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BRST Quantization and Gravity in Self-dual Variables

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Abstract

We examine the problem of constructing a real BRST charge for general relativity in the Ashtekar variables. In addition to reviewing the construction of Ashtekar, Mazur and Torre, we apply a method found previously by us for quantizing theories with complex constraints to gravity in Ashtekar's new variables, and we find real constraints expressed in terms of the Ashtekar variables. We find that although real BRST charges can be constructed, they are not polynomial and the polynomial BRST charges are not real.

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I. Introduction

It is ironic that while the gravitational interaction was the first of the fundamental forces to be understood classically, it has yet to be reconciled with quantum mechanics. Dirac wrestled with the problem of casting Einstein's General Theory of Relativity into Hamiltonian form in order to apply the rules of canonical quantization to it. Dirac reduced the problem of quantum gravity to finding solutions of the quantum constraints of general relativity. Progress in canonical quantum gravity was stymied by the non-polynomiality of the Dirac constraints. The perturbative approach to quantum gravity came to a dead end when general relativity, long known to be non-renormalizable, was shown to be infinite at two loops [1]. The failure of these more conservative approaches to quantum gravity is responsible in part for the enormous interest in supersymmetric string theory.

The difficulty in Dirac's formulation of quantum general relativity is in finding solutions to the quantum constraint equations. In 1987, Ashtekar published a canonical transformation of Einstein gravity to a new set of variables in which the constraints become polynomial [2]. The corresponding quantum constraint operators are second-order functional differential operators and are easier to solve. At least one solution of the full set of constraint equations is known [3] and there is great hope that many more solutions can be found to the Ashtekar constraints and that from them a quantum theory can be found.

A powerful method of quantization subsuming Dirac's was found in the late 1970's by several members of Fradkin's school [4] in the Soviet Union. This method, known by the initials of Becchi, Rouet, Stora, Tyutin, Batalin, Fradkin and Vilkovisky, or BRST-BFV, allows more freedom in finding physical quantum states and gives a general prescription for finding the inner product on physical states. We examine the construction of a BRST quantization of quantum gravity in the Ashtekar variables.

One unfortunate aspect of the the new variables found by Ashtekar is that they are necessarily complex in a spacetime of Lorentzian signature, leading to non-hermitian quantum constraint operators. Although there is probably a deep significance to the to the fact that the variables and the constraints are not real-valued, the non-hermiticity of the constraints presents us with great technical difficulties.

One aspect of these difficulties is that the construction of a BRST quantization of gravity using the Ashtekar variables is not straightforward. A useful BRST analysis of a physical theory requires the construction of a real BRST charge. Very soon after the discovery of the Ashtekar variables, BRST charges were constructed by Ashtekar, Mazur and Torre,[5] who did not consider the reality properties of the charge. We examine the complex structure of these constraints and show that, in fact, they are a complex mix of real constraints. Following the analysis of Ref. [6], we show that the BRST charges constructed by Ashtekar, Mazur and Torre [5] are complex. We investigate the construction of a new BRST charge which is real by using the technique developed in Ref. [6], namely, by adding the complex conjugates of the Ashtekar constraints to the original set of Ashtekar constraints and treating the combined set as a reducible set of constraints. We thus satisfy the criterion of reality, but we show that this is at the cost of polynomiality.

II. Complex structure of the Ashtekar constraints

In the Ashtekar formulation of general relativity, the constraints take on an especially simple form [2],

$$\begin{aligned} {}^{\pm}\mathcal{D}_a \widetilde{\sigma}^a{}_M{}^N &\approx 0 \\ \mathrm{Tr}(\widetilde{\sigma}^a\, {}^{\pm}F_{ab}) &\approx 0 \\ \mathrm{Tr}(\widetilde{\sigma}^a \widetilde{\sigma}^b\, {}^{\pm}F_{ab}) &\approx 0. \end{aligned} \tag{2.1}$$

Tensor indices are represented by lower case letters and spinor indices are represented by upper case letters. "Tr" indicates the trace over spinor indices. One generally chooses either the plus sign or the minus sign, each of which gives a full set of constraints, although the method we use in section 4 will use both sets. In this section, we examine the complex structure of the Ashtekar constraints. We separate them into their real and imaginary parts and write them as a complex combination of real constraints. The manifestly complex object in the Ashtekar constraints is the Ashtekar connection ${}^{\pm}\mathcal{D}_a$ or, more precisely, the spinor connection 1-form ${}^{\pm}A_{aA}{}^{B}$. Less obvious is the complex behavior of the $SU(2)$ spinors. Hermitian spinors behave like "real" numbers and anti-hermitian spinors behave like "imaginary" numbers under hermitian conjugation. We must also take care to examine the hermiticity properties of the spinors in the constraints. The hermiticity properties of $SU(2)$ spinors, and of the Ashtekar variables in particular, are discussed in the appendix. For convenience, we drop the $\pm$ signs from the Ashtekar variables, choosing $A_a := {}^{+}A_a$ in sections 2 and 3 of the paper.

We begin with the Ashtekar connection $\mathcal{D}_a$ and the connection 1-form $A_{aM}{}^{N}$. $\mathcal{D}_a$ is defined by

$$\mathcal{D}_a \lambda_{bM} = D_a \lambda_{bM} + \frac{i}{\sqrt{2}} \Pi_{aM}{}^{N} \lambda_{bN}, \tag{2.2}$$

where D_a is the covariant derivative operator that acts on both tensor and spinor indices. To separate the real and imaginary parts of $\mathcal{D}_a$, we take a closer look at its action on objects with tensor and spinor indices,

$$\mathcal{D}_a \lambda_{bM} = \partial_a \lambda_{bM} + \Gamma_{ab}{}^{c} \lambda_{cM} + \Gamma_{aM}{}^{N} \lambda_{bN} + \frac{i}{\sqrt{2}} \Pi_{aM}{}^{N} \lambda_{bN}. \tag{2.3}$$

Ignoring λ_{bM}, which is included only so that the indices match, we see that the first two terms on the right hand side, which involve only real spacetime operators, are manifestly real. The third term involves the spinorial connection 1-form $\Gamma_{aM}{}^{N} \equiv \Gamma_{ab} \sigma^{b}{}_{M}{}^{N}$. In this paper, we use the standard hermitian conjugate to define reality properties of matrices. The tensor Γ_{ab} is real and $\sigma^{a}{}_{M}{}^{N}$ is anti-hermitian (see appendix). Under complex conjugation, the third term goes to minus itself and therefore behaves like an imaginary number. The last term is hermitian because it is the product of i times the anti-hermitian spinor $\Pi_{aM}{}^{N} \equiv \Pi_{ab} \sigma^{b}{}_{M}{}^{N}$.

We rearrange the Ashtekar connection into "real" and "imaginary" parts,

$$\mathcal{D}_a\lambda_{bM} := \underbrace{\partial_a\lambda_{bM} + \Gamma_{ab}{}^c\lambda_{cM} + \frac{i}{\sqrt{2}}\Pi_{aM}{}^N\lambda_{bN}}_{\text{"real"}} + \underbrace{\Gamma_{aM}{}^N\lambda_{bN}}_{\text{"imaginary"}}. \tag{2.4}$$

The connection 1-form $A_{aM}{}^N$ consists of just the last two terms of $\mathcal{D}_a$. We can write it in terms of its "real" and "imaginary" parts,

$$A_{aM}{}^N = \underbrace{\frac{i}{\sqrt{2}}\Pi_{aM}{}^N}_{\text{"real"}} + \underbrace{\Gamma_{aM}{}^N}_{\text{"imaginary"}}. \tag{2.5}$$

We now consider the Gauss constraint. Using the definition of D_a, the fact that $D_a\widetilde{\sigma}^a{}_M{}^N \equiv D_a(q^{1/2}\sigma^a{}_M{}^N) = 0$ (because both $\sigma^a{}_M{}^N$ and q_{ab} are compatible with D_a), and the commutation relations of $\sigma^a_M{}^N$, we can rewrite the Gauss constraint in a number of equivalent forms,

$$\begin{aligned}\mathcal{D}_a\widetilde{\sigma}^a{}_M{}^N &\equiv \frac{i}{\sqrt{2}}[\Pi_a, \widetilde{\sigma}^a]_M{}^N \\ &\equiv \sqrt{2}iq^{1/2}\Pi_{[ab]}\sigma^b{}_M{}^P\sigma^a{}_P{}^N \\ &\equiv -iq^{1/2}\Pi_{[ab]}\epsilon^{abc}\sigma_{cM}{}^N\end{aligned} \tag{2.6}$$

The square brackets on the indices indicate antisymmetrization, and we use the rule $\Pi_{[ab]} = \frac{1}{2}(\Pi_{ab} - \Pi_{ba})$. It is also convenient to invert the last form of the Gauss constraint,

$$\Pi_{[ab]} = -\frac{i}{2}q^{-1/2}\epsilon_{abc}\sigma^c{}_N{}^M\mathcal{D}_d\widetilde{\sigma}^d{}_M{}^N. \tag{2.7}$$

Any of the forms of the Gauss constraint can be used to examine its reality properties, but the last of (2.6) is the simplest to use. The tensors $\Pi_{[ab]}$ and ϵ^{abc} are real, $\sigma^c{}_M{}^N$ is anti-hermitian, and the coefficient i makes the Gauss constraint overall hermitian. It thus behaves like a "real" number under complex conjugation,

$$\mathcal{D}_a\widetilde{\sigma}^a{}_M{}^N = \underbrace{-iq^{1/2}\Pi_{[ab]}\epsilon^{abc}\sigma_{cM}{}^N}_{\text{"real"}}. \tag{2.8}$$

To separate the vector constraint into real and imaginary parts, we first expand the curvature tensor F_{ab} using (2.2),

$$F_{abM}{}^{N} = R_{abM}{}^{N} + \sqrt{2}iD_{[a}\Pi_{b]M}{}^{N} - \Pi_{[aM}{}^{P}\Pi_{b]P}{}^{N}. \tag{2.9}$$

$F_{abM}{}^{N}$ is the spinorial curvature tensor of the Ashtekar connection, $(\mathcal{D}_a\mathcal{D}_b - \mathcal{D}_b\mathcal{D}_a)\lambda_M = F_{abM}{}^{N}\lambda_N$, and $R_{abM}{}^{N}$ is the spinorial curvature tensor of the covariant derivative operator, $(D_aD_b - D_bD_a)\lambda_M = R_{abM}{}^{N}\lambda_N$. Substituting (2.9) into the vector constraint gives

$$\mathrm{Tr}(\widetilde{\sigma}^a F_{ab}) = \underbrace{\frac{q^{1/2}}{\sqrt{2}}\Pi_{am}\Pi_{bn}\epsilon^{amn}}_{\text{real}} - \underbrace{\frac{iq^{1/2}}{\sqrt{2}}D^a(\Pi_{ba} - \Pi q_{ba})}_{\text{imaginary}}. \tag{2.10}$$

where we have indicated the real and imaginary parts of the vector constraint. The first term contains only the real tensors Π_{ab} and ϵ^{abc} and is manifestly real. Although D_a is in general "complex" because of the spinorial connection 1-form that it contains, the second term is purely imaginary because D_a is acting on a tensor. The spinorial piece of the connection does not enter and D_a behaves as a real operator.

Similarly, substituting (2.9) into the scalar constraint allows us to separate it into real and imaginary parts,

$$\mathrm{Tr}(\widetilde{\sigma}^a\widetilde{\sigma}^b F_{ab}) = \underbrace{\tfrac{1}{2}q(R + \Pi^2 - \Pi_{ab}\Pi^{ba})}_{\text{real}} - \underbrace{iq\epsilon^{abc}D_a\Pi_{bc}}_{\text{imaginary}}. \tag{2.11}$$

Equations (2.8), (2.10), and (2.11) explicitly show the real and imaginary parts of the Ashtekar constraints, but the real and imaginary parts cannot all be independent since we have seven complex constraint equations and only seven independent (real) constraints on phase space. The vector and scalar constraints implicitly contain the Gauss constraint. We wish to make this dependence on the Gauss constraint explicit. For convenience, we identify the Gauss constraint by

$$\mathcal{G}_M{}^N := \underbrace{\mathcal{D}_a\widetilde{\sigma}^a{}_M{}^N}_{\text{“real”}} \approx 0. \tag{2.12}$$

We consider first the vector constraint (2.10). By relabeling dummy indices and using the third of (2.6), we can rewrite the first term on the right side of (2.10) as

$$\frac{q^{1/2}}{\sqrt{2}}\Pi_{am}\Pi_{bn}\epsilon^{amn} = -\frac{i}{\sqrt{2}}\Pi_{bN}{}^{M}(\mathcal{D}_a\widetilde{\sigma}^a{}_M{}^N). \tag{2.13}$$

In the second term on the right side of (2.10) we need to separate Π_{ba} into its symmetric and antisymmetric components since the Gauss constraint is related to the antisymmetric part only. Using (2.7), we rewrite the second term as

$$\begin{aligned}\frac{iq^{1/2}}{\sqrt{2}}D^a(\Pi_{ba}-\Pi q_{ba}) &\equiv \frac{iq^{1/2}}{\sqrt{2}}D^a(\Pi_{[ba]}+\Pi_{(ba)}-\Pi q_{ba})\\ &= -\frac{1}{2\sqrt{2}}\epsilon_{bcd}\sigma^c{}_N{}^M D^d(\mathcal{D}_a\widetilde{\sigma}^a{}_M{}^N) + \frac{iq^{1/2}}{\sqrt{2}}D^a(K_{ab}-Kq_{ab}),\end{aligned} \tag{2.14}$$

where we have used $\Pi_{(ab)} = K_{ab}$. Combining these terms, we can rewrite the vector constraint as

$$\begin{aligned}\mathcal{V}_b := \mathrm{Tr}(\widetilde{\sigma}^a F_{ab}) = &\underbrace{-\frac{i}{\sqrt{2}}\Pi_{bN}{}^{M}(\mathcal{D}_a\widetilde{\sigma}^a{}_M{}^N)}_{\text{real}}\\ &+\underbrace{\frac{1}{2\sqrt{2}}\epsilon_{bcd}\sigma^c{}_N{}^M D^d(\mathcal{D}_a\widetilde{\sigma}^a{}_M{}^N) - \frac{iq^{1/2}}{\sqrt{2}}D^a(K_{ab}-Kq_{ab})}_{\text{imaginary}} \approx 0.\end{aligned} \tag{2.15}$$

Similarly, in the scalar constraint (2.11) we must separate $\Pi_{ab}\Pi^{ba}$ into symmetric and antisymmetric components. Using (2.7), the first of (A.5) and the first of (2.6) we find

$$\begin{aligned}\Pi_{ab}\Pi^{ba} &= \Pi_{(ab)}\Pi^{(ba)} + \Pi_{[ab]}\Pi^{[ba]}\\ &= K_{ab}K^{ab} - \tfrac{1}{2}q^{-1}(\mathcal{D}_b\widetilde{\sigma}^b{}_N{}^M)(\mathcal{D}_a\widetilde{\sigma}^a{}_M{}^N).\end{aligned} \tag{2.16}$$

The last term of (2.11) can be written in terms of the Gauss constraint by again using (2.7),

$$iq\epsilon^{abc}D_a\Pi_{bc} = \widetilde{\sigma}^a{}_N{}^M D_a(\mathcal{D}_b\widetilde{\sigma}^b{}_M{}^N). \tag{2.17}$$

The scalar constraint can thus be rewritten as

$$\begin{aligned}\mathcal{S} := \mathrm{Tr}(\widetilde{\sigma}^a\widetilde{\sigma}^b F_{ab}) = &\underbrace{\tfrac{1}{2}q(R+K^2-K_{ab}K^{ab}) + \tfrac{1}{4}(\mathcal{D}_b\widetilde{\sigma}^b{}_N{}^M)(\mathcal{D}_a\widetilde{\sigma}^a{}_M{}^N)}_{\text{real}}\\ &-\underbrace{\widetilde{\sigma}^b{}_N{}^M D_b(\mathcal{D}_a\widetilde{\sigma}^a{}_M{}^N)}_{\text{imaginary}} \approx 0.\end{aligned} \tag{2.18}$$

Equations (2.12), (2.15), and (2.18) are the forms of the Ashtekar constraints that we will find useful in the next sections.

Equations (2.12), (2.15), and (2.18) immediately lead to the reality conditions on the constraints,

$$\begin{aligned}
(\mathcal{G}_M{}^N)^\dagger &= \mathcal{G}_M{}^N \\
\mathcal{V}_b^* &= -\mathcal{V}_b - \sqrt{2}i\,\mathrm{Tr}(\Pi_b\mathcal{G}) \\
\mathcal{S}^* &= \mathcal{S} + 2\,\mathrm{Tr}(\widetilde{\sigma}^a D_a\mathcal{G}),
\end{aligned} \tag{2.19}$$

where $\dagger$ is hermitian conjugation and $*$ is ordinary complex conjugation. We observe that the reality conditions on the vector and scalar constraints have nonconstant coefficients. These reality conditions will prove to be useful in determining the reality of the BRST charges in the next sections.

III. The Ashtekar, Mazur, and Torre BRST charges

In 1987, Ashtekar, Mazur, and Torre [5] investigated the BRST structure of canonical general relativity in terms of the recently introduced new variables. They used methods developed by Henneaux [7] in which the constraints are assumed to be real, but they did not consider the consequences of the complex nature of the Ashtekar constraints. Ashtekar, Mazur, and Torre constructed three different BRST charges, one based on the original set of Ashtekar constraints and two others based on recombinations of the constraints. The recombinations were motivated by physical and computational arguments and were not related to the reality properties of the constraints. In this section, we review the BRST charges constructed by Ashtekar, Mazur, and Torre (AMT) and show that all three are intrinsically complex.

3.1. The original Ashtekar constraints

We consider first the BRST charge constructed from the standard Ashtekar constraints,

$$\begin{aligned} \mathcal{D}_a \tilde{\sigma}^a{}_A{}^B &\approx 0, \\ \mathrm{Tr}(\tilde{\sigma}^b F_{ab}) &\approx 0, \\ \mathrm{Tr}(\tilde{\sigma}^a \tilde{\sigma}^b F_{ab}) &\approx 0. \end{aligned} \tag{3.1}$$

This is not the case preferred by Ashtekar, Mazur, and Torre, but is logically the first case to consider.

The constraints are integrated against test functions to convert them to scalar-valued functions on the phase space,

$$\begin{aligned} U(\underline{N}) &= -i\sqrt{2} \int\limits_{\Sigma} \mathrm{Tr}\, \underline{N} \mathcal{D}_a \tilde{\sigma}^a \\ U(\vec{N}) &= -i\sqrt{2} \int\limits_{\Sigma} \mathrm{Tr}\, N^a \tilde{\sigma}^b F_{ab} \\ U(\underset{\sim}{N}) &= -i\sqrt{2} \int\limits_{\Sigma} \mathrm{Tr}\, \underset{\sim}{N} \tilde{\sigma}^a \tilde{\sigma}^b F_{ab}. \end{aligned} \tag{3.2}$$

The integration is over the spatial 3-manifold Σ and the fields $\underline{N}$, N^a, and $\underset{\sim}{N}$ are, respectively, a Lie-algebra-valued function on Σ, a vector field on Σ, and a scalar density of weight minus one on Σ.

Calculation of the Poisson brackets between the constraints yields the structure functions $U(\ ,\ |\ )$, where the two entries in the parentheses on the left of the vertical line take the place of the indices a and b of the structure function $U_{ab}{}^c$, and the entry to the right of the line takes the place of the index c. The only nonvanishing first-order structure functions are

$$
\begin{aligned}
U(\underline{N}, \underline{M} | \underline{\widetilde{L}}) &= -\int_\Sigma \mathrm{Tr}\, \underline{N M \widetilde{L}}, \\
U(\vec{N}, \vec{M} | \underline{\widetilde{L}}) &= \tfrac{1}{2} \int_\Sigma \mathrm{Tr}\, N^a M^b F_{ab} \underline{\widetilde{L}}, \\
U(\vec{N}, \vec{M} | \mathbf{L}) &= \tfrac{1}{2} \int_\Sigma (\pounds_{\vec{N}} M^a) \widetilde{L}_a, \\
U(\vec{N}, \underset{\sim}{\underline{M}} | \underline{\widetilde{L}}) &= \int_\Sigma \mathrm{Tr}\, \underset{\sim}{\underline{M}} N^b \widetilde{\sigma}^a F_{ab} \underline{\widetilde{L}}, \\
U(\vec{N}, \underset{\sim}{\underline{M}} | \widetilde{\widetilde{L}}) &= \tfrac{1}{2} \int_\Sigma (\pounds_{\vec{N}} \underset{\sim}{\underline{M}}) \widetilde{\widetilde{L}}, \\
U(\underset{\sim}{N}, \underset{\sim}{M}) | \mathbf{L}) &= \int_\Sigma (\underset{\sim}{N} \partial_a \underset{\sim}{M} - \underset{\sim}{M} \partial_a \underset{\sim}{N}) (\mathrm{Tr}\, \widetilde{\sigma}^a \widetilde{\sigma}^b) \widetilde{L}_b.
\end{aligned}
\tag{3.3}
$$

$\underline{\widetilde{L}}$ is a density of weight one with values in the $SU(2)$ Lie algebra (representing an index dual to $\underline{N}$), $\mathbf{L}$ is a covector field of weight one (representing an index dual to $\vec{N}$), and $\widetilde{\widetilde{L}}$ is a scalar density of weight two (representing an index dual to $\underset{\sim}{N}$).

The calculation of the second-order structure functions is quite tedious. Ashtekar, Mazur and Torre show that the only nonvanishing second-order structure functions are

$$
\begin{aligned}
U(\underset{\sim}{L}, \underset{\sim}{M}, \vec{K} | \widetilde{\mathbf{N}}, \underline{\widetilde{J}}) &= \frac{\sqrt{2} i}{6} \mathrm{Tr} \int_\Sigma (\underset{\sim}{M} \partial_a \underset{\sim}{L} - \underset{\sim}{L} \partial_a \underset{\sim}{M}) \widetilde{N}_b K^{(a} \widetilde{\sigma}^{b)} \underline{\widetilde{J}}, \\
U(\underset{\sim}{L}, \vec{M}, \vec{N} | \underline{\widetilde{K}}, \underline{\widetilde{J}}) &= \frac{\sqrt{2} i}{6} \mathrm{Tr} \int_\Sigma \underset{\sim}{L} N^a M^b F_{ab} \underline{\widetilde{K} \widetilde{J}}.
\end{aligned}
\tag{3.4}
$$

and then show that the third-order and fourth-order structure functions all vanish and that the theory is, therefore, rank-two. The BRST charge takes, as they put it, "the rather unwieldy form,"

$$
\begin{aligned}
Q'' = \int_\Sigma \mathrm{Tr}\Bigg[& \frac{\sqrt{2}}{i}\left(\underline{\eta}(\mathcal{D}_a\widetilde{\sigma}^a) + \eta^a\widetilde{\sigma}^b F_{ab} + \underset{\sim}{\eta}\widetilde{\sigma}^a\widetilde{\sigma}^b F_{ab}\right) + \underline{\eta\eta}\underline{\widetilde{\mathcal{P}}} - (\eta^b\partial_b\eta^a)\widetilde{\mathcal{P}}_a \\
& - (\eta^a\partial_a\underset{\sim}{\eta} + \underset{\sim}{\eta}\partial_a\eta^a)\widetilde{\widetilde{\mathcal{P}}} - 2\underset{\sim}{\eta}(\partial_a\underset{\sim}{\eta})(\mathrm{Tr}\widetilde{\sigma}^a\widetilde{\sigma}^b)\widetilde{\mathcal{P}}_b + 2\eta^a\underset{\sim}{\eta}\underline{\widetilde{\mathcal{P}}}\widetilde{\sigma}^b F_{ab} \\
& -\tfrac{1}{2}\eta^a\eta^b\underline{\widetilde{\mathcal{P}}}F_{ab} - 2i\sqrt{2}\underset{\sim}{\eta}(\partial_a\underset{\sim}{\eta})\eta^{(a}\widetilde{\sigma}^{b)}\widetilde{\mathcal{P}}_b\underline{\widetilde{\mathcal{P}}} - \frac{i\sqrt{2}}{2}\underset{\sim}{\eta}\eta^a\eta^b F_{ab}\underline{\widetilde{\mathcal{P}}\widetilde{\mathcal{P}}}\Bigg] .
\end{aligned}
\tag{3.5}
$$

We now wish to investigate the reality of this BRST charge. A BRST charge is an expansion in the antighost number, *i.e.*, in powers of the ghost momenta $\mathcal{P}$, and must be real at each antighost number for the overall charge to be real. The first three terms on the right side of (3.5) are the antighost number zero part, which we can rewrite as $-\sqrt{2}i\int_\Sigma(\mathrm{Tr}\,\underline{\eta}\mathcal{G} + \eta^a\mathcal{V}_a + \underset{\sim}{\eta}\mathcal{S})$. For this expression to be real we require that

$$
\begin{aligned}
i\,\mathrm{Tr}\,\underline{\eta}\mathcal{G} + i\eta^a\mathcal{V}_a + i\underset{\sim}{\eta}\mathcal{S} &= (i\,\mathrm{Tr}\,\underline{\eta}\mathcal{G} + i\eta^a\mathcal{V}_a + i\underset{\sim}{\eta}\mathcal{S})^* \\
&= -i\,\mathrm{Tr}\,\underline{\eta}^\dagger\mathcal{G}^\dagger - i\eta^{a*}\mathcal{V}_a^* - i\underset{\sim}{\eta}^*\mathcal{S}^* \\
&= i\,\mathrm{Tr}[(-\underline{\eta}^\dagger + \sqrt{2}i\eta^{a*}\Pi_a - 2\underset{\sim}{\eta}^*\widetilde{\sigma}^a D_a)\mathcal{G}] + i\eta^{a*}\mathcal{V}_a - i\underset{\sim}{\eta}^*\mathcal{S},
\end{aligned}
\tag{3.6}
$$

where the reality conditions (2.19) on the constraints have been used in the last step. Matching coefficients on the left and right sides and solving for the complex conjugate ghosts, we find reality conditions on the ghosts,

$$
\begin{aligned}
(\underline{\eta}_M{}^N)^\dagger &= -\underline{\eta}_M{}^N + \sqrt{2}i\Pi_{aM}{}^N\eta^a - 2\widetilde{\sigma}^a{}_M{}^N D_a\underset{\sim}{\eta}, \\
\eta^{a*} &= \eta^a, \\
\underset{\sim}{\eta}^* &= -\underset{\sim}{\eta}.
\end{aligned}
\tag{3.7}
$$

These, in turn, impose reality conditions on the ghost momenta,

$$
\begin{aligned}
\underline{\widetilde{\mathcal{P}}}^\dagger &= \underline{\widetilde{\mathcal{P}}}, \\
\widetilde{\mathcal{P}}_a^* &= -\widetilde{\mathcal{P}}_a + \sqrt{2}i\,\mathrm{Tr}(\Pi_a\underline{\widetilde{\mathcal{P}}}), \\
\widetilde{\widetilde{\mathcal{P}}}^* &= \widetilde{\widetilde{\mathcal{P}}} + 2\,\mathrm{Tr}(\widetilde{\sigma}^a D_a\underline{\widetilde{\mathcal{P}}}).
\end{aligned}
\tag{3.8}
$$

which are found by complex conjugating the fundamental Poisson brackets between the ghosts and their momenta and imposing the ghost reality conditions.

With the reality conditions (3.7) and (3.8), the antighost number zero part of the BRST charge (3.5) is real. The next term, $\underline{\eta\eta\widetilde{\mathcal{P}}}$, involves the trace of three $SU(2)$-valued ghosts. $\underline{\widetilde{\mathcal{P}}}$ is hermitian but, as (3.7) shows, $\underline{\eta}$ is generically neither hermitian nor anti-hermitian. The next term, $-(\eta^b\partial_b\eta^a)\widetilde{\mathcal{P}}_a$, is manifestly complex because η^a is real and $\widetilde{\mathcal{P}}_a$ has nonzero real and imaginary parts. The following term, $-(\eta^a\partial_a\underset{\sim}{\eta} + \underset{\sim}{\eta}\partial_a\eta^a)\widetilde{\widetilde{\mathcal{P}}}$, is also manifestly complex because η^a is real, $\underset{\sim}{\eta}$ is pure imaginary, and $\widetilde{\widetilde{\mathcal{P}}}$ has nonzero real and imaginary parts. Rather than continue, we need simply argue that the imaginary pieces in (3.5) do not cancel each other. We showed in Ref. [6] that the freedom to choose the reality properties of the ghosts and their momenta is exhausted at the antighost zero level, and that the appearance of complex terms at higher antighost numbers makes the BRST charge intrinsically complex. We have explicitly demonstrated here some of the intrinsically complex terms at antighost number one, and we conclude that the BRST charge (3.5) is intrinsically complex.

Having looked at one of the BRST charges constructed by Ashtekar, Mazur, and Torre (AMT) [5] in some detail, we will now look at the other two much more briefly, being satisfied to show explicitly that the constraints upon which they are built are intrinsically complex and concluding that the BRST charge is therefore also intrinsically complex.

3.2. Modified vector constraint

Next we look at the set of constraints and resulting BRST charge that was the primary focus of AMT. The constraints differ from those in (3.1) by the addition of a term to the vector constraint. The additional term is a multiple of the Gauss constraint and therefore preserves the weak equality of the system of constraints, *i.e.*, the modified constraints define the same constraint surface. The modified constraints are

$$\begin{aligned} \mathcal{D}_a\widetilde{\sigma}^a{}_A{}^B &\approx 0, \\ \mathrm{Tr}(\widetilde{\sigma}^b F_{ab} - A_a\mathcal{D}_b\widetilde{\sigma}^b) &\approx 0, \qquad (3.9) \\ \mathrm{Tr}(\widetilde{\sigma}^a\widetilde{\sigma}^b F_{ab}) &\approx 0. \end{aligned}$$

The extra term is added to the vector constraint both for physical and computational reasons. The physical reason is that the modified constraint is the generator of spatial diffeomorphisms and thus has a well-defined geometric meaning. The computational reason is that the Poisson bracket algebra is simplified by the addition of this constraint. Although the motivation of AMT was not to make the constraints real, they observe in a footnote that the addition of this term yields a hermitian function on the phase space.

We have already shown that the Gauss constraint (2.12) is hermitian and that the scalar constraint (2.18) has nonzero real and imaginary parts. This already is sufficient to make the BRST charge constructed from the constraints (3.9) intrinsically complex, but it is enlightening to examine the reality properties of the modified vector constraint and demonstrate that it is purely imaginary.

Since F_{ab} is antisymmetric in its tensor indices, the vector constraint (2.15) can be rewritten as

$$\begin{aligned}\mathrm{Tr}(\widetilde{\sigma}^b F_{ab}) = &\underbrace{\frac{i}{\sqrt{2}}\Pi_{aN}{}^M(\mathcal{D}_b\widetilde{\sigma}^b{}_M{}^N)}_{\text{real}} \\ &\underbrace{-\frac{1}{2\sqrt{2}}\epsilon_{abc}\sigma^b{}_N{}^M D^c(\mathcal{D}_d\widetilde{\sigma}^d{}_M{}^N) + \frac{iq^{1/2}}{\sqrt{2}}D^b(K_{ab} - Kq_{ab})}_{\text{imaginary}} \approx 0.\end{aligned} \tag{3.10}$$

The last term, involving the extrinsic curvature K_{ab}, is the independent physical constraint and cannot be removed by adding the Gauss or scalar constraints to it. Thus, in order to give the vector constraint well-defined reality properties, it is necessary to cancel the real part, which is a multiple of the Gauss constraint. We could simply subtract it off as it is, but it is nonpolynomial and would leave the resulting modified vector constraint nonpolynomial. Instead, we consider the term $\mathrm{Tr}(A_a\mathcal{D}_b\widetilde{\sigma}^b)$ and observe that we can use (2.5) to separate it into two terms,

$$A_{aN}{}^M(\mathcal{D}_b\widetilde{\sigma}^b{}_M{}^N) = \Gamma_{aN}{}^M(\mathcal{D}_b\widetilde{\sigma}^b{}_M{}^N) + \frac{i}{\sqrt{2}}\Pi_{aN}{}^M(\mathcal{D}_b\widetilde{\sigma}^b{}_M{}^N). \tag{3.11}$$

The second term in Eq. (3.11) is exactly the term we wish to cancel in the vector constraint and we have shown that it is real. Furthermore, using (2.6), we see that the first term,

$$
\begin{aligned}
\Gamma_{aN}{}^{M}(\mathcal{D}_b\widetilde{\sigma}^b{}_M{}^N) &= \Gamma_{ab}\sigma^b{}_N{}^M(\sqrt{2}iq^{1/2}\Pi_{[cd]}\sigma^d{}_M{}^P\sigma^c{}_P{}^N) \\
&= iq^{1/2}\Gamma_{ab}\Pi_{[cd]}\epsilon^{bcd}.
\end{aligned}
\tag{3.12}
$$

is purely imaginary. By subtracting the term (3.11) from the vector constraint, we simultaneously cancel the real part and add an imaginary part, leaving the modified vector constraint purely imaginary. The trivial step of multiplying the vector constraint by i turns it into a real constraint.

From the BRST point of view, the constraints (3.9) are an improvement over the constraints (3.1). Nevertheless, the BRST charge that Ashtekar, Mazur and Torre construct from them,

$$
\begin{aligned}
Q = \int_\Sigma \mathrm{Tr}\Bigg[&\frac{\sqrt{2}}{i}\left(\underline{\eta}(\mathcal{D}_a\widetilde{\sigma}^a) + \eta^a(\widetilde{\sigma}^b F_{ab} - A_a\mathcal{D}_b\widetilde{\sigma}^b) + \underset{\sim}{\eta}\widetilde{\sigma}^a\widetilde{\sigma}^b F_{ab}\right) \\
&+ \underline{\eta\eta}\underline{\widetilde{\mathcal{P}}} + (\eta^a\partial_a\underline{\eta})\underline{\widetilde{\mathcal{P}}} - (\eta^b\partial_b\eta^a)\widetilde{\mathcal{P}}_a - (\eta^a\partial_a\underset{\sim}{\eta} + \underset{\sim}{\eta}\partial_a\eta^a)\widetilde{\widetilde{\mathcal{P}}} \\
&-2\underset{\sim}{\eta}(\partial_a\underset{\sim}{\eta})(\mathrm{Tr}\widetilde{\sigma}^a\widetilde{\sigma}^b)(\widetilde{\mathcal{P}}_b - \mathrm{Tr}\,A_b\underline{\widetilde{\mathcal{P}}})\Bigg],
\end{aligned}
\tag{3.13}
$$

must necessarily be complex because the scalar constraint remains complex.

3.3. Modified scalar constraint

Having achieved some computational simplification by modifying the vector constraint, Ashtekar, Mazur and Torre then do the same, to some extent, by modifying the scalar constraint. The new constraints are

$$
\begin{aligned}
\mathcal{D}_a\widetilde{\sigma}^a{}_A{}^B &\approx 0, \\
\mathrm{Tr}(\widetilde{\sigma}^b F_{ab} - A_a\mathcal{D}_b\widetilde{\sigma}^b) &\approx 0, \\
\mathrm{Tr}[\widetilde{\sigma}^a\widetilde{\sigma}^b F_{ab} + 2\widetilde{\sigma}^a A_a(\mathcal{D}_b\widetilde{\sigma}^b)] &\approx 0.
\end{aligned}
\tag{3.14}
$$

To determine the reality properties, we investigate the added term. Using (2.5) and (2.6), we find

$$
\begin{aligned}
\mathrm{Tr}\left(\widetilde{\sigma}^a A_a(\mathcal{D}_b\widetilde{\sigma}^b)\right) &= \mathrm{Tr}\left[\widetilde{\sigma}^a(\Gamma_a + \frac{i}{\sqrt{2}}\Pi_a)(\sqrt{2}iq^{1/2}\Pi_{[dc]}\sigma^c\sigma^d)\right] \\
&= \underbrace{-q\Pi^{ab}\Pi_{[ab]}}_{\text{real}} + \underbrace{\sqrt{2}iq\Gamma^{ab}\Pi_{[ab]}}_{\text{imaginary}} .
\end{aligned}
\tag{3.15}
$$

The extra term has nonzero real and imaginary parts. Furthermore, comparing with (2.11), we see that the imaginary parts do not cancel. The scalar constraint remains intrinsically complex and we once again conclude that the BRST charge,

$$
\begin{aligned}
Q' = \int_\Sigma \mathrm{Tr}\Bigg[\frac{\sqrt{2}}{i}\Big(&\underline{\eta}(\mathcal{D}_a\widetilde{\sigma}^a) + \eta^a(\widetilde{\sigma}^b F_{ab} - A_a\mathcal{D}_b\widetilde{\sigma}^b) + \underset{\sim}{\eta}(\widetilde{\sigma}^a\widetilde{\sigma}^b F_{ab} \\
&+ 2\widetilde{\sigma}^a A_a\mathcal{D}_b\widetilde{\sigma}^b)\Big) + \underline{\eta\eta}\underline{\widetilde{\mathcal{P}}} + (\eta^a\partial_a\underline{\eta})\underline{\widetilde{\mathcal{P}}} - (\eta^b\partial_b\eta^a)\widetilde{\mathcal{P}}_a \\
&- (\eta^a\partial_a\underset{\sim}{\eta} + \underset{\sim}{\eta}\partial_a\eta^a)\widetilde{\widetilde{\mathcal{P}}} - 2\underset{\sim}{\eta}\widetilde{\sigma}^a(\partial_a\underline{\eta})\underline{\widetilde{\mathcal{P}}} - 2\underset{\sim}{\eta}(\partial_a\underset{\sim}{\eta})(\mathrm{Tr}\widetilde{\sigma}^a\widetilde{\sigma}^b)\widetilde{\mathcal{P}}_b\Bigg] ,
\end{aligned}
\tag{3.16}
$$

constructed from the constraints (3.14), must be intrinsically complex.

IV. The reducible formalism

In Ref. [6], we developed a technique for constructing a real BRST charge for a system with complex constraints which satisfy the condition that the constraints together with their complex conjugates are all first-class. We now apply this method to self-dual gravity. In this section, we make use of the symbols $+$ and $-$ to indicate quantities built from self-dual and anti-self-dual variables, respectively.

To the original Ashtekar constraints,

$$
\begin{aligned}
{}^{+}\mathcal{D}_a\widetilde{\sigma}^a{}_A{}^B &\approx 0, \\
\mathrm{Tr}(\widetilde{\sigma}^{b\,+}F_{ab}) &\approx 0, \\
\mathrm{Tr}(\widetilde{\sigma}^a\widetilde{\sigma}^{b\,+}F_{ab}) &\approx 0,
\end{aligned}
\tag{4.1}
$$

we add the complex conjugate constraints,

$$
\begin{aligned}
{}^{-}\mathcal{D}_a\widetilde{\sigma}^a{}_A{}^B &\approx 0, \\
\mathrm{Tr}(\widetilde{\sigma}^{b\,-}F_{ab}) &\approx 0, \\
\mathrm{Tr}(\widetilde{\sigma}^a\widetilde{\sigma}^{b\,-}F_{ab}) &\approx 0.
\end{aligned}
\tag{4.2}
$$

to obtain a reducible set of constraints. The reducibility conditions follow from (2.12), (3.10), and (2.18). The relations among the constraints are

$$
\begin{aligned}
({}^{+}\mathcal{D}_a\widetilde{\sigma}^a)^\dagger = {}^{+}\mathcal{D}_a\widetilde{\sigma}^a &= -{}^{-}\mathcal{D}_a\widetilde{\sigma}^a = -({}^{-}\mathcal{D}_a\widetilde{\sigma}^a)^\dagger, \\
\mathrm{Tr}(\widetilde{\sigma}^b\,{}^{+}F_{ab}) - \mathrm{Tr}({}^{+}A_a\,{}^{+}\mathcal{D}_b\widetilde{\sigma}^b) &= -\,\mathrm{Tr}(\widetilde{\sigma}^b\,{}^{-}F_{ab}) + \mathrm{Tr}({}^{-}A_a\,{}^{-}\mathcal{D}_b\widetilde{\sigma}^b), \\
\mathrm{Tr}(\widetilde{\sigma}^a\widetilde{\sigma}^b\,{}^{+}F_{ab}) + D_a[\mathrm{Tr}(\widetilde{\sigma}^{a\,+}\mathcal{D}_b\widetilde{\sigma}^b)] &= \mathrm{Tr}(\widetilde{\sigma}^a\widetilde{\sigma}^b\,{}^{-}F_{ab}) + D_a[\mathrm{Tr}(\widetilde{\sigma}^{a\,-}\mathcal{D}_b\widetilde{\sigma}^b)].
\end{aligned}
\tag{4.3}
$$

In the case of first-class complex constraints linearly dependent with their complex conjugates, we showed in Ref. [6] that there exists an hermitian BRST charge Ω satisfying the requirements,

$$
\{\Omega,\Omega\} = 0, \qquad \Omega = \eta^a G_a + \eta^{\bar a}G_{\bar a} + \phi^i(Z_i{}^a\mathcal{P}_a + Z_i{}^{\bar a}\mathcal{P}_{\bar a}) + \text{“more,”}
$$

where “more” means terms of higher antighost number. Here G_a and $G_{\bar a}$ are the constraints, η^a and $\eta^{\bar a}$ the ghosts, and $\mathcal{P}_a$ and $\mathcal{P}_{\bar a}$ are the ghost momenta. These quantities satisfy $G_a^\dagger = G_{\bar a}$, $\eta^{a\dagger} = \eta^{\bar a}$ and $\mathcal{P}_a^\dagger = -\mathcal{P}_{\bar a}$. The full set of constraints satisfies the reducibility conditions $Z_i{}^a G_a + Z_i{}^{\bar a}G_{\bar a} \equiv 0$. The field ϕ^i is a ghost-of-ghost, which is real and of opposite Grassmann parity to η. We assume that $Z_i{}^{a\dagger} = -Z_i{}^{\bar a}$.

The reducibility relations between the Ashtekar constraints and their hermitian conjugates are almost bi-polynomial (*i.e.* having each side be polynomial in either self- or anti-self-dual variables). A bi-polynomial BRST charge would be nearly as useful for quantization as a purely polynomial one. Even though the resulting BRST charge operator would not be an ordinary differential operator, the ghosts of the conjugate constraints could be played off against the original constraints, making available a large set of physical states.

The relation between the Gauss constraint and the vector constraint is promising, but the relation between the scalar constraint and the Gauss constraint would be bi-polynomial if the divergence were of a vector density instead of a vector double density. This limits the usefulness of the construction and indicates that a useful BRST quantization must entail a more radical reworking of the standard formalism.

V. Construction of real constraints

The method of the previous section produces a real BRST charge, but is rather cumbersome. In this section, motivated by the form of the reducibility relations (4.3), we now construct an irreducible set of *real* constraints for self-dual gravity. As we have already discussed, the Gauss constraint (2.12) is real and the modified vector constraint (3.9) is purely imaginary. We now describe how to make the scalar constraint real.

We consider the term $\mathrm{Tr}(\widetilde{\sigma}^a \mathcal{D}_a \mathcal{D}_b \widetilde{\sigma}^b)$. We expand the first Ashtekar derivative using $\mathcal{D}_a = D_a + \frac{i}{\sqrt{2}}\Pi_a$ to get

$$\begin{aligned}\mathrm{Tr}(\widetilde{\sigma}^a \mathcal{D}_a \mathcal{D}_b \widetilde{\sigma}^b) = \mathrm{Tr}(\widetilde{\sigma}^a D_a \mathcal{D}_b \widetilde{\sigma}^b) &+ \frac{i}{\sqrt{2}} \widetilde{\sigma}^a{}_B{}^A \Pi_{aA}{}^D \mathcal{D}_b \widetilde{\sigma}^b{}_D{}^B \\ &- \frac{i}{\sqrt{2}} \widetilde{\sigma}^a{}_B{}^A \Pi_{aD}{}^B \mathcal{D}_b \widetilde{\sigma}^b{}_A{}^D.\end{aligned} \tag{5.1}$$

We rewrite this as

$$\mathrm{Tr}(\widetilde{\sigma}^a \mathcal{D}_a \mathcal{D}_b \widetilde{\sigma}^b) = \mathrm{Tr}(\widetilde{\sigma}^a D_a \mathcal{D}_b \widetilde{\sigma}^b) + \frac{i}{\sqrt{2}} [\widetilde{\sigma}^a, \Pi_a]_B{}^A \mathcal{D}_b \widetilde{\sigma}^b{}_A{}^B, \tag{5.2}$$

and recognize the commutator as the Gauss constraint. The last term is then quadratic in the Gauss constraint,

$$\begin{aligned}\mathrm{Tr}(\widetilde{\sigma}^a \mathcal{D}_a \mathcal{D}_b \widetilde{\sigma}^b) &= \mathrm{Tr}(\widetilde{\sigma}^a D_a \mathcal{D}_b \widetilde{\sigma}^b) - (\mathcal{D}_a \widetilde{\sigma}^a{}_B{}^A)(\mathcal{D}_b \widetilde{\sigma}^b{}_A{}^B) \\ &\equiv \mathrm{Tr}(\widetilde{\sigma}^a D_a \mathcal{D}_b \widetilde{\sigma}^b) - \mathrm{Tr}[(\mathcal{D}_a \widetilde{\sigma}^a)(\mathcal{D}_b \widetilde{\sigma}^b)].\end{aligned} \tag{5.3}$$

The first term on the right side of (5.3) is exactly the imaginary part of the standard scalar constraint (2.18), while the second term is purely real. By adding (5.3) to the standard scalar constraint we cancel the imaginary part and add a real part, leaving the modified constraint,

$$\mathrm{Tr}(\widetilde{\sigma}^a \widetilde{\sigma}^b F_{ab}) + \mathrm{Tr}(\widetilde{\sigma}^a \mathcal{D}_a \mathcal{D}_b \widetilde{\sigma}^b) \approx 0, \tag{5.4}$$

purely real. It is also very nearly polynomial. The double derivative in the last term, when expanded, contains an unfortunate term, proportional to $\partial_a q^{1/2}$, that is not polynomial. The remaining terms are polynomial, as are the Gauss and vector constraints.

We now have a set of real constraints for the Ashtekar formulation of self-dual gravity,

$$\begin{aligned} \mathcal{D}_a \widetilde{\sigma}^a{}_A{}^B &\approx 0, \\ i\,\mathrm{Tr}(\widetilde{\sigma}^b F_{ab}) - i\,\mathrm{Tr}(A_a \mathcal{D}_b \widetilde{\sigma}^b) &\approx 0, \\ \mathrm{Tr}(\widetilde{\sigma}^a \widetilde{\sigma}^b F_{ab}) + \mathrm{Tr}(\widetilde{\sigma}^a \mathcal{D}_a \mathcal{D}_b \widetilde{\sigma}^b) &\approx 0. \end{aligned} \tag{5.5}$$

From a BRST viewpoint, we have returned to the realm of real constraints in which the standard BRST methods apply. A real BRST charge is therefore known to exist, as proven by Henneaux and Teitelboim [7].

In summary, all three sets of constraints upon which AMT have constructed their BRST charges are intrinsically complex. Their BRST charges are also complex and are therefore not viable precursors to a useful BRST quantization of self-dual gravity. We have given two methods by which a real BRST charge can be constructed for self-dual gravity. (1) Following the procedure we developed for complex extensions of real systems, we have extended the Ashtekar constraints to include their complex conjugates plus the resulting reducibility conditions. The BRST charge constructed by this approach is real but is not polynomial nor bi-polynomial. (2) By a judicious remixing of the original Ashtekar constraints, we have constructed a set of constraints which are real and very nearly polynomial. This again yields a real BRST charge, which is considerably simpler than in the reducible case.

While we have succeeded in constructing a set of constraints which are real it has been at the cost of sacrificing polynomiality. The difficulty that we have encountered in constructing a set of constraints which is both real and polynomial arises from the appearance of the double covariant derivative in the imaginary part of the scalar constraint. In the original form (2.1) of the (complex) constraints only a single covariant derivative appears, and there in the special form of a divergence of a vector density. The fortuitous cancellation of the action of the covariant derivative on the vector index by the action of the covariant derivative on the density factor is what allows the construction of a polynomial form of the constraints. This important feature does not carry over to attempts to make the scalar constraint real. This appears to be a serious impediment to achieving the goal of constructing a BRST charge for Ashtekar gravity which is both real and polynomial.

APPENDIX

Many of the rules of spinor algebra and spinor analysis can be found in chapter 5 and Appendix A of Ref. [8]. For convenience, we collect in this appendix the rules and notation used in this paper for calculating with spinors in Ashtekar gravity.

The standard representation of $SU(2)$ spinors is in terms of the Pauli matrices ${\tau^i}_A{}^B$, where i identifies the different Pauli matrices and the indices (A, B) identify the matrix elements of τ^i:

$$
{\tau^1}_A{}^B := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad {\tau^2}_A{}^B := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad {\tau^3}_A{}^B := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.1}
$$

The algebra of the Pauli matrices is given by:

$$
{\tau^i}_A{}^B {\tau^j}_B{}^C = i\epsilon^{ijk} \tau_{kA}{}^C + \delta^{ij} \delta_A{}^C. \tag{A.2}
$$

Given a real vector triad E^a_i, the $SU(2)$ soldering form ${\sigma^a}_A{}^B$ is defined by:

$$
{\sigma^a}_A{}^B \equiv -\frac{i}{\sqrt{2}} E^a_i {\tau^i}_A{}^B. \tag{A.3}
$$

The fundamental relation between $SU(2)$ spinors and the 3-metric q^{ab} follows from equations

(A.2) and (A.3):

$$\mathrm{Tr}\,\sigma^a\sigma^b \equiv \sigma^a{}_A{}^B\sigma^b{}_B{}^A = -q^{ab}. \tag{A.4}$$

A number of other useful relations also follow from equations (A.2) and (A.3):

$$\begin{aligned} [\sigma^a,\sigma^b]_A{}^B &= \sqrt{2}\epsilon^{abc}\sigma_{cA}{}^B, \\ \mathrm{Tr}(\sigma^a\sigma^b\sigma^c) &= -\frac{1}{\sqrt{2}}\epsilon^{abc}, \\ \mathrm{Tr}(\sigma^a\sigma^b\sigma^c\sigma^d) &= \tfrac{1}{2}(q^{ab}q^{cd}-q^{ac}q^{bd}+q^{ad}q^{bc}). \end{aligned} \tag{A.5}$$

$SU(2)$ spinor indices are raised and lowered with the (nowhere vanishing) antisymmetric matrices

$$\epsilon_{AB} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \epsilon^{AB} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{A.6}$$

where ϵ^{AB} is the inverse of ϵ_{AB} as defined by the relation $\epsilon^{AB}\epsilon_{AC} = \delta^B{}_C$. The conventions for raising and lowering spinor indices are:

$$\lambda^A = \epsilon^{AB}\lambda_B, \qquad \lambda_B = \lambda^A\epsilon_{AB}, \tag{A.7}$$

where care must be taken with the order of the indices because of the antisymmetry of ϵ_{AB}. A mnemonic device for remembering these conventions is to remember that spinor summations are "from upper left to lower right."

In considering the reality properties of expressions in spinor form, we need to consider the hermiticity properties of spinors. The Pauli spinors (A.1) are manifestly hermitian. By examining the representation of the $SU(2)$ soldering form $\sigma^a{}_A{}^B$ in terms of Pauli matrices and the real triad E^a_i,

$$\sigma^a{}_A{}^B \equiv -\frac{i}{\sqrt{2}}E^a_i\tau^i{}_A{}^B = \frac{1}{\sqrt{2}}\begin{pmatrix} -iE^a_3 & -E^a_2 - iE^a_1 \\ E^a_2 - iE^a_1 & iE^a_3 \end{pmatrix}, \tag{A.8}$$

we see that $\sigma^a{}_A{}^B$ is anti-hermitian. The $SU(2)$ connection $A_{aA}{}^B$ has a similar representation in terms of Pauli matrices, $A_{aA}{}^B = -\frac{i}{2}A^i_a\tau^i{}_A{}^B$, but the components A^i_a are complex, so that $A_{aA}{}^B$ does not have well-defined hermiticity properties; it is neither anti-hermitian nor hermitian. We recall, however, the definition of $A_{aA}{}^B$ in equation (2.5),

$$ {}^{\pm}A_{aA}{}^B = \Gamma_{aA}{}^B \pm \frac{i}{\sqrt{2}}\Pi_{aA}{}^B. \tag{A.9} $$

The "real" and "imaginary" parts of $A_{aA}{}^B$ are $\frac{i}{\sqrt{2}}\Pi_{aA}{}^B \equiv \frac{i}{\sqrt{2}}\Pi_{ab}\sigma^b{}_A{}^B$ and $\Gamma_{aA}{}^B \equiv \Gamma_{ab}\sigma^b{}_A{}^B$ respectively. The tensorial factors Γ_{ab} and Π_{ab} are real, so $\Gamma_{aA}{}^B$ and $\Pi_{aA}{}^B$ have the same hermiticity properties as $\sigma_{aA}{}^B$ and are therefore anti-hermitian.

The product of two $SU(2)$ matrices is, in general, neither hermitian nor anti-hermitian, even when the original matrices have well-defined hermiticity properties. However, the symmetrized and antisymmetrized products of hermitian and anti-hermitian $SU(2)$ matrices do have well defined hermiticity properties. We let $H_{aM}{}^N$ be an arbitrary hermitian matrix and $A_{aM}{}^N$ be an arbitrary anti-hermitian matrix,

$$ H_a^\dagger = H_a, \qquad A_a^\dagger = -A_a. \tag{A.10} $$

The symmetrized product of two hermitian matrices is hermitian,

$$ [H_{(a}H_{b)}]_M{}^N \equiv H_{aM}{}^P H_{bP}{}^N + H_{bM}{}^P H_{aP}{}^N = H_{cM}{}^N. \tag{A.11} $$

while the antisymmetrized product of two hermitian matrices is anti-hermitian,

$$ [H_{[a}H_{b]}]_M{}^N \equiv H_{aM}{}^P H_{bP}{}^N - H_{bM}{}^P H_{aP}{}^N = A_{cM}{}^N. \tag{A.12} $$

We state these and similar rules more concisely as

$$ \begin{aligned} H_{(a}H_{b)} &= H_c, \qquad & H_{[a}H_{b]} &= A_c, \\ H_{(a}A_{b)} &= A_c, \qquad & H_{[a}A_{b]} &= H_c, \\ A_{(a}A_{b)} &= H_c, \qquad & A_{[a}A_{b]} &= A_c. \end{aligned} \tag{A.13} $$

The square of an hermitian matrix is hermitian, as is the square of an anti-hermitian matrix,

$$ H_a H_a = H_b, \qquad A_a A_a = H_b. \tag{A.14} $$

Finally, we observe that the trace of a hermitian matrix is always real and that the trace of an anti-hermitian matrix is always purely imaginary,

$$\mathrm{Tr}\, H_a \equiv H_{aM}{}^{M} \in \mathbf{R}, \qquad \mathrm{Tr}\, A_a \equiv A_{aM}{}^{M} \in \mathbf{C}. \tag{A.15}$$

REFERENCES

1. M.H. Goroff and A. Sagnotti, *Phys. Lett.* **160B** (1985) 81;
   Nucl. Phys. **B266** (1986) 709.
2. A. Ashtekar, *Phys. Rev. Lett.*, **57** (1986) 2244;
   Phys. Rev. **D36** (1987) 1587.
3. B. Brügmann, R. Gambini, J. Pullin, *Nucl. Phys.* **B385** (1992) 587.
4. E. S. Fradkin and G. A. Vilkovisky, *Phys. Lett.* **55B** (1975) 224;
   I. A. Batalin and G. A. Vilkovisky, *Phys. Lett.* **69B** (1977) 309;
   E. S. Fradkin and T. E. Fradkina, *Phys. Lett.* **72B** (1978) 343.
5. A. Ashtekar, P. Mazur and C. Torre, *Phys. Rev.* **D36** (1987) 2955.
6. T.J. Allen and D.B. Crossley, "Reality Conditions, Reducibility and Ghosts," UW-Madison Preprint MAD/TH-94-10, hepth/95mmnnn.
7. M. Henneaux, *Phys. Rep.* **126**, 1 (1985); M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*, (Princeton University Press, Princeton, 1992)
8. A. Ashtekar, *Lectures on Non-Perturbative Canonical Gravity*, Notes prepared in collaboration with R.S. Tate, (World Scientific, Singapore, 1991)