 1986 there was surprising progress in the canonical approach to quantum gravity when Ashtekar published a canonical transformation of Einstein gravity to a new set of variables in which the quantum constraints are polynomial.[2] In the original work of Dirac, the constraints are not polynomial in the basic variables and, when quantized, become quite complicated, possibly pseudo-differential, operators. In the new variables, the quantum constraints are second order functional differential operators. We believe it is important to pursue this very conservative approach to quantum gravity before abandoning altogether the methods of quantum field theory for the very radical superstring theory.

The constraints of general relativity simplify enough in the new variables that there are now solutions known to the full set of constraints.[3] Still missing from the quantization program are an inner product on the space of physical states and a full set of solutions to the constraints. Between the time that Dirac developed his methods for quantizing constrained theories and the time that the new variables were found, a very powerful method for quantizing constrained theories was developed by Fradkin's Russian school[4]; the BRST-BFV (Becchi-Rouet-Stora-Tyutin–Batalin-Fradkin-Vilkovisky) method, or BRST method for short. Certain difficulties of Dirac's methods, such as operator ordering, are less severe in the BRST method. The BRST method also yields a natural measure for the inner product on the physical subspace and allows more freedom in the choice of physical states.

The application of the BRST formalism to the problems of finding physical states and an inner product on them is far from straightforward because the new variables are complex-valued and the quantum constraints are thus non-hermitian. The standard BRST formalism assumes that the constraints are real-valued in order that the quantum BRST charge, the central object of the formalism, is hermitian. Straightforwardly applying the standard prescription, we find that the non-hermiticity of the constraints forces the BRST charge to be non-hermitian, which prevents the decoupling of unphysical states from the spectrum and destroys unitarity. Although there is probably deep significance to the fact that the new variables are self-dual and therefore that they and the constraints are necessarily complex-valued, the non-hermiticity of the constraints presents us with challenging technical difficulties. In this paper we examine the options for incorporating non-hermitian constraints into the BRST formalism. We assume that the complex conjugates of the constraints together with the constraints themselves are together first-class. We have previously found a method for using complex constraints in the case that the complex conjugates of the constraints together with the constraints themselves are second-class.[5]

2. The Hermiticity of $\widehat{\Omega}$ in BRST quantization

The BRST quantization of a Hamiltonian dynamical system with gauge symmetries provides a useful system for finding physical quantum states and removing unphysical states, those having zero or negative norm, from the spectrum. The central object of this quantization is the BRST charge, $\widehat{\Omega}$. The quantum BRST charge is constructed from the constraints and auxiliary variables, called ghosts, to be nilpotent, $\widehat{\Omega}^2 = 0$. Physical states are defined to be those states annihilated by the BRST charge, $\widehat{\Omega}\,|\phi\rangle = 0$. Because of the nilpotency of $\widehat{\Omega}$, physical states are only defined up to the addition of BRST exact states, $|\phi\rangle \equiv |\phi\rangle + \widehat{\Omega}\,|\chi\rangle$. This ambiguity is removed by defining an equivalence class of states. In order to define these equivalence classes, it is necessary that two states which differ by a BRST exact state have the same inner products with all other physical states. We must require

$$(\langle\phi| + \langle\omega|\,\widehat{\Omega}^\dagger)(|\psi\rangle + \widehat{\Omega}\,|\chi\rangle) = \langle\phi|\psi\rangle\,, \tag{2.1}$$

for all states $|\omega\rangle$ and $|\chi\rangle$. In addition to being nilpotent, $\widehat{\Omega}^2 = 0$, and annihilating physical states, $\widehat{\Omega}\,|\psi\rangle = 0$, the BRST charge must have an hermitian conjugate that satisfies $\widehat{\Omega}^\dagger\widehat{\Omega} = 0$ and $\widehat{\Omega}^\dagger\,|\psi\rangle = 0$. This is accomplished if the BRST charge is Hermitian, $\widehat{\Omega}^\dagger = \widehat{\Omega}$. In the rest of this paper, we deal with the slightly simpler problem of finding a real *classical* BRST charge. This problem is simpler because we need not consider the extra difficulties that operator ordering introduces once real classical quantities are transcribed into operators. These are difficulties which one must face in any case, whether the constraints are real or complex.

3. Complex extensions of real rank zero and one theories

3.1. Constructing real BRST charges

We demonstrate the construction of real BRST charges for rank-zero and rank-one theories. In these cases, a real BRST charge can be constructed by letting the ghosts be complex and imposing reality conditions upon them. In order to have a simple system to analyze before going to the general case, we first consider a system of real constraints G°_a that are linearly recombined into an equivalent set of constraints,

$$G_a = C^b_a G^\circ_b, \tag{3.1}$$

with coefficients C^b_a that are, in general, complex quantities. Complex conjugation of Eq. (3.1),

$$G^*_a = C^{b*}_a G^\circ_b, \tag{3.2}$$

leads to reality conditions on the constraints,

$$G^*_a = C^{b*}_a (C^{-1})^d_b C^e_d G^\circ_e = C^{b*}_a (C^{-1})^d_b G_d = B^d_a G_d. \tag{3.3}$$

The complex conjugate of a constraint is some linear combination of the original constraints themselves. Since complex conjugation is an involution ($G^{**} = G$), it follows from

$$G^{**} = (BG)^* = B^* BG \equiv G, \tag{3.4}$$

that the coefficients B have the property

$$B^* = B^{-1}. \tag{3.5}$$

As a simple example of reality conditions, we consider the linear combination of constraints

$$\begin{aligned} G_1 &= G_1^\circ, \\ G_2 &= G_2^\circ + iAG_1^\circ, \end{aligned} \tag{3.6}$$

where G_1° and G_2° are real constraint functions. Complex conjugation of these constraints leads to the reality conditions,

$$\begin{aligned} G_1^* &= G_1, \\ G_2^* &= G_2 - 2iAG_1. \end{aligned} \tag{3.7}$$

In some cases, it is possible that the reality conditions on the constraints can be used to construct a real BRST charge by imposing corresponding reality conditions on the ghosts and their conjugate momenta. We consider separately the case of the coefficients B_a^b being constant on the phase space and the case of the coefficients being phase space functions.

A rank-zero theory has no nonzero structure functions. This is the abelian case,

$$\{G_a, G_b\} = 0, \tag{3.8}$$

and the BRST charge is given simply by

$$\Omega_{\text{abelian}} = \eta^a G_a. \tag{3.9}$$

We impose the condition that the BRST charge be real, $\Omega^* = \Omega$, and use the reality conditions (3.3) on the constraints to derive reality conditions on the ghosts,

$$\begin{aligned}\eta^a G_a &= (\eta^a G_a)^* \\ &= \eta^{a*} B_a^b G_b \\ &= \eta^{b*} B_b^a G_a.\end{aligned} \tag{3.10}$$

The last step is a relabeling of the dummy indices. Because the constraints G_a are linearly independent, the ghosts satisfy reality conditions

$$\eta^{a*} = \eta^b B_b^{a*}. \tag{3.11}$$

Complex conjugation of the fundamental ghost Poisson brackets, with the rule $\{A, B\}^* = -\{B^*, A^*\}$, yields reality conditions on the ghost momenta,

$$\mathcal{P}_a^* = -B_a^b \mathcal{P}_b. \tag{3.12}$$

It is easy to check that the BRST charge Ω and the Poisson bracket $\{\mathcal{P}_a, \eta^a\}$ are both preserved under complex conjugation by recalling Eq. (3.5).

A rank-one theory has first-order structure constants. In this case the constraints form a Lie algebra. The BRST charge for a rank-one theory is given by

$$\Omega_{\text{Lie}} = \eta^a G_a - \tfrac{1}{2}\eta^b \eta^c C_{cb}{}^a \mathcal{P}_a. \tag{3.13}$$

Exactly as in the abelian (rank zero) case, we assume the same reality conditions (3.1) on the constraints. The requirement that the antighost number zero part of (3.13) be real leads to the same reality conditions on the ghosts and their momenta as in the abelian case. The new element is the first-order structure functions C_{ab}^c. Reality conditions on the first-order structure functions follow from complex conjugation of the Poisson brackets between the constraints. The resulting reality conditions are

$$C_{ab}{}^{c*} = B_a^d B_b^e C_{de}{}^f (B^{-1})_f^c. \tag{3.14}$$

Using the rule for complex conjugating fermionic variables, $(AB)^* = B^* A^*$, we find it straightforward to check that the rank one term is also real,

$$(\eta^b \eta^a C_{ab}^c \mathcal{P}_c)^* = \eta^b \eta^a C_{ab}^c \mathcal{P}_c, \tag{3.15}$$

and therefore that the BRST charge (3.13) is real. In both the rank zero case and the rank one case, it has been essential that the B_a^b have zero Poisson brackets with each other (as is the case for constant B_a^b), in order to satisfy the requirement that the ghost Poisson brackets obey $\{\eta^{a*}, \mathcal{P}_b^*\} = -\{\mathcal{P}_b, \eta^a\}^*$.

3.2. Ghost Reality Conditions

In the case that the constraints form a true Lie algebra, either abelian or non-abelian, we find that we can force the BRST charge to be real by imposing reality conditions upon the ghosts.

Changing notation a bit, we suppose that the bosonic complex constraints $G_a \approx 0$ satisfy

$$Z_I^{\bar{a}} G_{\bar{a}} + Z_I^a G_a = 0, \tag{3.16}$$

where $G_{\bar{a}} \equiv G_a^*$ and Z_I^a and $Z_I^{\bar{a}}$ are invertible square matrices. The ghosts must satisfy reality conditions

$$\eta^{\bar{a}} Z_{\bar{a}}^I + \eta^a Z_a^I = 0, \quad Z_I^{\bar{k}} \mathcal{P}_{\bar{k}} + Z_I^c \mathcal{P}_c = 0. \tag{3.17}$$

The matrices Z_a^I and $Z_{\bar{a}}^I$ are the inverses to Z_I^a and $Z_I^{\bar{a}}$ respectively. The coefficients B_a^b are related to the Z_I^a as follows $G_{\bar{a}} \equiv G_a^* = B_a^b G_b = -Z_{\bar{a}}^I Z_I^b G_b$. We use $\eta^{\bar{\imath}}$ as another name for η^{i*} and $\mathcal{P}_{\bar{k}}$ for $-\mathcal{P}_k^*$ so that the Poisson bracket relation $\{\mathcal{P}_{\bar{\imath}}, \eta^{\bar{\jmath}}\} = -\{\mathcal{P}_i^*, \eta^{j*}\} = \{\mathcal{P}_i, \eta^j\}^* = -\delta_i^j$. This will be an important notational advantage later.

Although different from the standard real ghosts, complex ghosts do not present much of an obstacle to quantization. The standard inner product for the ghosts is given by the integral over fermi variables, in analogy with commuting variables. For bosonic functions $f(\eta) = f_0 + f_1\eta$, and $g(\eta) = g_0 + g_1\eta$, the standard inner product with a real ghost

$$\langle f|g\rangle = \int d\eta f^*(\eta)g(\eta) = f_1^* g_0 - f_0^* g_1, \tag{3.18}$$

is replaced by

$$\langle f|g\rangle = \int d\eta\, d\eta^*\, \delta(\bar{Z}\eta^* + Z\eta) f^*(\eta^*)\, g(\eta) = Z f_1^* g_0 + \bar{Z} f_0^* g_1, \tag{3.19}$$

when the ghosts are complex. We emphasize that η^* and η are not both dynamical variables. There is only one, complex, dynamical variable η and its conjugate momentum. The generalization to several ghosts and to mixed fermionic and bosonic ghosts is immediate.

4. Reality conditions with non-constant coefficients

We now consider constraints with reality conditions, (3.16), whose coefficients Z are general phase space functions. Under these conditions, it is in general impossible to preserve the Poisson bracket relation $\{\eta^{a*}, \mathcal{P}_b^*\} = -\{\mathcal{P}_b, \eta^a\}^*$ for the ghosts under the assumptions of Eq. (3.17). Because we cannot preserve the Poisson bracket relation, it is impossible to use the previous method to construct a real BRST charge.

We first demonstrate that the standard BRST treatment of a complexified theory in general yields a complex BRST charge and is therefore unacceptable. We then give an alternative BRST method by which a real BRST charge can be constructed. This is accomplished by extending the ghost phase space and including the complex conjugate constraints in addition to the original constraints. This expanded system of constraints is inherently reducible and is dealt with using the reducible BRST method for BRST quantization of systems with reducible constraints that is so well explained by Henneaux and Teitelboim.[6]

4.1. Standard BRST treatment

For our starting point, we consider a simple example of two abelian constraints G_1° and G_2°,

$$\{G_1^\circ, G_2^\circ\} = 0, \tag{4.1}$$

which are assumed to be real and bosonic. The BRST charge for this example is given by

$$\Omega^\circ = \eta_\circ^1 G_1^\circ + \eta_\circ^2 G_2^\circ, \tag{4.2}$$

and is manifestly real if the ghosts $\eta_\circ^1$ and $\eta_\circ^2$ are taken to be real, which we are free to do.

What we want to consider is the complex extension of this real theory. By this we mean the analytic continuation of the set of real functions on phase space to the set of complex functions on phase space, with a transformation that takes real constraints into complex constraints. As a concrete example, consider replacing the real constraints G_1° and G_2° by

$$\begin{aligned} G_1^\circ \quad &\rightarrow \quad G_1 = G_1^\circ \\ G_2^\circ \quad &\rightarrow \quad G_2 = G_2^\circ + iA(q,p)G_1^\circ, \end{aligned} \tag{4.3}$$

where we have added a linear multiple of the first constraint to the second. (If we had added a completely arbitrary imaginary term we would have introduced a third constraint, since both the real and imaginary parts must separately vanish, and we would have a different theory.) The coefficient $A = A(q,p)$ is an arbitrary real bosonic function of the phase space variables. We may think of this transformation as a "deformation" of the real constraints into complex constraints.

The Poisson bracket structure of the constraints becomes

$$\begin{aligned} &\{G_1, G_1\} = \{G_2, G_2\} = 0, \\ &\{G_1, G_2\} = \{G_1^\circ, G_2^\circ + iAG_1^\circ\} = i\{G_1, A\}G_1. \end{aligned} \tag{4.4}$$

There is only one nonzero first-order structure function,

$$C_{12}{}^{1} = i\{G_1, A\}, \tag{4.5}$$

and the second-order structure functions ${}^{(2)}U_{abc}{}^{de}$ necessarily vanish because they are antisymmetric in (abc) and we have only two indices available. The BRST charge (4.2) is thus deformed into

$$\Omega^{\circ} \quad \rightarrow \quad \Omega = \eta^1 G_1 + \eta^2 G_2 - i\eta^2\eta^1\{G_1, A\}\mathcal{P}_1. \tag{4.6}$$

We now want to investigate the reality properties of Ω. In particular, we want to see if Ω can remain real. If we hope to accomplish this we must allow the ghosts and ghost momenta to become complex, but there is no *a priori* reason why this should not be allowed. We complex conjugate Ω,

$$\Omega^* = G_1^*\eta^{1*} + G_2^*\eta^{2*} + i\mathcal{P}_1^*(-\{A^*, G_1^*\})\eta^{1*}\eta^{2*}, \tag{4.7}$$

and use the reality condition on G_2,

$$G_2^* = G_2 - 2iAG_1, \tag{4.8}$$

derived from the definition (4.3) and the reality of G_1°, G_2°, and A, to rearrange (4.7) into

$$\Omega^* = (\eta^{1*} - 2iA\eta^{2*})G_1 + \eta^{2*}G_2 - i\eta^{2*}\eta^{1*}\{G_1, A\}\mathcal{P}_1^*. \tag{4.9}$$

Requiring $\Omega^* = \Omega$, we find the reality properties of the ghosts η^1 and η^2 from the first two terms,

$$\begin{aligned} &\eta^{2*} = \eta^2 \\ \eta^{1*} - 2iA\eta^{2*} = \eta^1 \qquad &\text{or} \qquad \eta^{1*} = \eta^1 + 2iA\eta^2. \end{aligned} \tag{4.10}$$

A straightforward calculation yields the transformation of the original ghosts which is consistent with these reality properties,

$$\begin{aligned} \eta^1_\circ \quad &\rightarrow \quad \eta^1 = \eta^1_\circ - iA\eta^2_\circ, \\ \eta^2_\circ \quad &\rightarrow \quad \eta^2 = \eta^2_\circ, \end{aligned} \tag{4.11}$$

and requiring that the fundamental Poisson brackets between the ghosts be preserved ($\{\mathcal{P}_a, \eta^b\} = -\delta^b_a$) gives the corresponding transformation of the ghost momenta,

$$\begin{aligned} \mathcal{P}^\circ_1 \quad &\rightarrow \quad \mathcal{P}_1 = \mathcal{P}^\circ_1, \\ \mathcal{P}^\circ_2 \quad &\rightarrow \quad \mathcal{P}^2 = \mathcal{P}^\circ_2 + iA\mathcal{P}^\circ_1. \end{aligned} \tag{4.12}$$

In particular, we observe that $\mathcal{P}_1$ remains unchanged in the deformed theory and is therefore pure imaginary. The consequence of this is that the last term in Eq. (4.6) is purely imaginary because $\eta^2\eta^1$ is the same as $\eta^2_\circ\eta^1_\circ$, which, being the product of two real grassmann numbers, is imaginary. The last term is thus the product of three imaginary quantities. We find then, that performing a complex deformation of set of real constraints introduces an imaginary piece to the BRST charge. Therefore, the standard BRST treatment of a complexified theory will not work in quantum theory and another approach is required.

4.2. Inclusion of complex conjugate constraints

To eliminate the imaginary piece of the BRST charge, we can contemplate two approaches. The first is to transform the complex constraints into purely real constraints. This, however, simply returns us to the initial constraints G°_1 and G°_2. In self-dual gravity, the real constraints are not polynomial in the phase space variables causing the BRST charge to be non-polynomial as well. The whole reason for using the complex constraints is that they are polynomial in either the self-dual or anti-self-dual variables. This first approach is not useful for gravity in the self-dual Ashtekar variables.

The second approach to making the BRST charge real is to use the complex conjugates of the constraints along with the original set of constraints in the hope that the imaginary terms added to the BRST charge will then appear in complex conjugate pairs, making the BRST charge manifestly real. This procedure, however, introduces an additional complication. The complex conjugate constraints that we add are not independent of the original constraints and we therefore end up with a reducible set of constraints.

Before, when we had complex ghosts, we regarded the complex conjugates of the ghosts and their momenta not to be dynamical variables in their own right, but simply some linear recombinations of the original complex ghost and ghost momentum variables. Here, however, we are faced with introducing not only complex ghosts and ghost momenta, but also their complex conjugates as independent dynamical variables. First we give a simple example of the kind of construction needed for a real BRST charge, deferring the general case to the next section.

To see how this approach works, we continue with the example of the previous section and add the constraint $G_{\overline{2}}$, complex conjugate of G_2, to the constraints G_1 and G_2,

$$\begin{aligned} G_1 &= G_1^\circ, \\ G_2 &= G_2^\circ + iA(q,p)G_1^\circ, \\ G_{\overline{2}} &= G_2^\circ - iA(q,p)G_1^\circ. \end{aligned} \tag{4.13}$$

$A = A(q,p)$ is again assumed to be a real function on the phase space. However, to avoid the unnecessary complication of second-order structure functions, we assume that the Poisson brackets of A with the original constraints are constant and generically non-zero,

$$\{G_1^\circ, A\} := \Gamma_1 = \text{constant}, \qquad \{G_2^\circ, A\} := \Gamma_2 = \text{constant}. \tag{4.14}$$

The Poisson brackets of A with the modified constraints are then

$$\begin{aligned} \{G_1, A\} &= \Gamma_1, \\ \{G_2, A\} &= \Gamma_2 + iA\Gamma_1, \\ \{G_{\overline{2}}, A\} &= \Gamma_2 - iA\Gamma_1, \end{aligned} \tag{4.15}$$

and the nonconstant Poisson brackets among the constraints are

$$
\begin{aligned}
\{G_1, G_2\} &= i\Gamma_1 G_1, \\
\{G_1, G_{\overline{2}}\} &= -i\Gamma_1 G_1, \\
\{G_2, G_{\overline{2}}\} &= -2i\Gamma_2 G_1.
\end{aligned}
\tag{4.16}
$$

The nonzero first-order structure functions

$$
\begin{aligned}
C_{12}{}^{1} &= i\Gamma_1, \\
C_{1\overline{2}}{}^{1} &= -i\Gamma_1, \\
C_{2\overline{2}}{}^{1} &= -2i\Gamma_2.
\end{aligned}
\tag{4.17}
$$

follow directly from equations (4.16) above. Since the first-order structure functions are all constant, the second-order structure functions can be taken to vanish.

In addition to the constraint algebra, we also have the constraint reducibility condition

$$
Z := Z^a G_a = -2iAG_1 + G_2 - G_{\overline{2}} = 0, \tag{4.18}
$$

with reducibility coefficients

$$
Z^1 = -2iA, \qquad Z^2 = 1, \qquad Z^{\overline{2}} = -1. \tag{4.19}
$$

The last step before constructing the BRST charge Ω is to extend the phase space with a ghost and its canonically conjugate momentum for each constraint and for the reducibility condition,

$$
\begin{aligned}
&\eta^1,\ \mathcal{P}_1 && \text{(associated with } G_1\text{)}, \\
&\eta^2,\ \mathcal{P}_2 && \text{(associated with } G_2\text{)}, \\
&\eta^{\overline{2}},\ \mathcal{P}_{\overline{2}} && \text{(associated with } G_{\overline{2}}\text{)}, \\
&\phi,\ \pi && \text{(associated with } Z\text{)}.
\end{aligned}
\tag{4.20}
$$

The ghosts η^i and their momenta $\mathcal{P}_i$ are anticommuting (fermionic) variables as before. The ghost of ghost ϕ and its conjugate momentum π have statistics opposite those of the ghosts and are therefore commuting (bosonic) variables. The Poisson bracket structure among the ghosts can be taken to be canonical,

$$\begin{aligned} \{\mathcal{P}_i, \eta^j\} = \{\eta^j, \mathcal{P}_i\} = -\delta_i^j, \\ \{\pi, \phi\} = -\{\phi, \pi\} = -1, \end{aligned} \tag{4.21}$$

with all other brackets among the ghosts vanishing. In addition, we assume the brackets of the original phase space variables are unchanged and that the brackets between the ghosts and the original phase space variables vanish.

We now have all of the building blocks for the BRST charge, which we construct according to the rules for reducible gauge theories,

$$\begin{aligned} \Omega =& \eta^1 G_1 + \eta^2 G_2 + \eta^{\overline{2}} G_{\overline{2}} - i\Gamma_1 \eta^2 \eta^1 \mathcal{P}_1 + i\Gamma_1 \eta^{\overline{2}} \eta^1 \mathcal{P}_1 + 2i\Gamma_2 \eta^{\overline{2}} \eta^2 \mathcal{P}_1 \\ &+ \phi(-2iA\mathcal{P}_1 + \mathcal{P}_2 - \mathcal{P}_{\overline{2}}). \end{aligned} \tag{4.22}$$

There could, in principle, be additional terms to the BRST charge arising from the nonconstant reducibility coefficient Z^1, but a straightforward (though somewhat tedious) calculation shows that the BRST charge (4.22) is nilpotent, $\{\Omega, \Omega\} = 0$, and that it is therefore the complete BRST charge.

We now consider the reality of the BRST charge (4.22). For the sum of the zero-order terms $\eta^i G_i$ to be real, it is sufficient that the ghost η^1 be taken to be real and that the ghosts η^2 and $\eta^{\overline{2}}$ be complex conjugates,

$$(\eta^2)^* = \eta^{\overline{2}}. \tag{4.23}$$

Complex conjugation of the fundamental Poisson bracket between $\mathcal{P}_1$ and η^1 and between $\mathcal{P}_2$ and η^2 then requires that $\mathcal{P}_1$ be pure imaginary (as in the standard BRST treatment) and that $i\mathcal{P}_2$ and $i\mathcal{P}_{\overline{2}}$ be complex conjugates, since

$$-1 = \{\mathcal{P}_2, \eta^2\}^* = -\{\eta^{2*}, \mathcal{P}_2^*\} = -\{\mathcal{P}_2^*, \eta^{\overline{2}}\}, \tag{4.24}$$

implies

$$(\mathcal{P}_2)^* = -\mathcal{P}_{\overline{2}}. \tag{4.25}$$

Finally, we find that with the choice of reducibility coefficients (4.19), the ghost-of-ghost ϕ must be taken to be real. With these complex conjugation rules for the ghosts and their momenta, we can rewrite the BRST charge (4.22) in the form

$$\Omega = \left(\tfrac{1}{2}\eta^1 G_1 + \eta^2 G_2 - i\Gamma_1 \eta^2 \eta^1 \mathcal{P}_1 + i\Gamma_2 \eta^{\bar{2}} \eta^2 \mathcal{P}_1 - iA\phi\mathcal{P}_1 + \phi\mathcal{P}_2\right) + \text{c.c.}, \tag{4.26}$$

where c.c. stands for the complex conjugate of everything inside the parentheses. Thus, the BRST charge (4.22) contains terms which are either real or occur as sums of complex conjugate pairs and we have explicitly demonstrated that the BRST charge (4.22) is real.

It is clear that this procedure generalizes to an arbitrary complexification of a set of real constraints into a set of complex constraints of the form

$$G_i^\circ \quad \rightarrow \quad G_i = G_i^\circ + iA^{ij} G_j^\circ. \tag{4.27}$$

A real BRST charge for a system with an arbitrary set of complex first-class constraints which are also first-class with their complex conjugates can be constructed by adding to the complex constraints their complex conjugates and treating the extended system of constraints as a standard reducible set of constraints.

5. The General Case

In general, if an irreducible set of complex constraints, G_i, have complex conjugates, $G_{\bar{\imath}}$, which are linearly dependent upon them

$$Z_I := Z_I^{\bar{k}} G_{\bar{k}} + Z_I^j G_j \equiv 0, \tag{5.1}$$

then we follow a procedure very similar to that for reducible constraints. There is some room for redefinition of the coefficients Z_I^i and $Z_I^{\bar{\imath}}$. We remove some of this indefiniteness by requiring that they satisfy

$$Z_I^{i*} = -Z_I^{\bar{\imath}} \tag{5.2}$$

We introduce ghosts η^i, $\eta^{\bar{\imath}}$ and their canonical conjugates $\mathcal{P}_i$ and $\mathcal{P}_{\bar{\imath}}$ along with a constraint upon the ghost momenta

$$Z_I^{\bar{k}}\mathcal{P}_{\bar{k}} + Z_I^j\mathcal{P}_j \approx 0. \tag{5.3}$$

We can think of this constraint as only half of the reality conditions on the ghosts (3.17). If we reinterpret the reality conditions in Eq. (3.17) as constraints upon an *enlarged* ghost phase space (η^i, $\eta^{\bar{\imath}}$, $\mathcal{P}_j$, $\mathcal{P}_{\bar{\jmath}}$), we find that they are second-class constraints. What we are doing is analogous to replacing second-class constraints by half their number of first-class constraints.

In addition, we introduce a ghost-for-ghost, ϕ^I, for the reality condition (5.3) in order to eliminate the extra degrees of freedom we added when we introduced both ghosts η^i, $\eta^{\bar{\imath}}$. The ghost-for-ghost ϕ^I has statistics opposite those of the ghosts η^i, $\eta^{\bar{\imath}}$ and can be chosen real when the reducibility coefficients Z_I^i and $Z_I^{\bar{\imath}}$ satisfy (5.2).

Following the standard BRST procedure, we then introduce the new constraint (5.3) multiplied by its ghost, ϕ^I, into the general BRST charge.

The BRST charge begins with the terms

$$\begin{aligned}\Omega =&\eta^k G_k + \eta^{\bar{k}} G_{\bar{k}} + \phi^I (Z_I^{\bar{k}}\mathcal{P}_{\bar{k}} + Z_I^j \mathcal{P}_j) \\ &- \tfrac{1}{2}\eta^j\eta^i C_{ij}^k\mathcal{P}_k - \tfrac{1}{2}\eta^j\eta^{\bar{\imath}} C_{\bar{\imath}j}^k\mathcal{P}_k - \tfrac{1}{2}\eta^{\bar{\jmath}}\eta^i C_{i\bar{\jmath}}^k\mathcal{P}_k - \tfrac{1}{2}\eta^j\eta^i C_{ij}^{\bar{k}}\mathcal{P}_{\bar{k}} \\ &- \tfrac{1}{2}\eta^{\bar{\jmath}}\eta^i C_{i\bar{\jmath}}^{\bar{k}}\mathcal{P}_{\bar{k}} - \tfrac{1}{2}\eta^j\eta^{\bar{\imath}} C_{\bar{\imath}j}^{\bar{k}}\mathcal{P}_{\bar{k}} - \tfrac{1}{2}\eta^{\bar{\jmath}}\eta^{\bar{\imath}} C_{\bar{\imath}\bar{\jmath}}^k\mathcal{P}_k - \tfrac{1}{2}\eta^{\bar{\jmath}}\eta^{\bar{\imath}} C_{\bar{\imath}\bar{\jmath}}^{\bar{k}}\mathcal{P}_{\bar{k}} + \ldots,\end{aligned} \tag{5.4}$$

where the terms denoted by the elipses are chosen such that the charge is nilpotent and real. That terms can be chosen such that Ω is nilpotent has been shown by Henneaux.[6] The proof does not depend upon the reality properties of the dynamical variables. To show that the succeeding terms can all be chosen real requires us to look at the nilpotency condition itself.

We assume that the BRST charge can be expanded in a series indexed by the total antighost number, $\Omega = {}^{(0)}\Omega + {}^{(1)}\Omega + {}^{(2)}\Omega + \ldots$. The ghost momenta $\mathcal{P}_i$ and $\mathcal{P}_{\bar{\imath}}$ carry antighost number one and the ghost momentum $\mathcal{P}_{\phi I}$, conjugate to ϕ^I, carries antighost number two. The condition for nilpotency of Ω becomes the set of conditions

$$\begin{aligned}&2\{{}^{(p+1)}\Omega, {}^{(1)}\Omega\}_{\phi,\mathcal{P}_\phi} + 2\{{}^{(p+1)}\Omega, {}^{(0)}\Omega\}_{\eta,\mathcal{P}} = \\ &-\sum_{k=0}^{p}\{{}^{(p-k)}\Omega, {}^{(k)}\Omega\}_{\mathrm{orig}} - \sum_{k=0}^{p-1}\{{}^{(p-k)}\Omega, {}^{(k+1)}\Omega\}_{\eta,\mathcal{P}} - \sum_{k=0}^{p-2}\{{}^{(p-k)}\Omega, {}^{(k+2)}\Omega\}_{\phi,\mathcal{P}_\phi}\end{aligned} \tag{5.5}$$

for all p. The different subscripted brackets refer to the Poisson brackets with respect to the subscripted variables only. The subscript 'orig' refers to the original, non-ghost, variables. If we assume that all the terms ${}^{(k)}\Omega$ can be chosen real for all $0 \leq k \leq p$, then we see from Eq. (5.5) that ${}^{(p+1)}\Omega$ can be chosen real as well.

6. Conclusion

Motivated by the problem of performing a BRST quantization of general relativity in Ashtekar's new variables, we have demonstrated the construction of a real BRST charge using both the constraints and their complex conjugates. While it is necessary to use both the constraints and their complex conjugates and thus construct a charge that would not be strictly polynomial in either self-dual or anti-self-dual variables, such a BRST charge could still be quite useful for quantization. We would hope that each term would be polynomial in either self-dual or anti-self-dual variables multiplied by ghost variables. We defer discussion in detail of the application of this construction to gravity in Ashtekar variables to a forthcoming paper.

REFERENCES

1. M.H. Goroff and A. Sagnotti, *Phys. Lett.* **160B** (1985) 81;
   Nucl. Phys. **B266** (1986) 709.

2. A. Ashtekar, *Phys. Rev. Lett.*, **57** (1986) 2244;
   Phys. Rev. **D36** (1987) 1587.

3. B. Brügmann, R. Gambini, J. Pullin, *Nucl. Phys.* **B385** (1992) 587.

4. E. S. Fradkin and G. A. Vilkovisky, *Phys. Lett.* **55B** (1975) 224;
   I. A. Batalin and G. A. Vilkovisky, *Phys. Lett.* **69B** (1977) 309;
   E. S. Fradkin and T. E. Fradkina, *Phys. Lett.* **72B** (1978) 343.

5. T.J. Allen and D.B. Crossley, *Phys. Rev.* **D47** (1993) 5494.

6. M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*, (Princeton University Press, Princeton, NJ, 1992), Chap. 10