

Spin in Schrödinger-quantized Pseudoclassical Systems

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We examine the construction of the spin angular momentum in systems with pseudoclassical Grassmann variables. In constrained systems there are many different algebraic forms for the dynamical variables that will all agree on the constraint surface. For the angular momentum, a particular form of the generators is preferred, which yields superselection sectors of irreducible $\mathfrak{spin}^c(n)$ representations rather than reducible $\mathfrak{so}(n)$ representations when quantized in the Schrödinger realization.

I. INTRODUCTION

Quantization of the pseudoclassical actions^{1–7}, introduced more than forty years ago to describe spinning particles, has been done by the path integral method and by canonical quantization directly on the reduced phase space. Quantization in the Schrödinger picture was partially worked out by Barducci, Bordi, and Casalbuoni^{8,9} but, as far as we are aware, the full details have been worked out only recently¹⁰.

In the standard reduced phase space approach, the phase space has rotational covariance, which must be broken by choosing a particular splitting into coordinates and momenta—a “polarization”—in order to use the Schrödinger realization. Because the actions are first-order in velocities, in order to quantize without reducing the phase space first we must use Dirac’s constrained Hamiltonian quantization, but in exchange we gain a coordinate space that is rotationally covariant. In the reduced phase space approach, the Noether angular momentum directly gives the correct spin, while in the Dirac quantization, the Noether angular momentum is ambiguous and one particular form must be chosen to obtain the correct spin.

Construction of angular momentum as differential operators on functions of Grassmann variables is not new. Mankoč-Borštnik gave such a construction,^{11,12} though not in the context of the Schrödinger quantization of a constrained pseudoclassical mechanical system. What is new here is to examine angular momentum in the context of the constrained Schrödinger quantization of the simplest pseudoclassical systems.

In the Schrödinger realization, the full state space splits into orthogonal physical and ghost sectors that have positive-definite and negative-definite norms, respectively. Within those sectors, for dimensions greater than two, there are isomorphic superselection sectors, each of which corresponds to quantization directly on the reduced phase space and forms an irreducible representation of $\mathfrak{spin}^c(n)$.

The wave functions in the Schrödinger realization thus correspond to spinorial states, and can be directly

mapped^{10,13} to Kähler fermion^{14–17} differential form-valued wave functions.

In the following, we examine how the $\mathfrak{so}(n)$ algebra that acts on the physical state space becomes $\mathfrak{spin}^c(n)$. We use the Einstein summation convention that repeated indices are summed. Because our metric is the unit matrix, there is no distinction between upper and lower indices but we raise or lower them for notational convenience and to make the generalization to indefinite metrics nearly immediate.

II. PSEUDOCCLASSICAL ACTION

We consider systems of n anticommuting variables ξ^i described by the rotationally invariant action

$$S = \int d\tau \left[\frac{i}{2} \xi_i \dot{\xi}^i - H(\xi) \right], \quad (2.1)$$

as a Hamiltonian system with constraints, in the sense of Dirac. The canonical Poisson brackets of two phase space functions, $A(\pi, \xi)$ and $B(\pi, \xi)$ is given by

$$\{A, B\} = \frac{\partial^R A}{\partial \pi_i} \frac{\partial^L B}{\partial \xi^i} + \frac{\partial^R A}{\partial \xi^i} \frac{\partial^L B}{\partial \pi_i}, \quad (2.2)$$

where the R and L superscripts denote whether the derivative is to be taken from the right or the left.

III. DIRAC CONSTRAINT ANALYSIS

Because the action (2.1) is first-order in velocities, there are Dirac constraints^{18,19},

$$\varphi^i = \pi^i - \frac{i}{2} \xi^i \approx 0. \quad (3.1)$$

These constraints have constant, non-vanishing Poisson brackets with each other

$$\Delta^{ij} = \{\varphi^i, \varphi^j\} = -i\delta^{ij}, \quad (3.2)$$

and so are second-class. The matrix (3.2) has inverse

$$\Delta_{ij}^{-1} = i\delta_{ij}. \quad (3.3)$$

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The dynamical system can be reduced to the phase space defined by the constraints (3.1) with the (ortho-)symplectic form given by the Dirac bracket,

$$\{A, B\}_D = \{A, B\} - \{A, \varphi^i\} \Delta_{ij}^{-1} \{\varphi^j, B\}, \quad (3.4)$$

which allows second-class constraints to be taken to zero strongly because they will have vanishing Dirac brackets with any dynamical quantity,

$$\{A, \varphi^i\}_D \equiv 0. \quad (3.5)$$

The Dirac brackets satisfy the same (graded) Jacobi identity as the Poisson brackets.

Instead of using Dirac brackets, dynamical variables can be replaced by their “primed” versions¹⁹

$$A' = A - \{A, \varphi^i\} \Delta_{ij}^{-1} \varphi^j, \quad (3.6)$$

and the original Poisson bracket can be kept, at the possible cost of having certain relations holding only on the constraint surface, rather than the full phase space. The Poisson brackets of primed variables satisfy

$$\{A', B'\} \approx \{A', B\} \approx \{B, A'\} \approx \{A, B\}_D, \quad (3.7)$$

where we have used Dirac’s “weak equality,” \approx , to denote quantities whose difference vanishes on the constraint surface defined by the constraints, in this case Eqs. (3.1).

We have previously analyzed the system described by the action (2.1) in detail¹⁰ using the primed variables

$$\begin{aligned} \xi'^i &\equiv \xi^i - \{\xi^i, \varphi^a\} \Delta_{ab}^{-1} \varphi^b = \xi^i - i\varphi^i \\ &= \frac{1}{2}\xi^i - i\pi^i, \\ \pi'^i &\equiv \pi^i - \{\pi^i, \varphi^a\} \Delta_{ab}^{-1} \varphi^b = \pi^i - \frac{1}{2}\varphi^i \\ &= \frac{1}{2}\pi^i + \frac{i}{4}\xi^i = \frac{i}{2}\xi'^i. \end{aligned} \quad (3.8)$$

These variables have strongly vanishing Poisson brackets with the second-class constraints,

$$\begin{aligned} \{\xi'^i, \varphi^j\} &= 0, \\ \{\pi'_i, \varphi^j\} &= 0. \end{aligned} \quad (3.9)$$

After quantization, the Poisson brackets become anti-commutators and the set of $\hat{\xi}'^i$ then generate a Clifford algebra.¹⁰

IV. ANGULAR MOMENTUM

The conserved angular momentum of the system follows from Noether’s theorem applied to infinitesimal rotations of the variables

$$\delta_\omega \xi^i = \omega^i_j \xi^j, \quad \omega_{ji} = -\omega_{ij}. \quad (4.1)$$

When the action (2.1) is invariant under infinitesimal rotations, the definition and conservation of the spin angular momentum S^{ij} follow from the equations of motion and the invariance of the action under (4.1),

$$\delta_\omega S = \Delta \left(\frac{\partial^R L}{\partial \xi^i} \delta_\omega \xi^i \right) = \frac{1}{2} \omega_{ij} \Delta S^{ji} = 0, \quad (4.2)$$

where $\partial^R / \partial \xi^i$ denotes the derivative acting from the right, and Δ denotes the difference in values between final and initial times.

The angular momentum from (4.2) is

$$S^{ij} = -\xi^{[i} \pi^{j]} \equiv -\xi^i \pi^j + \xi^j \pi^i. \quad (4.3)$$

Even for non-invariant actions, the angular momentum still generates rotations of the canonical variables,

$$\delta_\omega z = \frac{1}{2} \omega^{ji} \{z, S_{ij}\}. \quad (4.4)$$

V. WEAKLY EQUIVALENT ANGULAR MOMENTA

First, we note the constraints (3.1) are vectors, and we have the Poisson brackets

$$\{S^{ij}, \varphi^k\} = \delta^{ik} \varphi^j - \delta^{jk} \varphi^i = \delta^{k[i} \varphi^{j]} \approx 0. \quad (5.1)$$

It is therefore consistent on the constraint surface to use the Poisson bracket rather than the Dirac bracket with the angular momentum.

From (5.1) we can calculate the Hanson-Regge-Teitelboim primed version of the angular momentum,

$$\begin{aligned} S'^{ij} &= S^{ij} - \{S^{ij}, \varphi^a\} \Delta_{ab}^{-1} \varphi^b \\ &= S^{ij} + i\varphi^{[i} \varphi^{j]}. \end{aligned} \quad (5.2)$$

We find the Poisson brackets

$$\begin{aligned} \{S'^{ij}, \varphi^k\} &= \{S^{ij}, \varphi^k\} + i\{\varphi^{[i} \varphi^{j]}, \varphi^k\} \\ &= \delta^{k[i} \varphi^{j]} + 2i\varphi^{[i} (-i\delta^{j]k}) \\ &= -\delta^{k[i} \varphi^{j]} \approx 0, \end{aligned} \quad (5.3)$$

which show that the primed version of the angular momentum (5.2) is no better than the original canonical version (4.3) in allowing the use of Poisson rather than Dirac brackets.

VI. WEAK AND STRONG LIE ALGEBRAS

Noting that under the primed generators S'^{ij} , the constraints transform (5.3) contravariantly rather than covariantly, as they do under the S^{ij} , we now consider the one-parameter family of weakly equal forms of the angular momentum

$$\Sigma_\epsilon^{ij} = S^{ij} + i\epsilon \varphi^{[i} \varphi^{j]}, \quad (6.1)$$

and examine whether their Lie algebras close strongly or only weakly.

The generators J^{ij} of $\mathfrak{so}(n)$ (and also $\mathfrak{spin}^c(n)$), satisfy the Poisson bracket algebra

$$\{J^{ij}, J^{mn}\} = -\delta^{jm} J^{in} + \delta^{jn} J^{im} - \delta^{in} J^{jm} + \delta^{im} J^{jn}. \quad (6.2)$$

It is straightforward to check that the canonical Noether generators (4.3) satisfy the relations (6.2) strongly.

From the Poisson brackets

$$\begin{aligned} \{S^{ij}, S^{mn}\} &= -\delta^{jm} S^{in} + \delta^{jn} S^{im} - \delta^{in} S^{jm} + \delta^{im} S^{jn}, \\ \{S^{ij}, \varphi^{[m} \varphi^{n]}\} &= -\delta^{jm} \varphi^{[i} \varphi^{n]} + \delta^{jn} \varphi^{[i} \varphi^{m]} \\ &\quad - \delta^{in} \varphi^{[j} \varphi^{m]} + \delta^{im} \varphi^{[j} \varphi^{n]}, \end{aligned} \quad (6.3)$$

$$\begin{aligned} \{\varphi^{[i} \varphi^{j]}, S^{mn}\} &= -\{S^{mn}, \varphi^{[i} \varphi^{j]}\} \\ &= \{S^{ij}, \varphi^{[m} \varphi^{n]}\}, \end{aligned}$$

and

$$\begin{aligned} \{i\varphi^{[i} \varphi^{j]}, i\varphi^{[m} \varphi^{n]}\} &= 2\delta^{jm} i\varphi^{[i} \varphi^{n]} - 2\delta^{jn} i\varphi^{[i} \varphi^{m]} \\ &\quad + 2\delta^{in} i\varphi^{[j} \varphi^{m]} - 2\delta^{im} i\varphi^{[j} \varphi^{n]}, \end{aligned} \quad (6.4)$$

we can determine the Poisson brackets of the quantities (6.1) as

$$\begin{aligned} \{\Sigma^{ij}, \Sigma^{mn}\} &= -\delta^{jm} \left(S^{in} + 2(1-\epsilon)i\epsilon\varphi^{[i} \varphi^{n]} \right) \\ &\quad + \delta^{jn} \left(S^{im} + 2(1-\epsilon)i\epsilon\varphi^{[i} \varphi^{m]} \right) \\ &\quad - \delta^{in} \left(S^{jm} + 2(1-\epsilon)i\epsilon\varphi^{[j} \varphi^{m]} \right) \\ &\quad + \delta^{im} \left(S^{jn} + 2(1-\epsilon)i\epsilon\varphi^{[j} \varphi^{n]} \right), \end{aligned} \quad (6.5)$$

and see that the generators (6.1) will satisfy the Lie algebra relations (6.2) strongly only for

$$\epsilon = 0 \quad \text{or} \quad \epsilon = \frac{1}{2}. \quad (6.6)$$

For other values of ϵ the algebra (6.2) is only satisfied weakly. We observe that the Poisson bracket of the generator Σ^{ij} and the constraint φ^k ,

$$\begin{aligned} \{\Sigma^{ij}, \varphi^k\} &= \{S^{ij}, \varphi^k\} + i\epsilon\{\varphi^{[i} \varphi^{j]}, \varphi^k\} \\ &= \delta^{ki} \varphi^{[j]} + 2i\epsilon\varphi^{[i} (-i\delta^{j]k}) \\ &= (1-2\epsilon)\delta^{ki} \varphi^{[j]} \approx 0, \end{aligned} \quad (6.7)$$

also vanishes strongly if and only if $\epsilon = 1/2$. It's also instructive to note that when $\epsilon = 1/2$,

$$\Sigma^{ij} = -\xi'^{[i} \pi'^{j]} = -\frac{i}{2} \xi'^{[i} \xi'^{j]}, \quad (6.8)$$

which immediately implies $\{\Sigma^{ij}, \varphi^k\} \equiv 0$.

Eq. (6.4) shows that the quantities $-\frac{i}{2}\varphi^{[i} \varphi^{j]}$ also satisfy the Poisson bracket relations (6.2), and with Eq. (6.8), implies that the canonical $\mathfrak{so}(n)$ generators (4.3) are a sum of two independent, commuting $\mathfrak{spin}^c(n)$ generators,

$$S^{ij} = -\frac{i}{2} \xi'^{[i} \xi'^{j]} - \frac{i}{2} \varphi^{[i} \varphi^{j]}. \quad (6.9)$$

VII. QUANTUM MECHANICS

The Schrödinger realization¹⁰ of quantum mechanics for theories described by the action (2.1) with n Grassmann variables consists of replacement of the canonical variables by the operators

$$\begin{aligned} \hat{\xi}_i \psi(\xi) &= \xi_i \psi(\xi), \\ \hat{\pi}_i \psi(\xi) &= i \frac{\partial^L}{\partial \xi_i} \psi(\xi), \end{aligned} \quad (7.1)$$

that act on wave functions $\psi(\xi)$ of all n Grassmann variables. The Schrödinger inner product is given by the integral over the full configuration space,

$$\langle \phi | \psi \rangle = i^{\lfloor \frac{n}{2} \rfloor} \int \phi^* \psi d\xi_1 d\xi_2 \dots d\xi_n = \langle \psi | \phi \rangle^*, \quad (7.2)$$

under which the coordinates variables $\hat{\xi}_i$ are self-adjoint, and the momentum variables $\hat{\pi}_i$ are anti-self-adjoint.

The second-class constraints $\hat{\varphi}_i \approx 0$ are imposed by the generalized Gupta-Bleuler condition that all physical matrix elements of the constraints vanish,

$$\langle \phi_{\text{phys}} | \hat{\varphi}_i | \psi_{\text{phys}} \rangle = 0. \quad (7.3)$$

This condition splits the full state space into two orthogonal subspaces: positive norm physical states and negative norm unphysical (ghost) states. Note that if the opposite sign is taken in (7.2), the physical and ghost states are interchanged. The constraints map physical states into ghost states, and vice-versa.¹⁰

If all dynamical variables have weakly vanishing commutators or anti-commutators with the constraints, then the constraints can be taken to be strongly zero and the Hilbert space restricted to physical states. Replacing the Grassmann operators $\hat{\xi}_i$ by $\hat{\xi}'_i$ and $\hat{\pi}_i$ by $\hat{\pi}'_i$ in all dynamical variables built from them will accomplish this.

A. Two dimensions

In a system with just two anticommuting coordinates, the physical (positive norm) states are spanned by the orthonormal states¹⁰

$$\begin{aligned} |0\rangle &= e^{\frac{i}{2}\xi_1\xi_2} = 1 + \frac{i}{2}\xi_1\xi_2, \\ |1\rangle &= \frac{1}{\sqrt{2}} (\xi_1 + i\xi_2) e^{\frac{i}{2}\xi_1\xi_2} = \frac{1}{\sqrt{2}} (\xi_1 + i\xi_2), \end{aligned} \quad (7.4)$$

while the ghost (negative norm) states are spanned by the orthonormal states

$$\begin{aligned} |\bar{0}\rangle &= e^{-\frac{i}{2}\xi_1\xi_2} = 1 - \frac{i}{2}\xi_1\xi_2, \\ |\bar{1}\rangle &= \frac{1}{\sqrt{2}} (\xi_1 - i\xi_2) e^{-\frac{i}{2}\xi_1\xi_2} = \frac{1}{\sqrt{2}} (\xi_1 - i\xi_2). \end{aligned} \quad (7.5)$$

Denoting the general state of the system by

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = a|0\rangle + b|1\rangle + c|\bar{0}\rangle + d|\bar{1}\rangle, \quad (7.6)$$

we can write the matrix representation of the canonical position and momentum operators on the general state as

$$\begin{aligned} \hat{\xi}_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_1 & -i\sigma_2 \\ -i\sigma_2 & \sigma_1 \end{pmatrix}, & \hat{\xi}_2 &= \frac{1}{\sqrt{2}} \begin{pmatrix} -\sigma_2 & -\sigma_2 \\ \sigma_2 & \sigma_2 \end{pmatrix}, \\ \hat{\pi}_1 &= \frac{i}{2\sqrt{2}} \begin{pmatrix} \sigma_1 & i\sigma_2 \\ i\sigma_2 & \sigma_1 \end{pmatrix}, & \hat{\pi}_2 &= \frac{1}{2\sqrt{2}} \begin{pmatrix} -i\sigma_2 & i\sigma_2 \\ -i\sigma_2 & i\sigma_2 \end{pmatrix}, \end{aligned} \quad (7.7)$$

where σ_1 and σ_2 are standard Pauli matrices. The diagonal pieces map physical states to physical states and unphysical to unphysical, while the off-diagonal pieces map physical to unphysical states or vice-versa. In this matrix representation, the canonical Noether angular momentum is

$$\hat{S}_{12} = \hat{\xi}_2 \hat{\pi}_1 - \hat{\xi}_1 \hat{\pi}_2 = \frac{1}{2} \begin{pmatrix} \mathbb{1} - \sigma_3 & 0 \\ 0 & -\mathbb{1} + \sigma_3 \end{pmatrix}. \quad (7.8)$$

Including the term containing the second-class constraints and with $\epsilon = 1/2$, we find

$$\hat{\Sigma}_{12} = \hat{S}_{12} + i\hat{\phi}_1 \hat{\phi}_2 = \frac{1}{2} \begin{pmatrix} -\sigma_3 & 0 \\ 0 & +\sigma_3 \end{pmatrix}. \quad (7.9)$$

B. Three dimensions

Adding one more anticommuting coordinate to the two-dimensional system creates two superselection sectors of physical states and two of ghost states. These are superselection sectors for the Hanson-Regge-Teitelboim $\hat{\xi}'_i$ and $\hat{\pi}'_i$ operators, and are denoted by unprimed and primed states.¹⁰

The positive norm physical states are spanned by the orthonormal basis

$$\begin{aligned} |0\rangle &= \frac{1}{\sqrt{2}} \left(1 + \frac{i}{2} \xi_1 \xi_2 \right) \left(1 + \frac{\xi_3}{\sqrt{2}} \right), \\ |1\rangle &= \frac{1}{\sqrt{8}} (\xi_1 + i\xi_2) \left(1 + \frac{\xi_3}{\sqrt{2}} \right), \\ |0'\rangle &= \frac{1}{\sqrt{2}} \left(1 - \frac{i}{2} \xi_1 \xi_2 \right) \left(1 - \frac{\xi_3}{\sqrt{2}} \right), \\ |1'\rangle &= \frac{1}{\sqrt{8}} (\xi_1 - i\xi_2) \left(1 - \frac{\xi_3}{\sqrt{2}} \right), \end{aligned} \quad (7.10)$$

while the negative norm ghost states are denoted by barred states and spanned by the orthogonal anti-normal

basis

$$\begin{aligned} |\bar{0}\rangle &= \frac{1}{\sqrt{2}} \left(1 - \frac{i}{2} \xi_1 \xi_2 \right) \left(1 + \frac{\xi_3}{\sqrt{2}} \right), \\ |\bar{1}\rangle &= \frac{1}{\sqrt{8}} (\xi_1 - i\xi_2) \left(1 + \frac{\xi_3}{\sqrt{2}} \right), \\ |\bar{0}'\rangle &= \frac{1}{\sqrt{2}} \left(1 + \frac{i}{2} \xi_1 \xi_2 \right) \left(1 - \frac{\xi_3}{\sqrt{2}} \right), \\ |\bar{1}'\rangle &= \frac{1}{\sqrt{8}} (\xi_1 + i\xi_2) \left(1 - \frac{\xi_3}{\sqrt{2}} \right). \end{aligned} \quad (7.11)$$

In this full basis, where the general state of the system is denoted by

$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{pmatrix} = a|0\rangle + b|1\rangle + c|\bar{0}\rangle + d|\bar{1}\rangle + e|0'\rangle + f|1'\rangle + g|\bar{0}'\rangle + h|\bar{1}'\rangle, \quad (7.12)$$

the coordinate and momentum operators are represented, using the standard Pauli matrices, by the 8×8 matrices

$$\hat{\xi}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_1 & -i\sigma_2 & 0 \\ -i\sigma_2 & \sigma_1 & 0 \\ 0 & 0 & \sigma_1 & -i\sigma_2 \\ & & -i\sigma_2 & \sigma_1 \end{pmatrix}, \quad (7.13)$$

$$\hat{\xi}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sigma_2 & -\sigma_2 & 0 \\ \sigma_2 & \sigma_2 & 0 \\ 0 & 0 & \sigma_2 & \sigma_2 \\ & & -\sigma_2 & -\sigma_2 \end{pmatrix}, \quad (7.14)$$

$$\hat{\xi}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_3 & 0 & 0 & \sigma_3 \\ 0 & \sigma_3 & \sigma_3 & 0 \\ 0 & -\sigma_3 & -\sigma_3 & 0 \\ -\sigma_3 & 0 & 0 & -\sigma_3 \end{pmatrix}, \quad (7.15)$$

$$\hat{\pi}_1 = \frac{1}{2\sqrt{2}} \begin{pmatrix} i\sigma_1 & -\sigma_2 & 0 \\ -\sigma_2 & i\sigma_1 & 0 \\ 0 & 0 & i\sigma_1 & -\sigma_2 \\ & & -\sigma_2 & i\sigma_1 \end{pmatrix}, \quad (7.16)$$

$$\hat{\pi}_2 = \frac{1}{2\sqrt{2}} \begin{pmatrix} -i\sigma_2 & i\sigma_2 & 0 \\ -i\sigma_2 & i\sigma_2 & 0 \\ 0 & 0 & i\sigma_2 & -i\sigma_2 \\ & & i\sigma_2 & -i\sigma_2 \end{pmatrix}, \quad (7.17)$$

$$\hat{\pi}_3 = \frac{1}{2\sqrt{2}} \begin{pmatrix} i\sigma_3 & 0 & 0 & -i\sigma_3 \\ 0 & i\sigma_3 & -i\sigma_3 & 0 \\ 0 & i\sigma_3 & -i\sigma_3 & 0 \\ i\sigma_3 & 0 & 0 & -i\sigma_3 \end{pmatrix}, \quad (7.18)$$

which allow for a more standard and reliable computation tool than Grassmann operators.

It is straightforward to check that the representation of the canonical angular momentum algebra of the $\hat{S}_{ij} = -\hat{\xi}_{[i}\hat{\pi}_{j]}$, splits the four physical states into the reducible $\mathfrak{so}(3)$ representation $\mathbf{1} \oplus \mathbf{3}$, and similarly splits the four ghost states. The splitting mixes the superselection sectors; the $\hat{S}^2 = 2$, $\hat{S}_{12} = 0$ state is

$$|S = 1, S_z = 0\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |0'\rangle), \quad (7.19)$$

while the singlet state is

$$|S = 0, S_z = 0\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |0'\rangle), \quad (7.20)$$

and similarly for the ghost sectors.

Once the constraint modification is made in Eq. (6.1) with $\epsilon = 1/2$, the $\hat{\Sigma}_{ij}$ exactly commute with all second-class constraints, making the constraints scalars under rotation, and the physical and ghost spaces each split up into superselection sectors as two irreducible $\mathfrak{spin}^c(3)$ representations, $\mathbf{2} \oplus \mathbf{2}$. This is implied by the relation Eq. (6.8), and the fact that the $\hat{\xi}'_i$ do not mix superselection sectors¹⁰.

C. General case

With n anticommuting coordinates, there are 2^n total states, half of which are physical and half of which are ghost. A ghost state differs from a physical state by a reflection in one of the ξ_i in its wave function, which changes the sign of its norm. A rotation cannot change just one sign, so $SO(n)$ rotations will preserve the sign of the norm.

The 2^{n-1} physical states fall into $N_s = 2^{\lfloor \frac{n-1}{2} \rfloor}$ superselection sectors of the $\hat{\xi}'_i$ operators, and the same is true of the ghost states. Each superselection sector, physical or ghost, has dimension $2^{\lfloor \frac{n}{2} \rfloor}$. For odd n , these superselection sectors are irreducible representations of $\mathfrak{spin}^c(n)$. For even n , because the generators (6.8) have even Grassmann parity, these superselection sectors are a sum of two irreducible spinor representations of $\mathfrak{spin}^c(n)$ having opposite Grassmann parity. However, products of odd numbers of the $\hat{\xi}'_i$ operators will mix them. Additional Grassmann degrees of freedom added to the theory may prevent these irreducible spinor representations from being quantum mechanical superselection sectors. From the construction of states as functions of the Grassmann variables ξ_i given in Ref. 10, we can see that under the rotations generated by the \hat{S}_{ij} , the superselection sectors will mix, giving in general a sum of irreducible scalar, vector, and antisymmetric tensor representations of $SO(n)$ with at most $\lfloor \frac{n}{2} \rfloor$ indices. Explicit constructions of these states is given in Appendix B.

For dimensions smaller than three there is just one superselection sector. For $n = 1$ there is a single scalar state and for $n = 2$ there is one superselection sector consisting of a scalar and a half-vector, while for $n = 5$, for example, the four physical $\mathfrak{spin}^c(5)$ spinor $\mathbf{4}$ superselection sectors will mix under $\mathfrak{so}(5)$ into the reducible representation $\mathbf{1} \oplus \mathbf{5} \oplus \mathbf{10}$, and for $n = 6$, the four physical $\mathfrak{spin}^c(6)$ reducible spinor $\mathbf{8}$ superselection sectors will mix under $\mathfrak{so}(6)$ into the reducible representation $\mathbf{1} \oplus \mathbf{6} \oplus \mathbf{15} \oplus \mathbf{10}$. The same is true for the ghost sectors.

In constructing isospin operators to describe relativistic particles interacting with a non-Abelian gauge field, Balachandran *et al.* examined constructions of isospin for any representation of an arbitrary gauge group in terms of both commuting and anticommuting variables.⁷ Two of these constructions are relevant to our construction of spin from the fundamental representation of $\mathfrak{so}(n)$. When the spin is constructed—*e.g.*, in a reduced phase space quantization—from an abstract n -dimensional Clifford algebra as in Eq. (6.8), a single $\mathfrak{spin}^c(n)$ representation results.¹⁰ Because the Schrödinger realization constructs the Clifford algebra generators $\hat{\xi}'^i$ in Eq. (3.8) as operators on wave functions of all n of the Grassmann coordinates, repeated $\mathfrak{spin}^c(n)$ representations occur as identical superselection sectors.¹⁰ A consistent quantization can be obtained by restricting to just one of these superselection sectors, although restricting to more than one superselection sector is also consistent, and in the language of Kähler fermions has been proposed as the origin of fermion families.²⁰ When the angular momentum is constructed from two n -dimensional Clifford algebras, as in the decomposition of the Noether angular momentum in Eq. (6.9), one obtains a $\mathfrak{spin}^c(2n)$ representation⁷ that reduces under $\mathfrak{so}(n)$. In Appendix B we give a tensor construction for the reduction under $\mathfrak{so}(n)$, different from the method given in Ref. 7.

VIII. CONCLUSIONS

The unique generators of the family (6.1) that have vanishing Poisson brackets with the constraints become the generators of $\mathfrak{spin}^c(n)$ upon quantization and the physical state space splits up into superselection sectors that are identical $\mathfrak{spin}^c(n)$ representations. A consistent quantization can be obtained by restricting physical states to one or more of these superselection sectors.

The unique generators of that same family (6.1) under which the constraints transform as vectors become the generators of $\mathfrak{so}(n)$ upon quantization and the physical $\mathfrak{spin}^c(n)$ superselection sectors mix to become a sum of antisymmetric tensor $\mathfrak{so}(n)$ irreducible representations. One cannot generally make a useful quantization using the Noether form of the angular momentum because the Grassmann position and momentum operators must be modified by the Hanson-Regge-Teitelboim procedure so that they do not mix physical and ghost states; mixing of

ghost and physical states would ruin the consistent removal of the ghost states from the theory. With the modified ξ' and π' operators, the natural states are in $\text{spin}^c(n)$ superselection sectors, which do not mix under the ξ' , unlike the $\mathfrak{so}(n)$ irreps. To use the Noether form of the angular momentum, the full 2^{n-1} -dimensional $\text{spin}^c(2n)$ physical state space, meaning all the $\mathfrak{so}(n)$ scalar, vector, and antisymmetric tensor states, would have to be kept. Such a quantization does not seem to have relevance to physics.

We also found, in the quantum version of Eq. (6.9), that the canonical $\mathfrak{so}(n)$ generators (4.3) are the sum of two independent commuting $\text{spin}^c(n)$ generators:

$$\hat{S}^{ij} = -\hat{\xi}^{[i} \hat{\pi}^{j]} = -\frac{i}{2} \hat{\xi}'^{[i} \hat{\xi}'^{j]} - \frac{i}{2} \hat{\phi}^{[i} \hat{\phi}^{j]}. \quad (8.1)$$

After setting the constraints $\hat{\phi}_i$ strongly to zero and restricting the Hilbert space to the physical Hilbert space, the Noether $\mathfrak{so}(n)$ angular momentum generators (8.1) become the $\text{spin}^c(n)$ generators, Eq. (8.2).

The selection of (6.1) with $\epsilon = 1/2$,

$$\hat{\Sigma}^{ij} = -\hat{\xi}^{[i} \hat{\pi}^{j]} + \frac{i}{2} \hat{\phi}^{[i} \hat{\phi}^{j]} = -\frac{i}{2} \hat{\xi}'^{[i} \hat{\xi}'^{j]}, \quad (8.2)$$

as the correct form of the spin angular momentum depends on only two things. One is that the second-class constraints $\varphi^i \approx 0$ transform as vectors under the canonical Noether angular momentum (4.3) and the other is the specific constant value of the rotationally invariant Poisson brackets of the constraints given in Eq. (3.2). Our results thus apply to somewhat more general actions than the simple action (2.1) that we have considered here.

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DATA AVAILABILITY STATEMENT

Data sharing not applicable – no new data generated.

Appendix A: Operator representations in three dimensions

We gather here for reference the matrix representations for various quantities in the basis (7.12), expressed as tensor products of 2×2 matrices. The canonical coordinates are given by

$$\begin{aligned} \sqrt{2} \hat{\xi}_1 &= \mathbb{1} \otimes \mathbb{1} \otimes \sigma_1 - i \mathbb{1} \otimes \sigma_1 \otimes \sigma_2, \\ \sqrt{2} \hat{\xi}_2 &= -\sigma_3 \otimes \sigma_3 \otimes \sigma_2 - i \sigma_3 \otimes \sigma_2 \otimes \sigma_2, \\ \sqrt{2} \hat{\xi}_3 &= \sigma_3 \otimes \mathbb{1} \otimes \sigma_3 + i \sigma_2 \otimes \sigma_1 \otimes \sigma_3, \end{aligned}$$

and the canonical momenta are given by

$$\begin{aligned} 2\sqrt{2} \hat{\pi}_1 &= i \mathbb{1} \otimes \mathbb{1} \otimes \sigma_1 - \mathbb{1} \otimes \sigma_1 \otimes \sigma_2, \\ 2\sqrt{2} \hat{\pi}_2 &= -i \sigma_3 \otimes \sigma_3 \otimes \sigma_2 - \sigma_3 \otimes \sigma_2 \otimes \sigma_2, \\ 2\sqrt{2} \hat{\pi}_3 &= i \sigma_3 \otimes \mathbb{1} \otimes \sigma_3 + \sigma_2 \otimes \sigma_1 \otimes \sigma_3. \end{aligned}$$

The Hanson-Regge-Teitelboim primed operators $\hat{\xi}'_i$ are given by

$$\begin{aligned} \sqrt{2} \hat{\xi}'_1 &= \mathbb{1} \otimes \mathbb{1} \otimes \sigma_1, \\ \sqrt{2} \hat{\xi}'_2 &= -\sigma_3 \otimes \sigma_3 \otimes \sigma_2, \\ \sqrt{2} \hat{\xi}'_3 &= \sigma_3 \otimes \mathbb{1} \otimes \sigma_3, \end{aligned}$$

and the constraint operators $\hat{\phi}_i$ are given by

$$\begin{aligned} \sqrt{2} \hat{\phi}_1 &= -\mathbb{1} \otimes \sigma_1 \otimes \sigma_2, \\ \sqrt{2} \hat{\phi}_2 &= -\sigma_3 \otimes \sigma_2 \otimes \sigma_2, \\ \sqrt{2} \hat{\phi}_3 &= \sigma_2 \otimes \sigma_1 \otimes \sigma_3. \end{aligned}$$

The canonical angular momentum components are

$$\begin{aligned} \hat{S}_{23} &= \frac{1}{2} (-\mathbb{1} \otimes \sigma_3 \otimes \sigma_1 + \sigma_1 \otimes \sigma_3 \otimes \sigma_1), \\ \hat{S}_{31} &= \frac{1}{2} (\sigma_3 \otimes \mathbb{1} \otimes \sigma_2 + \sigma_2 \otimes \mathbb{1} \otimes \sigma_1), \\ \hat{S}_{12} &= \frac{1}{2} (\sigma_3 \otimes \sigma_3 \otimes \mathbb{1} - \sigma_3 \otimes \sigma_3 \otimes \sigma_3), \end{aligned}$$

and the spin generators are

$$\begin{aligned} \hat{\Sigma}_{23} &= -\frac{1}{2} \mathbb{1} \otimes \sigma_3 \otimes \sigma_1, \\ \hat{\Sigma}_{31} &= \frac{1}{2} \sigma_3 \otimes \mathbb{1} \otimes \sigma_2, \\ \hat{\Sigma}_{12} &= -\frac{1}{2} \sigma_3 \otimes \sigma_3 \otimes \sigma_3. \end{aligned}$$

The bilinear constraint operators $-\frac{i}{2} \hat{\phi}_{[i} \hat{\phi}_{j]}$ are

$$\begin{aligned} -i \hat{\phi}_2 \hat{\phi}_3 &= \frac{1}{2} \sigma_1 \otimes \sigma_3 \otimes \sigma_1, \\ -i \hat{\phi}_3 \hat{\phi}_1 &= \frac{1}{2} \sigma_2 \otimes \mathbb{1} \otimes \sigma_1, \\ -i \hat{\phi}_1 \hat{\phi}_2 &= \frac{1}{2} \sigma_3 \otimes \sigma_3 \otimes \mathbb{1}. \end{aligned}$$

Appendix B: Orthonormal basis of $\mathfrak{so}(n)$ irreps

Here we exhibit an orthonormal basis for the physical states and ghost states as antisymmetric irreducible representations of $\mathfrak{so}(n)$ built from the set of 2^n possible components of a function on the n -dimensional configuration space. To simplify the expressions, we use the scaled variables, $\zeta^\mu = \xi^\mu / \sqrt{2}$, as suggested by the form of

the physical states¹⁰ as given, for example, by Eq. (7.10) for $n = 3$.

A basis of the full physical plus ghost state space is the set of all unique monomials

$$\zeta^{\mu_1} \zeta^{\mu_2} \dots \zeta^{\mu_k}, \quad (\text{B1})$$

where k runs from 0 to n . When $k = 0$ the monomial is defined to be 1. This basis is neither orthogonal nor normalized, nor are the states purely physical or ghost, though they are either purely real or purely imaginary.

Monomials with more than $\lfloor \frac{n}{2} \rfloor$ indices may be labeled with $\lfloor \frac{n}{2} \rfloor$ indices or fewer by using the $\mathfrak{so}(n)$ invariant tensor $\epsilon^{\mu_1 \mu_2 \dots \mu_n}$ ($\epsilon^{12 \dots n} = \epsilon_{12 \dots n} = 1$) to define

$$*(\zeta^{\mu_1} \dots \zeta^{\mu_k}) = \frac{1}{(n-k)!} \epsilon^{\mu_1 \dots \mu_k \rho_1 \dots \rho_{n-k}} \zeta^{\rho_1} \zeta^{\rho_2} \dots \zeta^{\rho_{n-k}}. \quad (\text{B2})$$

Orthonormal bases for the physical and ghost state spaces are the set of unique linear combinations of monomials in Eqs. (B1) and (B2) with at most $\lfloor \frac{n}{2} \rfloor$ indices given by

$$|\mu_1, \mu_2, \dots, \mu_k, \pm\rangle = \alpha_k [\zeta^{\mu_1} \dots \zeta^{\mu_k} \pm \beta_k *(\zeta^{\mu_1} \dots \zeta^{\mu_k})], \quad (\text{B3})$$

where

$$\beta_k = i^{\lfloor \frac{n}{2} \rfloor} (-1)^{\lfloor \frac{k}{2} \rfloor}, \quad (\text{B4})$$

and

$$\alpha_k = 2^{\frac{n-2}{4}}. \quad (\text{B5})$$

With the positive sign, the set of unique states of form (B3) is an orthonormal basis for the physical states. With the negative sign, that set is an (anti)-orthonormal basis of the ghost states. For $n = 2$, these states are given in Eqs. (7.4) and (7.5) respectively, while for $n = 1$, these two states are

$$|\pm\rangle = \frac{1}{\sqrt{2}} (1 \pm \zeta). \quad (\text{B6})$$

For even $n = 2m$, the physical (or ghost) states with m indices satisfy duality relations

$$|\mu_1, \dots, \mu_m, \pm\rangle = \pm \frac{\beta_m}{m!} \epsilon^{\mu_1 \dots \mu_m \rho_1 \dots \rho_m} |\rho_1, \dots, \rho_m, \pm\rangle. \quad (\text{B7})$$

The set of unique states (B3), those with $\mu_1 < \mu_2 < \dots < \mu_k$, for a given k forms an irreducible representation of $\mathfrak{so}(n)$, which can be seen by acting the Noether angular momentum generators (4.3) on these states, which either produce zero or another state of the same form.

It is immediate that the unique states of form (B3) with different k are orthogonal under the inner product (7.2) and that the set of all unique states of form (B3) is linearly independent.

The number unique states of form (B3) is 2^n , half of them physical and half of them ghost. Thus we obtain the result that they must form a basis for both the physical and ghost states.

A useful identity for verifying Eq. (B4) is

$$\lfloor \frac{n}{2} \rfloor + \lfloor \frac{k}{2} \rfloor + \lfloor \frac{n-k}{2} \rfloor + (n-k)k \equiv 1 \pmod{2}, \quad (\text{B8})$$

for integers $0 \leq k \leq n$.

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