Vibrational modes of a rotating string

Theodore J. Allen
Department of Physics, University of Wisconsin-Madison,
1150 University Avenue, Madison, WI 53706 USA
and
Department of Physics, Eaton Hall
Hobart and William Smith Colleges
Geneva, NY 14456 USA

Jeffrey R. Schmidt*
Department of Physics, University of Wisconsin-Parkside,
Kenosha, WI 53141 USA

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*Corresponding author. Fax: (414) 595-2056. Phone: (414) 595-2134. Email: jeff@rustam.uwp.edu

Abstract

Motivated by non-relativistic models of a QCD string, we examine the system of a non-relativistic string in uniform rotational motion with one end fixed and with a mass (quark) attached to the other end. A QCD string has no purely longitudinal modes so some constraint must be imposed upon the non-relativistic system to exclude these modes. Accordingly, we examine the cases that the string is either inextensible or purely transverse. For each case we solve first the discretized string and then do the continuum case. We find the small amplitude oscillatory motions and frequencies of oscillation of the string and mass. We show that the assumption of a node at the end of the inextensible string produces frequencies that are very close to the actual frequencies. For the transverse string, we show that keeping centrifugal terms for the motion of the masses comprising the string leads to an unstable mode.

1 Introduction

Recent lattice simulations [1, 2] show that QCD leads to the formation of a chromoelectric flux tube between two quark color sources [3]. The motion of a quark-antiquark-flux tube system is usually investigated by starting from the static quark configuration and then performing a Born-Oppenheimer quantization of the resulting static potential [4, 5]. Here we investigate the motion of a string with a single mass on the end in a uniformly rotating frame as a model of the motion of a heavy quark on the end of a flux tube. We consider two cases in which the motion of the string excludes purely longitudinal motion. In the first two sections we analyze the inextensible string in both the discrete and continuum formalisms. In the third and fourth sections we analyze a string in which the displacements from a straight line are purely transverse. This is the kinematical model considered by Merlin and Paton [5]. Our analysis is purely non-relativistic and classical. We discuss briefly the connection to the quantum theory as well as to the full relativistic motion of a string with very heavy ends.

2 Discrete Inextensible String Model

Following the spirit of the analysis of a non-relativistic string as formulated by Isgur and Paton [4], we first construct the string action as a discrete set of N masses m, and one terminal mass M, connected by $N+1$ massless links. In our case, we take the length of the links to be fixed and the total length to be ℓ. This has the effect of removing purely longitudinal motion of the string. This inextensible string has a Lagrangian that is the sum of the individual kinetic energies

$$L_{\text{string}} = \frac{m}{2}\sum_{i=1}^{N} \dot{\mathbf{x}}_i^2, \tag{1}$$

subject to the fixed length constraint

$$\sum_{i=1}^{N+1} |\mathbf{x}_i - \mathbf{x}_{i-1}| = \ell. \tag{2}$$

In uniformly rotating coordinates, this Lagrangian becomes

$$L_{\text{string}} = \frac{m}{2}\sum_{i=1}^{N} \left\{ \dot{\mathbf{x}}_i^2 - 2\omega \cdot \dot{\mathbf{x}}_i \times \mathbf{x}_i + \omega^2 \mathbf{x}_i^2 \right\}. \tag{3}$$

We consider vibrations in the plane of rotation and re-express the positions $\mathbf{x}_n$ of the masses in terms of their angles θ_i from a straight, uniformly rotating string. Using the angles

θ_i eliminates the fixed length condition of the string. Figure 1 shows the relationship of the angles to the positions of the masses.

$$x_n = \frac{\ell}{N+1}\sum_{i=1}^{n}\cos\theta_i, \qquad y_n = \frac{\ell}{N+1}\sum_{i=1}^{n}\sin\theta_i. \tag{4}$$

Substituting the relations (4) in second-order approximation into the Lagrangian (3), we find

$$\begin{aligned} L_{\mathrm{string}} &= \frac{m\ell^2}{2}\sum_{n=1}^{N}\left[\frac{1}{N+1}\sum_{i=1}^{n}\dot{\theta}_i\right]^2 + \frac{m\omega^2\ell^2}{2}\sum_{n=1}^{N}\left[\left(\frac{1}{N+1}\sum_{i=1}^{n}(1-\frac{\theta_i^2}{2})\right)^2 + \left(\frac{1}{N+1}\sum_{i=1}^{n}\theta_i\right)^2\right] \\ &= \frac{m\ell^2}{2}\dot{\theta}\cdot\mathbf{K}\cdot\dot{\theta} + \frac{m\ell^2\omega^2}{2}\theta\cdot[\mathbf{R}+\mathbf{K}]\cdot\theta, \end{aligned} \tag{5}$$

where the matrices $\mathbf{K}$ and $\mathbf{R}$ are given by

$$K_{i,j} = \frac{1}{(N+1)^2}\min(N+1-i, N+1-j), \qquad R_{i,j} = \frac{(i-N-1)(i+N)}{2(N+1)^2}\delta_{i,j} \tag{6}$$

The Lagrangian for the terminating mass M similarly becomes

$$L_{\mathrm{M}} = \frac{M\ell^2}{2}\left[\frac{1}{N+1}\sum_{i}^{N+1}\dot{\theta}_i\right]^2 + \frac{M\omega^2\ell^2}{2}\left[\left(\frac{1}{N+1}\sum_{i}^{N+1}(1-\frac{\theta_i^2}{2})\right)^2 + \left(\frac{1}{N+1}\sum_{i}^{N+1}\theta_i\right)^2\right] \tag{7}$$

$$= \frac{M\ell^2}{2}\dot{\theta}\cdot\mathbf{U}\cdot\dot{\theta} + \frac{M\ell^2\omega^2}{2}\theta\cdot[-\frac{1}{N+1}\mathbf{1}+\mathbf{U}]\cdot\theta, \tag{8}$$

with

$$U_{i,j} = \frac{1}{(N+1)^2}. \tag{9}$$

Computation of the normal modes and frequencies Ω by diagonalization of the energy matrices in the case that the mass on the end is three times the string mass yields the frequencies in Table 1. The first mode is a purely rotating string without any waves. The other modes have frequencies that are proportional to the angular velocity ω of the bulk rotation. The frequencies could alternatively be given in terms the tension b_0 at the end of the string, the mass M, and the radius r of the orbit through $\omega = \sqrt{\frac{b_0}{Mr}}$. The indices ν, defined by the relation

$$\frac{\nu(\nu+1)}{2} = \frac{\Omega^2+\omega^2}{\omega^2}, \tag{10}$$

that occurs in the analytical solution of the continuum case given below, have differences that tend to a constant.

3 Continuum Inextensible String

The Lagrangian for an elastic string

$$L = \frac{1}{2}\int_0^\ell ds\, \Big(\rho(s)\dot{\mathbf{X}}^2(s) - E\mathbf{X}'^2(s)\Big)\,, \tag{11}$$

in the case that the string is inextensible,

$$\int_0^\ell ds\, \sqrt{\mathbf{X}'^2(s)} = \ell, \tag{12}$$

yields the continuum Lagrangian,

$$L = \frac{1}{2}\int_0^\ell \rho(s)\dot{\mathbf{X}}^2(s) ds, \tag{13}$$

corresponding to the discrete case, Eq. (1).

To examine vibrational motion in the plane of rotation, we pass to a rotating frame and define new variables $x(s)$ and $y(s)$ as

$$X(s) = x(s)\cos\omega t - y(s)\sin\omega t, \qquad Y(s) = x(s)\sin\omega t + y(s)\cos\omega t. \tag{14}$$

The general Lagrangian with arbitrary mass density becomes

$$L = \frac{1}{2}\int_0^\ell ds'\, \rho(s')\, \Big\{\dot{x}^2(s') + \dot{y}^2(s') - 2\omega(\dot{x}(s)y(s) + x(s)\dot{y}(s)) + \omega^2(x^2(s') + y^2(s'))\Big\}\,. \tag{15}$$

We will later assume that the mass density along the string is uniformly ρ_0 and that there is a mass M on the end.

We solve the constraints (12) by introducing the collective coordinates

$$x(s,t) = \int_0^s \cos\theta(s',t)ds', \qquad y(s,t) = \int_0^s \sin\theta(s',t)ds'. \tag{16}$$

Notice that the first term in the Coriolis contribution is cubic in theta and its derivatives, and the second portion of the Coriolis term can be neglected in the limit of small amplitude motions.

With the aid of the Heaviside function Θ

$$\Theta(s-u) = \begin{cases} 1 & s > u \\ \frac{1}{2} & s = u \\ 0 & s < u. \end{cases} \tag{17}$$

we compute the functional derivatives

$$\begin{aligned}
\frac{\delta}{\delta\theta(u)} x(s,t) &= -\Theta(s-u)\sin\theta(u), &(18)\\
\frac{\delta}{\delta\theta(u)} y(s,t) &= \Theta(s-u)\cos\theta(u), &(19)\\
\frac{\delta}{\delta\dot\theta(u)} \dot x(s,t) &= -\Theta(s-u)\sin\theta(u), &(20)\\
\frac{\delta}{\delta\dot\theta(u)} \dot y(s,t) &= \Theta(s-u)\cos\theta(u), &(21)
\end{aligned}$$

and obtain

$$\frac{\delta}{\delta\theta(u)} L = -\int_u^\ell ds\,\rho(s)\,\omega^2\,\left[\sin\theta(u)x(s) - \cos\theta(u)y(s)\right], \tag{22}$$

and

$$\frac{\delta}{\delta\dot\theta(u)} L = -\int_u^\ell ds\,\rho(s)\,\left[\sin\theta(u)\dot x(s) - \cos\theta(u)\dot y(s)\right]. \tag{23}$$

The Coriolis terms have functional derivatives that are quadratic in θ and its time derivative and can be neglected. The only terms in the equations of motion that survive as the small angle limit is performed lead to the equation of motion

$$\int_u^\ell ds\,\rho(s)\,\ddot y(s) = -\omega^2 \int_u^\ell ds\,\rho(s)\,[\sin\theta(u)x(s) - \cos\theta(u)y(s)]. \tag{24}$$

Evaluating the right hand side in the small angle limit $x(s)\to s$, $\cos\theta(s)\to 1$, $\sin\theta(s)\approx \tan\theta(s) = \frac{dy(s)}{ds}$, we find

$$\int_u^\ell ds\,\rho(s)\,\ddot y(s) = -\omega^2 \int_u^\ell ds\,s\,\rho(s)\,\frac{dy(u)}{du} + \omega^2 \int_u^\ell ds\,\rho(s)\,y(s). \tag{25}$$

Inserting the explicit density function

$$\rho(s) = \rho_0 + M\delta(s-\ell), \tag{26}$$

we obtain

$$\begin{aligned}
-\int_u^\ell ds\,\rho_0\,\ddot y(s) - M\,\ddot y(\ell) &= \omega^2 \int_u^\ell ds\,s\,\rho_0\,\frac{dy(u)}{du} - \omega^2 \int_u^\ell ds\,\rho_0\,y(s)\\
&\quad + \omega^2 M\ell\,\frac{dy(u)}{du} - \omega^2 M\,y(\ell), &(27)
\end{aligned}$$

which is an equation of motion

$$-\int_u^\ell ds\,\rho_0\,\ddot y(s) = \frac{\omega^2\rho_0}{2}(\ell^2 - u^2)\,\frac{dy(u)}{du} - \omega^2\,\rho_0 \int_u^\ell y(s)\,ds + \omega^2 M\ell\,\frac{dy(u)}{dt}, \tag{28}$$

and a boundary condition

$$-\ddot{y}(\ell) = \omega^2 \ell \, \frac{dy(\ell)}{du} - \omega^2 \, y(\ell). \tag{29}$$

Harmonic solutions with frequency Ω,

$$y(s,t) = y(s) \, e^{i\Omega t}, \tag{30}$$

then satisfy

$$-\Omega^2 \rho_0 y(u) = \frac{\rho_0 \omega^2}{2} \, \frac{d}{du} \left[(\ell^2 - u^2) \frac{dy}{du} \right] + \rho_0 \omega^2 \, y(u) + M \ell \omega^2 \, \frac{d^2 y}{du^2}, \tag{31}$$

with

$$\frac{(\Omega^2 + \omega^2)}{\ell \omega^2} \, y(\ell) = \frac{dy(\ell)}{du}. \tag{32}$$

In terms of the variables

$$u = \sqrt{\ell^2 + 2 \frac{M \ell}{\rho_0}} \, \sigma, \tag{33}$$

we obtain the Legendre equation,

$$\frac{d}{d\sigma} \left[(1 - \sigma^2) \frac{dy}{d\sigma} \right] + 2 \frac{\Omega^2 + \omega^2}{\omega^2} \, y = 0, \tag{34}$$

in which the end of the string is located at σ value

$$\sigma_e = \sqrt{\frac{1}{1 + 2 \frac{M_e}{M_s}}}, \tag{35}$$

with M_s being the mass of the string and M_e being the terminating mass.

Solutions of Eq. (34) will be of the form

$$y_\nu = A P_\nu(\sigma) + B Q_\nu(\sigma), \tag{36}$$

where ν satisfies

$$\frac{\Omega^2 + \omega^2}{\omega^2} = \frac{\nu(\nu + 1)}{2}. \tag{37}$$

In terms of the variable $\phi = \cos \sigma$, the end of the string is at $\phi_e = \cos \sigma_e$ and the boundary condition becomes

$$-y(\phi_e) \, \frac{\nu(\nu + 1)}{2} \, \tan \phi_e = \frac{dy(\phi_e)}{d\phi}. \tag{38}$$

The Legendre functions in general do not vanish at the origin for non-integral ν. The values of the Legendre functions and their derivatives at the origin are

$$\begin{aligned}
P_\nu(0) &= \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi}\Gamma(\frac{\nu}{2}+1)} \cos\frac{\nu\pi}{2}, \\
Q_\nu(0) &= -\frac{\sqrt{\pi}\Gamma(\frac{\nu+1}{2})}{2\Gamma(\frac{\nu}{2}+1)} \sin\frac{\nu\pi}{2}, \qquad (39)\\
P'_\nu(0) &= \frac{2\Gamma(\frac{\nu}{2}+1)}{\sqrt{\pi}\Gamma(\frac{\nu+1}{2})} \sin\frac{\nu\pi}{2}, \\
Q'_\nu(0) &= \frac{\sqrt{\pi}\Gamma(\frac{\nu}{2}+1)}{\Gamma(\frac{\nu+1}{2})} \cos\frac{\nu\pi}{2}. \qquad (40)
\end{aligned}$$

Odd parity solutions that vanish at the origin are of the form

$$y_\nu^{odd}(\sigma) = A(\pi \sin\frac{\pi\nu}{2} P_\nu(\sigma) + 2\cos\frac{\pi\nu}{2} Q_\nu(\sigma)). \qquad (41)$$

These functions have simple Fourier representations

$$y_\nu^{odd}(\phi) = A' \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k(\nu+1)_k}{k!(\nu+\frac{3}{2})_k} \cos[(\nu+2k+1)\phi - \frac{\nu\pi}{2}]. \qquad (42)$$

The transcendental equation that determines the spectrum is quite difficult to solve analytically due to the complicated nature of the solutions, but a numerical computation gives the frequencies and indices in Table 2. The numerical analysis of the functions P_ν and Q_ν are complicated by the fact that their Fourier series are very slowly converging. This is especially true of Q_ν, given that $Q_\nu(z)$ is defined in terms of P_ν,

$$Q_\nu(z) = \frac{-\pi\, P_\nu(-z) + \pi\cos\nu\pi\ P_\nu(z)}{2\ \sin\nu\pi}, \qquad (43)$$

which in turn is defined by the series

$$P_\nu(z) = \sum_{n=0}^{\infty} \frac{(-\nu)_n(\nu+1)_n}{(n!)^2} \left(\frac{1-z}{2}\right)^n, \qquad (44)$$

leaving Q_ν poorly converging near the end of the string.

In the limit of large indices, $\nu \to \infty$, however, the Legendre functions become much simpler

$$\begin{aligned}
P_\nu(\phi) &\to \frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{3}{2})}\sqrt{\frac{2}{\pi\sin\phi}} \cos[(\nu+\frac{1}{2})\phi - \frac{\pi}{4}], \\
Q_\nu(\phi) &\to \frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{3}{2})}\sqrt{\frac{\pi}{2\sin\phi}} \sin[(\nu+\frac{1}{2})\phi + \frac{\pi}{4}], \qquad (45)
\end{aligned}$$

which can be seen by computing the asymptotic limit of their well-known Fourier representations [6].

In Fig. 2 we see that for a large terminating mass ($M_e = 3M_s$) there is almost a node at the end of the string for each of the first three modes. The boundary condition

$$y_\nu^{odd}(\sigma_e) \to 0, \tag{46}$$

is very nearly correct and becomes a better approximation for higher modes. This approximate boundary condition can be solved in the asymptotic limit;

$$1 = \tan\left[(\nu + \frac{1}{2})\phi_e - \frac{\pi\nu}{2}\right], \tag{47}$$

or

$$\nu_n = \frac{(4n+1)\pi + 2\phi_e}{4(\frac{\pi}{2} - \phi_e)}. \tag{48}$$

The ν values for consecutive modes become evenly spaced

$$\Delta\nu \to \frac{\pi}{|\phi_e - \frac{\pi}{2}|}. \tag{49}$$

The values of ν and the corresponding frequencies Ω computed from Eq. (37) are displayed in Table 3 for $\phi_e = 1.1832$. The first listed mode is the one for which the string is straight.

We see that there is quite good agreement between the exact frequencies and the approximate one computed by assuming a node at the end.

4 Discrete Transverse String

We investigate the motion of a string whose vibrations are forced to be purely transverse. Following Merlin and Paton [5], we examine a string with mass points placed at positions determined by the equilibrium position $\mathbf{r}$ of the mass on the end

$$\mathbf{r}_n = \frac{n\mathbf{r}}{N+1} + \mathbf{q}_n, \tag{50}$$

with the vectors $\mathbf{q}_n$ orthogonal to $\mathbf{r}$. The action

$$\frac{1}{2}M\dot{\mathbf{r}}^2 - b_0 r - c_0 + \frac{1}{2}m\sum_{n=1}^{N}\dot{\mathbf{r}}_n^2 - \frac{b_0(N+1)}{2r}\sum_{n=1}^{N+1}(\mathbf{q}_n - \mathbf{q}_{n-1})^2, \tag{51}$$

when restricted to the $\mathbf{q}_n$ and specialized to motion with fixed r in the orbital plane, becomes

$$\begin{aligned} L &= \frac{1}{2}m\sum_{n=1}^{N}\left[(\omega\, y_n)^2 + (\omega\, x_n + \dot{y}_n)^2\right] - \frac{b_0(N+1)}{2r}\sum_{n=1}^{N+1}(y_n - y_{n-1})^2, \\ &\equiv \frac{m}{2}\left(\dot{\mathbf{y}}\cdot\mathbf{T}\cdot\dot{\mathbf{y}} + \omega^2\mathbf{y}\cdot\mathbf{T}\cdot\mathbf{y} - \beta\mathbf{y}\cdot\mathbf{U}\cdot\mathbf{y}\right) \end{aligned} \tag{52}$$

with $x_n = \frac{nr}{N+1}$, $\beta = \frac{b_0(N+1)}{mr}$, and $\mathbf{T}$ and $\mathbf{U}$ given by

$$\mathbf{T} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \frac{M}{m} \end{pmatrix}, \qquad \mathbf{U} = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 1 \end{pmatrix}. \tag{53}$$

It is important to note that in the model considered by Merlin and Paton [5], the centrifugal terms $\frac{1}{2}\omega^2 y_n^2$ are omitted.

We may solve the model by introducing the $K \times K$ matrices

$$V_K = \beta \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}_{K\times K}, \tag{54}$$

and the determinants

$$D_K = \det(V_K - \lambda \mathbf{1}). \tag{55}$$

We find the secular equation for the model considered by Merlin and Paton to be

$$(1 - \frac{M}{m}\lambda) = \frac{D_{N-1}}{D_N} = \frac{\sin(N\theta)}{\sin((N+1)\theta)}, \tag{56}$$

where λ is the square of the frequencies Ω, $\lambda = \Omega^2$ and θ is defined by $\cos\theta = 1 - \frac{\lambda}{2}$. This last relation is found by solving the three term recursion relation for D_K that follows from expanding the determinant in minors,

$$D_K = (2-\lambda)D_{K-1} - D_{K-2}, \qquad D_0 = 1, \qquad D_1 = 2 - \lambda, \tag{57}$$

and solving by assuming that $D_K = c_+\xi_+^K + c_-\xi_-^K$ where $\xi_\pm$ are the solutions to the quadratic $\xi^2 - (2-\lambda)\xi + 1 = 0$.

In the limit of an infinite mass M, the values of θ and λ that follow from Eq. (56) are

$$\theta = \frac{m\pi}{N+1}, \quad m = 1, 2, \ldots, N+1, \tag{58}$$

$$\lambda = 2\left(1 - \cos(\frac{n\pi}{N+1})\right) \equiv \frac{\Omega^2}{\beta}. \tag{59}$$

If the centrifugal terms are kept, the result is modified only by the subtraction of ω^2 from the square of the vibrational frequencies:

$$\lambda = 2\left(1 - \cos(\frac{n\pi}{N+1})\right) = \frac{\Omega^2 + \omega^2}{\beta}. \tag{60}$$

5 Continuum Transverse String

The continuum action corresponding to the discrete action (52) is

$$L = \frac{1}{2}\int \left[\rho(s)\dot{y}^2 + \rho(s)\omega^2 y^2 - b_0 y'^2\right], \tag{61}$$

where we will again choose the density along the string to be given by Eq. (26). The equations of motion are the familiar ones,

$$\begin{aligned} \rho_0\, \ddot{y} - \rho_0\, \omega^2 y - b_0\, y'' &= 0, \\ M\ddot{y}(\ell) - M\omega^2\, y(\ell) + b_0 y'(\ell) &= 0, \end{aligned} \tag{62}$$

leading to the solution

$$\begin{aligned} y(s,t) &= y_0 \sin(\varpi s)\, e^{i\Omega t}, \\ \varpi^2 &= \frac{\rho_0(\Omega^2+\omega^2)}{b_0}, \end{aligned} \tag{63}$$

with the vibrational frequencies Ω determined by the transcendental equation following from the boundary condition. This boundary equation,

$$\varpi \tan(\varpi \ell) = \frac{\rho_0}{M}, \tag{64}$$

is nearly identical to that obtained in the static case [7]. In the case that the mass M is infinite, the continuum boundary condition (64) and the discrete secular equation (56) yield the same answers for the frequencies for modest numbers of string masses N. In the case of the Merlin-Paton string model, everything is the same except that ω^2 is removed from all formulas, just as in the discrete case considered above.

6 Discussion

We have examined the non-relativistic motion of two different classical models for strings with a massive end. The physically realistic model of an inextensible string is exactly soluble in terms of Legendre functions and the modes strongly resemble sinusoids. For end masses that are only a few times the mass of the rest of the string, approximation of the end conditions by assuming a node at the end mass is very good. The same result holds for the Merlin-Paton string, as can be seen from Table 4. We have applied both continuum and discrete methods to the solutions of the string models. The effect of the Brillouin zone on the spectrum is evident in both string models (Tables 1 and 2 and Tables 4 and 5) but the continuum and discrete frequencies of the inextensible string show better agreement for higher modes when the fixed-end boundary conditions are imposed (Tables 2 and 3).

The transverse rotating string studied by Merlin and Paton is the non-relativistic limit of the Nambu-Goto string. In the Merlin-Paton model, the centrifugal terms are dropped for all of the masses comprising the string but is kept for the mass on the end. In principle the radius r of the orbit is dynamically determined by the model, though we have only considered essentially circular orbits. Dropping the centrifugal terms makes it possible to satisfy the Virasoro condition to lowest order in the displacements y_n,

$$0 = \dot{\mathbf{X}}(r_n) \cdot \mathbf{X}'(r_n) = -\omega y_n r_n + O(y_n^2), \tag{65}$$

in the non-relativistic limit and and also avoids an unstable mode with $\Omega^2 < 0$ for high rotational velocities ω^2, as can be seen from Eqs. (63) and (64). As for the inextensible string, the boundary condition for the Merlin-Paton string is well approximated by the requirement of a node on the end for end masses of three times the string mass.

Quantization of any of these models in the limit of small vibration amplitude is straightforward and could be accomplished along the lines laid out by Lambiase and Nesterenko [7]. All that is needed are the modes and the frequencies because the equations of motion and boundary conditions are linear, so superposition holds.

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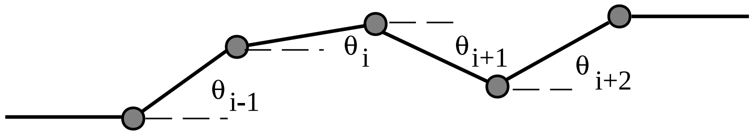

Figure 1: The angles θ_i.

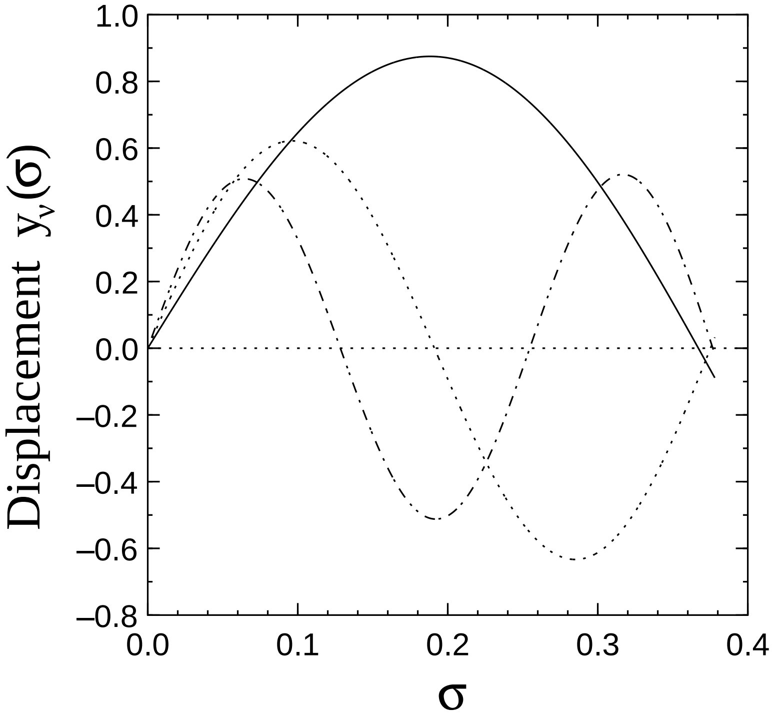

Figure 2: Displacement vs. position of the first three modes of an inextensible string with $M_e = 3M_s$.

Table 1: Frequencies and indices of the first twelve modes on an inextensible string with twenty masses and a terminating mass $M_e = 3M_s$.

Ω^2/ω^2	ν	Ω^2/ω^2	ν	Ω^2/ω^2	ν	Ω^2/ω^2	ν
0	1	276.91	23.081	1050.3	45.356	2167.4	65.356
31.973	7.6360	485.21	30.687	1394.4	52.330	2579.0	71.334
124.69	15.362	744.94	38.128	1769.6	59.010	2994.9	76.908

Table 2: Frequencies and indices of the first twelve modes on an inextensible string in the continuum model having a terminating mass $M_e = 3M_s$, as solved with the Runge-Kutta method.

Ω^2/ω^2	ν	Ω^2/ω^2	ν	Ω^2/ω^2	ν	Ω^2/ω^2	ν
0	1	296.43	23.895	1183.2	48.168	2661.0	72.468
33.678	7.843	526.32	31.979	1610.1	56.267	3285.1	80.57
132.23	15.831	821.92	40.072	2102.7	64.367	3974.7	88.672

Table 3: Frequencies and indices of an inextensible string computed by assuming a node at the end and using asymptotic solutions.

Ω^2/ω^2	ν	Ω^2/ω^2	ν	Ω^2/ω^2	ν	Ω^2/ω^2	ν
0	1	294.50	23.815	1181.4	48.132	2659.5	72.448
31.723	7.6053	524.44	31.921	1608.4	56.237	3283.6	80.553
130.26	15.710	820.07	40.026	2101.1	64.343	3973.4	88.658

Table 4: Frequencies of the first twelve modes of an $N = 20$ discrete Merlin-Paton string with $M_e = 3M_s$ and as computed by assuming a node at the end. $\omega_0^2 = N(N+1)b_0/m\ell$

Ω^2/ω_0^2	$\Omega^2_{\text{fixed}}/\omega_0^2$	Ω^2/ω_0^2	$\Omega^2_{\text{fixed}}/\omega_0^2$	Ω^2/ω_0^2	$\Omega^2_{\text{fixed}}/\omega_0^2$	Ω^2/ω_0^2	$\Omega^2_{\text{fixed}}/\omega_0^2$
10.03679	9.3821	146.57	145.96	420.50	420.00	777.59	777.23
37.97248	37.3188	224.82	224.24	533.57	533.11	903.08	902.77
83.82344	83.1862	316.81	316.27	653.49	653.08	1027.18	1026.92

Table 5: Frequencies of the first twelve modes of a continuum Merlin-Paton string with $M_e = 3M_s$ using the boundary condition in Eq. (64). $\omega_0^2 = b_0/\rho_0\ell^2$

Ω^2/ω_0^2	Ω^2/ω_0^2	Ω^2/ω_0^2	Ω^2/ω_0^2
10.524	158.58	484.28	987.63
40.142	247.41	632.32	1194.9
89.492	355.98	800.11	1421.9