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The Canonical Structure of the Manifestly Supersymmetric String*

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Abstract

Both the Green-Schwarz and Siegel strings are presented in canonical form. Both systems are shown to describe the same number of physical degrees of freedom. The apparent extra symmetries of the Siegel string are not true symmetries but are combinations of second-class constraints. A formal quantization procedure is outlined and the problems of quantization are discussed.

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1. Introduction

There is an elegant formalism, known as superspace, in which the global spacetime supersymmetry of a theory is manifest. The Neveu-Schwarz-Ramond (NSR) form of the superstring has spacetime supersymmetry in addition to the manifest superconformal symmetry on the worldsheet. Consequently, although any string field theory built from the NSR form will have spacetime supersymmetry, its supersymmetry will not be manifest. To construct a string field theory with manifest spacetime supersymmetry, one needs a first-quantized theory which has superspace coordinates as the fundamental fields on the two-dimensional worldsheet. A classical Lagrangian having global supersymmetry and a local fermionic worldsheet symmetry has been constructed by Green and Schwarz [1]. This action can be quantized in light-cone gauge and is the same as the NSR string in that gauge [2].

While the Green-Schwarz action is free in light-cone gauge, in general gauges it is an interacting two-dimensional theory. This, in part, is why the covariant quantization of the theory is difficult. Because the action is free in light-cone gauge, one might expect that the action could be quantized straightforwardly in a covariant gauge, and that the covariantly quantized action would have a simple form. A formal quantization of the Green-Schwarz action shows that, on the contrary, there are difficulties even in passing from the Hamiltonian to a Lagrangian description.

The possibility should be kept in mind, therefore, that the Green-Schwarz action is not the appropriate action for quantization and may need to be amended in whole or in part. One such possible action has been proposed by Siegel [3]. The motivation for the Siegel string is to incorporate the smallest algebra containing the generators of reparametrizations and of local fermionic transformations as the invariance algebra of the string. Unfortunately, the phase space constraints of the Siegel string are equivalent to those of the Green-Schwarz string in generic regions of phase space, as I will show. It seems then that one is forced to use the Green-Schwarz action, or some other action not yet formulated, in order to obtain a covariantly first-quantized string with manifest spacetime supersymmetry.

Even though there are difficulties in constructing a covariantly quantized string with manifest spacetime supersymmetry, such a construction is a necessary ingredient of the corresponding superstring field theory. The dynamical variables (including ghosts and auxiliary fields) of the first-quantized string become the coordinates on which the string field depends. And, as Siegel has explained [3], the constraints of the first-quantized theory will determine the free dynamics of the string field. The free field Lagrangian is $\Psi\hat{H}\Psi$, with $\hat{H}$ the first-quantized Hamiltonian (operator) and Ψ the string field.

A manifestly supersymmetric string field theory may make the possibilities for, or necessity of, supersymmetry breaking more apparent and may put some constraints on the allowable vacua.

The ghost structure of the theory is likely to yield as much insight into string physics as it has for the bosonic and NSR strings [4, 5].

The first step on the road to string field theory is the construction of a first-quantized theory. The classical string actions are singular systems (in the sense of Dirac [6]) and their quantizations begin with the analysis of their constraints. The analysis and quantization of constrained systems were first investigated by Dirac [6] who considered canonical quantization only. A path integral quantization eliminates some of the problems of ordering in the canonical formalism and also yields the ghost parts of the action [7]. The path integral quantization of constrained Hamiltonian systems was fully worked out by Fradkin and his school [8, 9, 10]. It is through the Fradkin formalism that one may formally quantize the Green-Schwarz action. The difficulties of, and remaining technical steps in this quantization are discussed in section 4. The Siegel string will be shown to be essentially equivalent to the Green-Schwarz string, and thus is shown to have no particular advantage over the Green-Schwarz string in the canonical formalism.

2. The Green-Schwarz String

The quantities which appear in the Green-Schwarz action are the two-dimensional metric $g_{\alpha\beta}$, a ten-dimensional position X^μ, and two anticommuting Majorana-Weyl spinors $\theta^A, A = 1,2$. Both X^μ and θ^A transform as scalars under worldsheet reparametrizations. When light-cone gauge is fixed, the remaining pieces of θ^A together transform as a spinor on the world-sheet, with the label A becoming a two-dimensional spinor index. The covariant classical Green-Schwarz action is [1]

$$
\begin{aligned}
I_{GS} = \frac{1}{\pi} \int d^2\sigma \sqrt{-g} \Big\{ & -\frac{1}{2} g^{\alpha\beta}\, \Pi_\alpha \cdot \Pi_\beta \\
& - i\epsilon^{\alpha\beta} \Pi_\alpha \cdot [\bar\theta^1 \gamma\, \partial_\beta\, \theta^1 - \bar\theta^2 \gamma\, \partial_\beta\, \theta^2] \\
& - \epsilon^{\alpha\beta} \bar\theta^1 \gamma\, \partial_\alpha \theta^1 \bar\theta^2 \gamma\, \partial_\beta \theta^2 \Big\}
\end{aligned}
\tag{2.1}
$$

where

$$
\Pi^\mu_\alpha := \partial_\alpha X^\mu - i \sum_A \bar\theta^A \gamma^\mu\, \partial_\alpha \theta^A. \tag{2.2}
$$

Just as the Brink-Schwarz superparticle has primary constraints [11] there are similar primary constraints for the Green-Schwarz action [12, 13]:

$$
P_g \approx 0,
$$

$$
\bar D^A := \zeta^A + i\bar\theta^A\, \gamma_\mu (P^\mu + (-)^A\, X^{\mu\prime} - (-)^A i \bar\theta^A\, \gamma^\mu \theta^{A\prime}) \approx 0. \tag{2.3}
$$

Here P_g is canonically conjugate to g, prime denotes derivative with respect to σ, and ζ^A is the conjugate momentum to θ^A satisfying the (symmetric) canonical Poisson bracket

$$
\Big\{ \zeta^A(\sigma)\, ,\, \theta^B(\sigma') \Big\} = h^A\, \delta^{AB}\, \delta(\sigma - \sigma') \tag{2.4}
$$

where h^A is the chirality projector for the spinor $\theta^A : h^A \theta^A = \theta^A$. The second relation in (2.3) defines the momentum, $\zeta^A := \partial_R L / \partial_R \dot\theta^A$, which is the right derivative of the

Lagrangian with respect to the velocity of θ^A. Already one can see that there is something about the θ^A which is peculiar for scalar fields. The momentum conjugate to the field θ^A is constrained to be a function of fields other than the velocity of θ^A. This is more the behavior of a two-dimensional spinor field.

Two more constraints need to be imposed in order to conserve the first constraint of (2.3). These two constraints are the vanishing of the (traceless) stress tensor $T_{\alpha\beta}$:

$$0 \approx T_{\alpha\beta} := \dot{P}_{g\alpha\beta} = \sqrt{-g}\Big(\tfrac{1}{2} g_{\alpha\beta} g^{\gamma\delta} - \delta^\gamma_\alpha \delta^\delta_\beta\Big)\Pi_\gamma \cdot \Pi_\delta. \tag{2.5}$$

The Π^μ_α in eq. (2.5) are the expressions (2.2) expressed in canonical variables. In conformal coordinates the constraints (2.5) are particularly simple. Writing

$$\Pi^\mu_A := \Pi^\mu_0 + (-)^A \Pi^\mu_1 = (P^\mu + (-)^A X^{\mu\prime}) - 2i(-)^A \bar{\theta}^A \gamma^\mu \theta^{A\prime}, \tag{2.6}$$

one obtains $\Pi^2_A = 0$ for the constraints (2.5).

One may check that there are no more constraints which need to be imposed in addition to (2.3) and (2.5). Upon examining the constraints one finds that some of them are second-class. Specifically, one computes the Poisson bracket of the fermionic constraints with themselves and finds

$$\Big\{\bar{D}^{Aa}(\sigma),\, \bar{D}^{Bb}(\sigma')\Big\} = 2i\delta^{AB}\,\delta(\sigma - \sigma')(\gamma^0 h_A \gamma^\mu)^{ab}\Pi_{A\mu}(\sigma). \tag{2.7}$$

(Roman minuscules are ten-dimensional spinor indices.) Because Π^μ_A is null (from (2.5)), exactly half of the components of $\bar{D}^A \approx 0$ are second-class and half are first-class. The null vectors Π^μ_A are useful for separating these constraints covariantly [12, 13]. One may check that the first-class constraints are separated from the second-class constraints by

$$F^A := \bar{D}^A \gamma_\mu \Pi^\mu_A \approx 0\,, \quad A = 1, 2, \tag{2.8a}$$

$$G^A := \bar{D}^A \gamma_\mu \Pi^\mu_{\bar{A}} \approx 0\,, \quad A = 1, 2. \tag{2.8b}$$

The bar over the label A in (2.8b) is to denote the other value A may take; that is $\bar{1} = 2, \bar{2} = 1$. The $F^A(\sigma)$ are first-class and the $G^A(\sigma)$ are second-class. The choice of second-class constraints is not unique. One could choose *any* null vector V_A with $V_A \cdot \Pi_A \not\approx 0$ and define a new second-class constraint $\tilde{G}^A = \bar{D}^A \gamma_\mu V^\mu_A \approx 0$. One is forced into choosing the null vector V_A to be $\Pi^\mu_{\bar{A}}$ because the choice of any null vector not given by the theory itself would break manifest covariance. It turns out [12] that the generator of reparametrizations is not purely Π^2_A but is $\frac{1}{2}\Pi^2_A + 2(-)^A \bar{D}^A \theta^{A\prime}$. The full set of constraints for the Green-Schwarz string is

$$
\begin{aligned}
&(P_g)_{\alpha\beta} \approx 0, \\
&T_A := \tfrac{1}{2}\Pi^2_A + 2(-)^A \bar{D}^A \theta^{A\prime} \approx 0 \ , \quad A = 1,2, \\
&F_A := \bar{D}^A \gamma_\mu \Pi^\mu_A \approx 0 \qquad\qquad\quad\ , \quad A = 1,2, \\
&G_A := \bar{D}^A \gamma_\mu \Pi^\mu_{\bar{A}} \approx 0 \qquad\qquad\quad\ , \quad A = 1,2.
\end{aligned}
\tag{2.9}
$$

The first two of these constraints generate Weyl rescalings of the metric and two-dimensional reparametrizations of the world-sheet. The constraints $F^A \approx 0$ generate the local fermionic κ-symmetry and together with the first two constraints are the first-class constraints of the theory. The conditions $G^A \approx 0$ are second-class and must be treated differently from the rest of the constraints in (2.9). Before analyzing the Siegel string, let us count the degrees of freedom of the Green-Schwarz string. Ignoring the metric degrees of freedom one has twenty bosonic and sixty-four fermionic phase space variables on which there are two bosonic first-class, sixteen fermionic first-class and sixteen fermionic second-class constraints. The first-class constraints each fix out two degrees of freedom while the second-class constraints each fix out a single degree of freedom. (This counting works irrespective of the choice of gauge fixing conditions.) Thus there are sixteen bosonic and sixteen fermionic physical phase space degrees of freedom at each point along the string.

3. The Siegel String

The Siegel string is motivated by a desire not to have the whole of $\bar{D}^A$ fixed to zero. This is reasonable because the troublesome second-class constraints are contained in $\bar{D}^A$. If only three quarters of the components of $\bar{D}^A$ were constrained to vanish, then there would be the correct number of degrees of freedom, and the constraints would all be first-class. Further, the quantity $\bar{D}^A$ is part of the fermion emission vertex and its vanishing seems to be at odds with its use in the fermion emission vertex [3]. There is no problem with this because the full emission vertex does not vanish by the constraints.

The two-dimensional fields used to construct the Siegel string are, in addition to the fields of the Green-Schwarz string, a ten-dimensional and worldsheet vector P^μ_α, two ten-dimensional Majorana-Weyl spinors D^A_α which are also vectors on the worldsheet, and three auxiliary fields $\psi^{A\ \beta}_{\alpha}$, $\chi^{A\alpha}_{\ \ \beta\mu\nu\rho}$, and $\phi^{A\alpha\beta}_{\ \ \gamma\mu}$. The ψ^A are two Majorana-Weyl spinors in ten dimensions while χ^A and ϕ^A are an antisymmetric tensor and vector respectively. The full Weyl invariant classical action is [3]

$$
\begin{aligned}
I_S = \int d^2\sigma\sqrt{-g}\Big\{ & g^{\alpha\beta}\Big(\tfrac{1}{2}P_\alpha\cdot P_\beta + P_\alpha\cdot(\partial_\beta X - i\sum_A \bar{\theta}^A\gamma\partial_\beta\theta^A)\Big) \\
& + i\epsilon^{\alpha\beta}\,\partial_\alpha X\cdot(\bar{\theta}^2\gamma\partial_\beta\theta^2 - \bar{\theta}^1\gamma\,\partial_\beta\theta^1) + \epsilon^{\alpha\beta}\bar{\theta}^1\gamma\partial_\alpha\theta^1\cdot\bar{\theta}^2\gamma\partial_\beta\theta^2 \\
& + i\sum_A \bar{D}^A_\alpha\partial_\beta\theta^A\Pi^{A\alpha\beta} + \sum_A \Pi^{A\eta\alpha}\Pi^{A\delta}_{\ \ \beta}\,\psi^{A\ \beta}_{\ \alpha}\not{P}_\eta D^A_\delta \\
& + \sum_A \chi^{A\alpha}_{\ \ \ \beta\,\mu\nu\rho}\,\Pi^{A\delta}_{\ \ \alpha}\Pi^{A\gamma\beta}\bar{D}^A_\delta\gamma^{\mu\nu\rho}D^A_\gamma \\
& + \sum_A \phi^{A\alpha\beta}_{\ \ \ \gamma\mu}\,\Pi^{A\delta}_{\ \ \alpha}\Pi^{A\epsilon}_{\ \ \beta}\Pi^{A\gamma\eta}\bar{D}^A_\delta\gamma^\mu\partial_\eta D^A_\epsilon\Big\}.
\end{aligned}
\tag{3.1}
$$

The quantities $\Pi^{A\alpha\beta}$ are projection operators

$$
\Pi^{A\alpha\beta} := \tfrac{1}{2}(g^{\alpha\beta} + (-)^A\epsilon^{\alpha\beta}) \tag{3.2}
$$

and are not related to the expression (2.2) even though they are, unfortunately, denoted by the same symbol. The Dirac analysis proceeds analogously to the Siegel superparticle [11]. The canonical phase space has the conjugate pairs (X, P_X), (P, P_P), (θ, ζ), (D, B), (g, P_g), (ψ, P_ψ), (χ, P_χ) and (ϕ, P_ϕ) as canonical variables.

The definition of momenta leads directly to the primary constraints.

$$
\begin{aligned}
\phi_1 &:= \zeta^A + i\bar{\theta}^A\gamma_\mu(P_\alpha{}^\mu g^{\alpha 0} + (-)^A X^{\mu\prime} - i(-)^A\bar{\theta}^{\bar{A}}\gamma^\mu\theta^{\bar{A}\prime}) - i\bar{D}^A_\beta\Pi^{A\beta 0} \approx 0, \\
\phi_2 &:= P_X^\mu - P_\alpha{}^\mu g^{\alpha 0} - i\sum_B(-)^B\bar{\theta}^B\gamma^\mu\theta^{B\prime} \approx 0, \\
\phi_3 &:= B^{A\eta} - \phi^{A\alpha\beta}_{\gamma\mu}\bar{D}^A_\delta\gamma^\mu\Pi^{A\delta}_\alpha\Pi^{A\eta}{}_\beta\Pi^{A\gamma 0} \approx 0, \\
\phi_4 &:= P_P{}^{\alpha\mu} \approx 0, \\
\phi_5 &:= P_\psi{}^{A\alpha}{}_\beta \approx 0, \\
\phi_6 &:= P_\chi{}^{A}{}_\alpha{}^{\beta\mu\nu\rho} \approx 0, \\
\phi_7 &:= P_{g\alpha\beta} \approx 0, \\
\phi_8 &:= P_\phi{}^{A}{}_{\alpha\beta}{}^{\gamma\mu} \approx 0.
\end{aligned}
\tag{3.3}
$$

In these primary constraints the variables are all mixed up in a complicated fashion, but it is possible to see that ϕ_1, ϕ_2, ϕ_4 and pieces of ϕ_3 are second-class. Whether or not the rest are first-class is less clear, but one must suspect that $\phi_{5,6,7,8}$ are first-class as they shift the Lagrange multiplier fields. In order to simplify the analysis one may fix these suspected gauge invariances with further constraints, and then must check that there are no inconsistencies that follow from the imposition and conservation of the extra constraints. With this caveat, set

$$
\begin{aligned}
\omega_1 &:= g^{\alpha\beta} - \eta^{\alpha\beta} \approx 0, \\
\omega_2 &:= \chi^A \approx 0, \\
\omega_3 &:= \phi^A \approx 0, \\
\omega_4 &:= \psi^A \approx 0, \\
\omega_5 &:= D^A_\alpha\Pi^{\bar{A}\alpha 1} \approx 0,
\end{aligned}
\tag{3.4}
$$

and require their conservation. It is important to note that the imposition of these covariant gauge conditions in no way changes the counting of the independent degrees of freedom.

Conservation of the constraints (3.3) and (3.4) requires the additional constraints

$$
\begin{aligned}
\phi_9 &:= P_\gamma^{\ \mu}\gamma_\mu\, D^A_\epsilon\, \Pi^{A\gamma}_{\ \ \alpha}\, \Pi^{A\epsilon}_{\ \ \beta} \approx 0, \\
\phi_{10} &:= \tfrac{1}{2}\bar{D}^A_\gamma \gamma^{\mu\nu\rho}\, D^A_\delta \Pi^{A\gamma}_{\ \ \alpha}\Pi^{A\delta\beta} \approx 0, \\
\phi_{11} &:= \bar{D}^A_\delta \gamma^\mu\, D^{A'}{}_\epsilon\, \Pi^{A\delta}_{\ \ \alpha}\, \Pi^{A\epsilon}_{\ \ \beta} \approx 0, \\
\phi_{12} &:= T_{\alpha\beta} \approx 0, \\
\phi_{13} &:= P_1^{\ \mu} + X^{\mu'} - i\sum_A \bar{\theta}^A\gamma^\mu\theta^{A'} \approx 0,
\end{aligned}
\tag{3.5}
$$

where

$$
\begin{aligned}
T_{00} = T_{11} &= \tfrac{1}{2}(P_0^2 + P_1^2) - i\sum_A \bar{D}_1^A\theta^{A'}, \\
T_{01} &= P_0 \cdot P_1 - i\sum_A \bar{D}_0^A\theta^{A'}
\end{aligned}
\tag{3.6}
$$

is the stress tensor and is the same as the modified stress tensor in (2.9). The set of constraints (3.3), (3.4), and (3.5) are conserved without the imposition of any further constraints. The set of fixing conditions (3.4) is consistent and leaves an algebra [3, 14] of constraints generated by $\phi_{9,10,11,12}$. The Hamiltonian which preserves the constraints,

$$
\begin{aligned}
\mathcal{H} = &-\frac{1}{2}\eta^{\alpha\beta}P_\alpha\cdot P_\beta - P_1\cdot(X' - i\sum_B \bar{\theta}^B\gamma\theta^{B'}) \\
&- i\sum_B \bar{\theta}^B\gamma_\mu\theta^{B'}((-)^B P_0^{\ \mu} - X^{\mu'} + i\bar{\theta}^{\bar{B}}\gamma^\mu\theta^{\bar{B}'}) \\
&+ P_X^2 + P_X\cdot(P_0 - i\sum_B (-)^B\bar{\theta}^B\gamma\theta^{B'}) \\
&+ \sum_A (-)^A(\zeta^A\theta^{A'} + B^{A\eta}D^{A'}{}_\eta) - \epsilon_{\alpha\beta}P_P^\alpha\cdot P^{\beta'},
\end{aligned}
\tag{3.7}
$$

is unique up to the addition of first-class constraints and is equal to T_{00} upon setting the second-class constraints to zero strongly. (That is, taking the second-class constraints to vanish *identically* and replacing the Poisson bracket by the Dirac bracket [6] so that no contradictions will result from taking the constraints to vanish identically.) Since the constraints $\phi_{1,\ldots,8,13}$ and $\omega_{1,\ldots,5}$ together are second-class, one must think of $\bar{D}^A_\alpha$ and ${P_\alpha}^\mu$ as derived quantities given by their expressions in (3.3, 3.4, 3.5). When this is done, the only independent variables left are X, P_X, θ and ζ, which must still satisfy $\phi_{9,10,11,12} \approx 0$. Two of these constraints have *identical* counterparts in the Green-Schwarz theory. The constraint ϕ_9 is F^A and ϕ_{12} is T_A. The Green-Schwarz string has eight additional second-class constraints $G^A \approx 0$ while the Siegel string has instead twenty-nine additional constraints which have vanishing Poisson brackets with all other constraints on the constraint surface. One might be tempted to call these twenty-nine constraints first-class, but if they were first-class then there would be a mismatch in the number of physical degrees of freedom between the Green-Schwarz and Siegel string. In fact, both theories have the same number of physical degrees of freedom and have the same second-class constraints in generic regions of phase space. To see this, one must analyze the constraints $\phi_{9,10,11,12}$ carefully. First, it is useful to have a simple notation. Set A $=1$ because the case A $=2$ is analogous. Let D^A_α become D because A $=1$ and α has only one non-zero component by (3.4). Similarly, let $P = P_\alpha \Pi^{A\alpha\beta}$. The constraints (3.5) are now easily written as

$$\tfrac{1}{2}P^2 + 2\bar{D}\theta' \approx 0, \qquad \not{P}D \approx 0, \qquad \bar{D}\gamma^{\mu\nu\rho}D \approx 0, \qquad \bar{D}\gamma^\mu D' \approx 0. \tag{3.8}$$

Because $\not{P}$ is invertible for $P^2 \neq 0$, the first two of these together imply that

$$P^2 \approx 0 \qquad \text{and} \qquad \bar{D}\theta' \approx 0 \tag{3.9}$$

separately.

Here we must assume that P^2 is not a nilpotent commuting supernumber. Although the variable P_μ is *a priori* a commuting supernumber, we may restrict it to be real. It is permissible for us to do this because we will define quantum mechanics through the use of the path integral and, as shown in the appendix, an integral over all commuting supernumbers is equal to the same integral restricted to be over purely real numbers. Because the constraints are imposed on the path integral, we may analyze the constraints using this same assumption. It is also important that the variable

P_μ is a global supersymmetry invariant so that the global supersymmetry will not be ruined by this restriction. It is clear that the first constraint of (3.8) then implies that P^2 and $\bar{D}\theta'$ must vanish separately, because $\bar{D}\theta'$ has no real number piece. (In the terminology of DeWitt [16], $\bar{D}\theta$ has no body and P^2 has no soul.)

Alternatively, we may argue in the following fashion. We still restrict the variable P_μ to be real. Because $\not{P}$ is invertible for $P^2 \neq 0$, the first two constraints of (3.8) together imply that

$$P^2 \approx 0 \qquad \text{and} \qquad \bar{D}\theta' \approx 0 \tag{3.10}$$

separately. The argument proceeds by multiplying the second constraint by $\not{P}$ and then dividing by P^2 if it is non-zero.

The relevant constraints on the derived quantity D are

$$\not{P}D \approx 0, \qquad \bar{D}\theta' \approx 0, \qquad \bar{D}\gamma^{\mu\nu\rho} D \approx 0, \qquad \bar{D}\gamma^\mu D' \approx 0. \tag{3.11}$$

The solution of these constraints, $D = f(\theta', \not{P})$ will be the analogous constraints to $\bar{D} = 0$ in the Green-Schwarz theory. For any f, except $f = (\not{P} - i\bar{\theta}\gamma\theta'\gamma)\theta +$ *constant*, the constraints $D = f(\theta', \not{P})$ are obviously second-class. One may dispose of the possibility $f = (\not{P} - i\bar{\theta}\gamma\theta'\gamma)\theta$ by showing that it is not a solution.

One may show that the only solution for generic θ' is $f = 0$ or that (3.11) are equivalent to $\bar{D} \approx 0$. The third constraint is most easily solved. It implies that

$$D^a(\sigma) = \lambda^a(\sigma)\epsilon(\sigma) \tag{3.12}$$

where $\lambda^a(\sigma)$ is a commuting spinor function and $\epsilon(\sigma)$ is an anticommuting scalar. The second equation implies

$$\epsilon(\sigma) \propto (\bar{\lambda}\theta'), \tag{3.13}$$

while the last requires that $\epsilon\epsilon' = 0$ or, equivalently,

$$\epsilon(\sigma) \propto \epsilon'(\sigma). \tag{3.14}$$

Generically, all of the components of θ' are independent Grassmann numbers and have zeros as functions of σ. Eq. (3.14) requires $\lambda(\sigma)$ to have poles of the same order as the zeros of these generic θ configurations unless ϵ is identically zero. The expression D, which is tacitly assumed to be a differentiable function of σ, is expressed through (3.12, 3.13) as

$$D^a = \beta\lambda^a(\bar{\lambda}\theta'). \tag{3.15}$$

Therefore D^a has poles as a function of σ for generic $\theta(\sigma)$ field configurations unless it vanishes.

There is one loop-hole in the above argument. There is a way to solve the constraints (3.11) which is not of the form (3.12). The last two constraints can be solved by setting D proportional to a constant nilpotent commuting number (such as $\epsilon_1\epsilon_2$, ϵ_1 and ϵ_2 both Grassmann). That is, the expression (3.15) satisfies the last three constraints if $\beta^2 = 0$. It also satisfies the first constraint if the commuting spinor λ is annihilated by $\not{P}$. These solutions must be considered "pathological."

These pathological solutions are assumed to be unimportant for the quantum theory. To illuminate the peculiar nature of solutions involving nilpotent commuting numbers, consider a constraint $P_\mu P^\mu = 0$. Any P_μ of the form $P_\mu = \beta M_\mu$ with M arbitrary and $\beta^2 = 0$ satisfies the constraints. The finite dimensional analog of the path integral measure over such a constrained surface is $d^nP\,\delta(P^2)$ which becomes $\beta^n\delta(\beta^2)d^nM/M^2$ upon replacement of P_μ by βM_μ. From the rules in the appendix, we would define $\beta^n\delta(\beta^2)$ as $\beta^{n+2}\delta'(0)$ which is ambiguous but should be set to zero because β is nilpotent. These pathological solutions can be eliminated if we define the integrals over these subspaces to vanish.

The existence of these pathological solutions is of secondary importance to the fact that they, like $D = 0$, are also second-class constraints.

It is peculiar that the algebra of constraints (3.8) hides second-class constraints. Usually one believes that constraints which form an algebra are first-class and generate symmetries. An analogous, though simpler, model of this situation can be made. Suppose there is a system with constraints $p \approx 0$ and $q \approx 0$. These constraints cannot be imposed simultaneously on the system because their Poisson brackets do not vanish; $\{q, p\} = 1$. These are second-class constraints. An *equivalent* set of constraints may be imposed on the system. Consider the set of constraints

$$p^2 \approx 0, \qquad q^2 \approx 0, \qquad pq \approx 0. \tag{3.16}$$

The constraints (3.16) are equivalent to $p \approx 0, q \approx 0$ in that the hypersurfaces defined by both sets of constraints are identical. The difference is that (3.16) form an algebra:

$$\{q^2, p^2\} = 4pq, \qquad \{p^2, pq\} = -2p^2, \qquad \{q^2, pq\} = 2q^2. \tag{3.17}$$

Thus the quadratic constraints (3.16) appear to be first-class. When they are solved (written in a form linear in the dynamical variables) one can see they are actually second-class. This simple example shows how algebras of non-linear quantities may hide second-class constraints.

The quantization of theories with quadratic constraint algebras is not straightforward. If we insist on dealing with the constraints in their non-linear form, we will be unable to obtain any states at all, despite the fact that the constraints form an algebra. This can be demonstrated with the simple example (3.16) above. First, we transcribe the constraints into operators. In order to preserve the algebra (3.17), we must order the constraints as follows.

$$\begin{aligned} p^2 &\rightarrow \hat{p}^2, \\ q^2 &\rightarrow \hat{q}^2, \\ pq &\rightarrow \tfrac{1}{2}(\hat{p}\hat{q} + \hat{q}\hat{p}). \end{aligned} \tag{3.18}$$

Imposing these operators on a wavefunction leads to the conclusion that the wavefunction must vanish. Of course, these constraints may be treated in the Gupta-Bleuler fashion, but even second-class constraints may be successfully imposed this way. The

BFV formalism may not be applied to the system (3.16) directly because the constraints are reducible. When using the reducibility conditions

$$
\begin{aligned}
0 &= q(p^2) + p(pq), \\
0 &= q(pq) + p(q^2).
\end{aligned}
\tag{3.19}
$$

in the BFV formalism for reducible theories, one finds that one cannot obtain a consistent BRS charge Ω.

Because the solutions of the constraints (3.8) are second-class, the "symmetries" of the Siegel string system generated by the last two constraints of (3.8) are not true symmetries. The Siegel string, because it has the same constraints as the Green-Schwarz string, also has sixteen bosonic and sixteen fermionic physical phase space degrees of freedom. As classical theories the two formulations of the string are equivalent. For quantization the (linear) Green-Schwarz constraints are more suitable.

4. The Quantization of the Manifestly Supersymmetric String

The bosonic string and the NSR superstring are both covariantly quantized through the Faddeev-Popov procedure in which the canonical structure of the theories need never enter. Instead, the integration over metric degrees of freedom is rewritten to factor out the diffeomorphisms explicitly through a change of variables. The resulting Jacobian becomes the ghost action once the ghosts are introduced.

Theories which have complicated phase space structure, such as nontrivial second-class constraints, or algebras of first-class constraints which have phase-space dependent structure constants, cannot be quantized using a Lagrangian path integral with the Faddeev-Popov method. Theories with phase-space dependent structure constants in the first-class constraint algebra have a more complicated BRS charge which leads to a Lagrangian containing ghost-ghost interactions [10, 7]. Complicated second-class constraints require a modification of the path integral measure and a modification of the Poisson brackets [10]. It is unfortunate that the manifestly supersymmetric string has both complications.

In order to quantize covariantly one of these complicated theories, the constraints must first be separated into first- and second-classes. The Poisson bracket is redefined so that the second-class constraints have vanishing brackets with any function on phase space. The measure of the path integral is modified by the introduction of delta functions of the second-class constraints multiplied by the square root of the superdeterminant of the matrix of Poisson brackets of all second-class constraints.

$$\text{Measure factor} = \delta^N(\chi_i)\sqrt{\text{sdet}\{\chi_i, \chi_j\}_P} \tag{4.1}$$

Next, the first-class constraints must be considered. The first-class constraints may be used to construct a BRS symmetry generator which will later be used to fix out the first-class symmetries. One starts by enlarging the phase space. For each first-class constraint $\phi \approx 0$ a Lagrange multiplier λ and its conjugate momentum π are introduced. A ghost η, antighost $\tilde{\eta}$ and their conjugate momenta ρ and $\tilde{\rho}$ round out the set additional phase space variables needed for each first-class constraint ϕ. The variables λ and π have the same statistics as ϕ while the statistics of $\eta, \tilde{\eta}, \rho$ and $\tilde{\rho}$ are the opposite of ϕ. There is also a new constraint one must introduce, which is $\pi \approx 0$.

From these constraints one constructs a BRS generator [7]

$$\Omega = \tilde{\eta}\pi + \eta\phi + \text{more} \tag{4.2}$$

to satisfy

$$\{\Omega, \Omega\}_{\text{Dirac}} = 0. \tag{4.3}$$

This condition is nontrivial to satisfy because Ω is fermionic and the Dirac bracket (4.3) is symmetric for fermionic functions and does not vanish identically.

The correct generating functional for the system is

$$Z_\Psi = \int DPDQ\, \delta[\chi_i]\sqrt{\text{sdet}\{\chi_i, \chi_j\}_P}\, \exp i \int dt(P\dot{Q} - H + \{\Psi, \Omega\}_D). \tag{4.4}$$

with Ψ any imaginary fermionic function on the extended phase space of original variables and ghost, antighost and Lagrange multiplier degrees of freedom, with ghost number -1. The Fradkin-Vilkovisky theorem [8] states that the generating functional is independent of the gauge fixing function Ψ.

The Fradkin method cannot be straightforwardly applied to systems with quadratic constraints which appear to be first-class. In fact, by incorporating the model quadratic constraints (3.16) in a BRS generator and applying the rule (4.4), one may show that the correct measure factor, $\delta(p)\delta(q)$, cannot be obtained. Similarly, the treatment of the constraints (3.11) as first-class constraints in the Fradkin formalism will not yield the correct result (i.e., the result one gets when the second-class constraints are separated out explicitly as in (2.9)). Perhaps there is a modification of the Fradkin formalism which allows more flexibility in the treatment of second-class constraints, but the replacement of second-class constraints by quadratic first-class constraints does not work.

Without possessing a more flexible formalism one is forced to treat the system (2.9) according to the rules of the Fradkin formalism. Thus one can write down, at least formally, the most general quantum version of the manifestly supersymmetric string.

The measure factor is

$$\delta[G^{Aa}(\sigma)]\Big(\det{}'(\{G^{Aa}(\sigma), G^{Bb}(\sigma')\}_P)\Big)^{-\frac{1}{2}} \tag{4.5}$$

where

$$\{G^{Aa}(\sigma), G^{Bb}(\sigma')\}_P \approx 4i\delta(\sigma-\sigma')\delta^{AB}\Pi_A \cdot \Pi_{\bar{A}}(\gamma^0 h_A \gamma_\mu \Pi^\mu_{\bar{A}})^{ab}. \tag{4.6}$$

Next, one must construct the BRS charge Ω to satisfy $\{\Omega, \Omega\}_D = 0$ and show that the quantum BRS charge only squares to zero for ten spacetime dimensions. This still has yet to be done, but there is no reason to doubt that it can be done.

There is, perhaps, little calculational power to be gained in continuing the quantization in this fashion because the Poisson bracket (4.6) is cubic in the momentum P^μ. The elevation of this bracket from the measure to the action with appropriate "second-class ghosts" will yield an action cubic in momenta. The momentum integrals cannot be done explicitly to yield a conventional Lagrangian, but one could consider this momentum space path integral as the configuration space path integral of a first-order Lagrangian, which is, unfortunately, not free. All that is needed to use this formal quantization is the explicit form of the BRS operator Ω whose quantum analog is nilpotent. This quantization could be used for any (world sheet) perturbative calculation.

To conclude this section, I resolve the puzzle of why the above remarks do not apply to the light-cone gauge

$$X^+ + P^+\tau \approx 0, \qquad P^{+'} \approx 0, \qquad \gamma^+\theta^A \approx 0, \qquad g_{\alpha\beta} \approx \eta_{\alpha\beta}. \tag{4.7}$$

In other words, why can the light-cone gauge fixed string be quantized so easily and why is it free? The answer crucially depends upon the observation that the constraints (4.7) may be treated on the same footing with the constraints (2.3) and (2.5). The constraints (4.7) must also be conserved and so fix the Hamiltonian to be [12]

$$\mathcal{H} = \frac{1}{2}(P^2 + X'^2) + \zeta^2\theta^{2'} - \zeta^1\theta^{1'}. \tag{4.8}$$

The whole set of constraints (4.7), (2.3), and (2.5) are all together second-class constraints as they must be since (4.7) fix the gauge completely. The generating functional (4.4) may be used to quantize the theory. Because there are no first-class constraints there is no BRS charge. The superdeterminant in the measure factor is not field dependent, and the delta functions may be solved easily. When the momentum integral is done, one is left with a free theory for the transverse modes. The main point of this is that the theory is much simpler if one does not have to separate the constraints into first- and second-classes. It also helps that the gauge conditions (4.7) are simple and linear.

5. Discussion

The main result of this paper is the demonstration that the Siegel string is essentially equivalent to the Green-Schwarz string.

The extra symmetries of the Siegel string are not actually symmetries at all but hide second-class constraints. Because they are not symmetries, they do not need to be fixed out through a gauge choice and do not properly belong in the BRS generator of the theory. The existence of a formalism for quantizing a theory with a quadratic algebra of constraints which are actually second-class is an open question. The existing formalism for quantization requires the explicit identification of the second-class constraints and therefore the Green-Schwarz form is the most appropriate for quantization. A formal quantization of the Green-Schwarz system has been given. The construction of the BRS charge which has zero Dirac brackets with itself and the demonstration that the quantum mechanical BRS charge is nilpotent, are necessary to complete the quantization. The important question of how treat the reducibility of the constraints and therefore how to define the determinant in (4.5) must also be resolved in order to complete the quantization. This formal quantization does not possess the most useful attribute of the NSR and bosonic strings; the freedom of the world sheet σ-model.

Nevertheless, one knows how to begin constructing the associated string field theory. The set of fields on which the wave functional depends is the original set X^μ, θ^A, the ghosts for the diffeomorphisms and local supersymmetry, the "second-class ghosts," and the fields used to elevate the delta functional in the measure factor (4.5) to the action. Much less clear is the explicit form for the dynamics of the free string field theory.

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APPENDIX

Calculus of Commuting and Anticommuting Grassmann Numbers [15]

Following DeWitt, we define *supernumbers* by starting with an infinite dimensional Grassmann algebra with basis ζ^a, $a = 1, 2, \ldots$ satisfying only the relations

$$\begin{aligned} \zeta^a \zeta^b &= -\zeta^b \zeta^a, \\ (\zeta^a)^2 &= 0. \end{aligned} \tag{A.1}$$

We denote this algebra over a base field $\mathbf{F}$ by $\Lambda_\infty(\mathbf{F})$. We shall be concerned mostly with $\Lambda_\infty(\mathbf{R})$. Any supernumber in $\Lambda_\infty(\mathbf{R})$ may be split into its *body* and *soul*

$$\begin{aligned} &x \in \Lambda_\infty(\mathbf{R}), \quad x = x_B + x_S \\ &x_B \in \mathbf{R}, \quad x_S = \sum_{n=1}^{\infty} \frac{1}{n!} c_{\alpha_1 \ldots \alpha_n} \zeta^{\alpha_n} \cdots \zeta^{\alpha_1} \\ &c_{\alpha_1 \ldots \alpha_n} \in \mathbf{R} \end{aligned} \tag{A.2}$$

Functions on $\Lambda_\infty(\mathbf{R})$ may be defined by extending any infinitely differentiable real function by the *formal* series

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_B) x_S^n. \tag{A.3}$$

Because the series (A.3) is purely formal, there is no problem with convergence.

More important for physics is the distinction between *even* and *odd* (that is, commuting and anticommuting) supernumbers. Any supernumber x can be split into two pieces, $x_c \in \mathbf{R}_c$ and $x_a \in \mathbf{R}_a$.

$$\begin{aligned} x &= x_c + x_a; \\ x_c &= x_B + \sum_{n=1}^{\infty} \frac{1}{(2n)!} c_{\alpha_1 \ldots \alpha_{2n}} \zeta^{\alpha_{2n}} \cdots \zeta^{\alpha_1}, \\ x_a &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} c_{\alpha_1 \ldots \alpha_{2n+1}} \zeta^{\alpha_{2n+1}} \cdots \zeta^{\alpha_1}. \end{aligned} \tag{A.4}$$

Analytic functions of a single anti-commuting variable are precisely the linear functions

$$f(x_a) = a + bx_a, \tag{A.5}$$

because x_a is nilpotent, $x_a^2 = 0$.

Functions of a real commuting variable, obtained from infinitely differentiable real functions, are defined by the formal series (A.3).

A definite integral along a path in $\mathbf{R}_c$ of a function of a function of a commuting variable is given by

$$\int_a^b f(x)\, dx = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{a_B}^{b_B} f^{(n)}(x_B(t)) x_S^n(t) \left[\frac{dx_B}{dt} + \frac{dx_S}{dt} \right] dt \tag{A.6}$$

where t is a real number. Here we take the parametrization to be $x_B(t) = t$. The striking thing about the integral (A.6) is that it is independent of the path $(x_B(t), x_S(t))$ used to define it. This fact is easily demonstrated. First we rewrite the integral (A.6) as

$$\int_a^b f(x)\, dx = \sum_{n=0}^{\infty} \int_{a_B}^{b_B} f^{(n)}(t) \left[\frac{1}{n!} x_S^n(t) + \frac{1}{(n+1)!} \frac{d}{dt} x_S^{n+1}(t) \right] dt \tag{A.7}$$

Next we split the sums apart and integrate by parts.

$$\begin{aligned}
\int_a^b f(x)\, dx = &\int_{a_B}^{b_B} f(t) \left[1 + x_S'(t)\right] dt \\
&+ \sum_{n=1}^{\infty} \frac{1}{n!} \left[f^{(n-1)}(b_B) b_S^n - f^{(n-1)}(a_B) a_S^n \right] \\
&+ \sum_{n=1}^{\infty} \int_{a_B}^{b_B} \left[-\frac{1}{n!} f^{(n-1)} \frac{dx_S^n(t)}{dt} + \frac{1}{(n+1)!} f^{(n)}(t) \frac{dx_S^{n+1}}{dt} \right] dt.
\end{aligned} \tag{A.8}$$

We obtain the desired result

$$\int_{a}^{b} f(x)\,dx = \int_{a_B}^{b_B} f(t)\,dt + \sum_{n=1}^{\infty} \frac{1}{n!} \left[f^{(n-1)}(b_B) b_S^n - f^{(n-1)}(a_B) a_S^n \right] \\ = F(b) - F(a), \tag{A.9}$$

as long as f and all of its derivatives are finite at the bodies a_B and b_B. A corollary of this is that the improper integral over $\mathbf{R}_c$ is the same as that over $\mathbf{R}$. This follows from the fact that

$$\lim_{t\to\infty} F(t + x_S) = \lim_{t\to\infty} F(t) \tag{A.10}$$

holds for all smooth functions F and all finite $x_S \in \mathbf{R}_c$, if the limit exists.

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