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Non-Gaussian Density Perturbations in Inflationary Cosmologies*

T. J. Allen, B. Grinstein and Mark B. Wise†

California Institute of Technology, Pasadena, CA 91125

Abstract

The primordial mass density fluctuations may have arisen from quantum fluctuations in a (massless) scalar field that occurred during an inflationary era. We show that it is possible for primordial mass density fluctuations, which arose in this way, to be highly non-Gaussian. We also show that the "bad" infrared properties of the propagator for a massless scalar field in de Sitter space can translate itself into a power spectrum, for the the two-point spatial correlation of objects that do not trace the mass, which behaves like k^{-3}, at small wave numbers k.

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† Sloan Fellow.

Limits on the anisotropy of the microwave background radiation restrict the primordial fluctuations in the mass density to be small at early times. However, the fluctuations do grow in the matter-dominated era because of gravitational instability. Presumably the primordial fluctuations in the mass density determined the large-scale structure of the universe that is observed today. Understanding the nature of these fluctuations is one of the important issues in physics. If the length scales associated with the physical processes that generated the primordial fluctuations are very small, compared to astrophysically relevant length scales, then they probably can be neglected in the correlations of the mass density. This naturalness requirement of scale invariance restricts the power spectrum for the two-point correlation of the mass density fluctuations to have the Harrison-Zeldovich form,[1] but allows for many types of connected higher point correlations.

A gaussian probability distribution is the simplest scale invariant choice for the primordial mass density fluctuations. This choice also has considerable predictive power. Provided there is no new physics occurring at very late times, once the type of fluctuations (*e.g.*, adiabatic) and the type of dark matter are specified, only the normalization of the two-point correlation of the mass density fluctuations is left undetermined (we assume an $\Omega = 1$ universe with vanishing cosmological constant). It appears that this simplest choice is encountering difficulties explaining the observed large-scaled structure of the universe. For example, with cold dark matter and adiabatic fluctuations, it is hard to explain the significant correlations of rich clusters of galaxies that occur at large distances.[2] This problem has lead to speculation that the primordial mass density fluctuations are not gaussian.[3] Since primordial fluctuations in the mass density with wavelengths that are less than the horizon length (*i.e.*, $1/H$) today had wavelengths greater than the horizon length at early times, it is difficult to come up with reasonable ways to generate these fluctuations. In fact, at the present time, there are only two schemes. In one it is the presence of topological defects called cosmic strings that generate the fluctuations.[4] The other scheme has the fluctuations in the energy density arising from quantum fluctuations in a scalar field during an inflationary era.[5][6] The smallness of the primordial fluctuations restricts the field that is driving the inflation to be very weakly coupled so that its quantum fluctuations (and the corresponding fluctuations in the mass density) are approximately gaussian.[7] The purpose of this letter is to show that it is possible to have cosmologies where quantum fluctuations in a field during an inflationary era give rise to scale invariant non-gaussian primordial fluctuations in the mass density. We shall do this by constructing a model where quantum fluctuations in an invisible axion field[8] [9] give rise to significantly non-gaussian isocurvature mass density fluctuations.[10]

Assuming that the Peccei-Quinn symmetry[11] is already broken, we take the interactions of the axion field during the inflationary era to be described by the lagrangian density

$$\mathcal{L} = \tfrac{1}{2} g^{\mu\nu}\partial_\mu a \partial_\nu a + \frac{\lambda}{4! f^4}(g^{\mu\nu}\partial_\mu a \partial_\nu a)^2, \tag{1}$$

where the de Sitter space metric is

$$ds^2 = \frac{1}{H^2\tau^2}(d\tau^2 - d\mathbf{x}^2). \tag{2}$$

In eq. (2) we have adopted conformally flat spacetime coordinates and H is the Hubble constant during the inflationary era. It is correlations of the Fourier transform of the axion field, $\tilde{a}(\mathbf{k},\tau)$, defined by

$$a(\mathbf{x},\tau) = \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{a}(\mathbf{k},\tau) \tag{3}$$

that determine the correlations of the mass density fluctuations. Since we are interested in fluctuations of the mass density on astrophysically relevant scales, we can restrict our attention to the correlations of $\tilde{a}(\mathbf{k},\tau)$ at $k\tau << 1$. Treating the coupling λ as small, the leading contributions to the connected two- and four-point correlations of the Fourier transform of the axion field (in de Sitter space) are given for very small $k\tau$ by

$$\langle \tilde{a}(\mathbf{k}_1,\tau)\tilde{a}(\mathbf{k}_2,\tau)\rangle_c \simeq \frac{H^2}{2k_1^3}(2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2), \tag{4}$$

$$\begin{gathered}\langle \tilde{a}(\mathbf{k}_1,\tau)\ldots\tilde{a}(\mathbf{k}_4,\tau)\rangle_c \simeq \frac{H^4}{4(k_1k_2k_3k_4)^{3/2}}\lambda(\frac{H}{f})^4 g(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) \\ (2\pi)^3\delta^3(\mathbf{k}_1+\ldots+\mathbf{k}_4),\end{gathered} \tag{5}$$

with

$$\begin{aligned} g(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) &= \frac{1}{(k_1k_2k_3k_4)^{1/2}}\Big\{ \frac{4}{k_T^5}[(k_1k_2 - \mathbf{k}_1\cdot\mathbf{k}_2)(k_3k_4 - \mathbf{k}_3\cdot\mathbf{k}_4)] \\ &- \frac{1}{k_T^4}[\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2(k_1+k_2)(k_3k_4-\mathbf{k}_3\cdot\mathbf{k}_4) + \hat{\mathbf{k}}_3\cdot\hat{\mathbf{k}}_4(k_3+k_4)(k_1k_2-\mathbf{k}_1\cdot\mathbf{k}_2)] \\ &- \frac{1}{3k_T^3}[\hat{\mathbf{k}}_3\cdot\hat{\mathbf{k}}_4(k_1k_2-\mathbf{k}_1\cdot\mathbf{k}_2) + \hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2(k_3k_4-\mathbf{k}_3\cdot\mathbf{k}_4) - (\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2)(\hat{\mathbf{k}}_3\cdot\hat{\mathbf{k}}_4) \\ &\cdot (k_1+k_2)(k_3+k_4)] + \frac{1}{3k_T}(\hat{\mathbf{k}}_3\cdot\hat{\mathbf{k}}_4)(\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2)\Big\} + (2\leftrightarrow 3) + (2\leftrightarrow 4) \end{aligned} \tag{6}$$

where $k_T = k_1 + k_2 + k_3 + k_4$, $k_j = |\mathbf{k}_j|$ and $\hat{\mathbf{k}}_j = \mathbf{k}_j/k_j$. In addition to these correlations, the axion field has some expected value which is of order the axion decay constant

$$\langle a(\mathbf{x}, \tau) \rangle \sim f. \tag{7}$$

After the Universe has exited the de Sitter era and cooled to a temperature of about 800 MeV, the axion develops significant non-derivative interactions because the Peccei-Quinn symmetry is not respected by non-perturbative strong interactions. The fluctuations in the axion field generated in the inflationary era are converted into fluctuations in the mass density as they enter the horizon. Local energy conservation ensures that for wavelengths large compared to the horizon length, the fluctuations in the axion energy density are compensated by fluctuations in the energy density of the other matter and radiation fields so that there are no net energy density fluctuations (*i.e.*, the fluctuations are isocurvature). Linearizing in the axion fluctuations, the two-point and connected four-point correlations of the Fourier transform of the mass density fluctuations $\tilde{\delta}(\mathbf{k})$ that result from eqs. (4) and (5) are

$$\left\langle \tilde{\delta}(\mathbf{k}_1)\tilde{\delta}(\mathbf{k}_2) \right\rangle = \frac{N^2 T^2(k_1)}{k_1^3} (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2) \tag{8}$$

$$\begin{aligned} \left\langle \tilde{\delta}(\mathbf{k}_1) \ldots \tilde{\delta}(\mathbf{k}_4) \right\rangle_c = & \frac{N^4 T(k_1) \ldots T(k_4)}{(k_1 k_2 k_3 k_4)^{3/2}} \lambda (\frac{H}{f})^4 \\ & \cdot g(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)(2\pi)^3 \delta^3(\mathbf{k}_1 + \ldots + \mathbf{k}_4), \end{aligned} \tag{9}$$

where $g(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ is given in eq. (6). In eqs. (8) and (9), $T(k)$ is the transfer function appropriate to isocurvature fluctuations.[10] At small k, $T(k) \sim k^2$ so that the correlations in eqs. (8) and (9) are scale invariant. For large k, $T(k)$ goes to a constant. The normalization constant N is such that the two-point correlation implies that averaged over the horizon volume, the mass density fluctuations are of order

$$\left\langle \left(\int\limits_{\substack{\text{horizon}\\ \text{volume}}} d\mathbf{x} \;\; \delta(\mathbf{x}) \right)^2 \right\rangle^{1/2} \sim \left(\frac{H\Omega_a}{f} \right). \tag{10}$$

In eq. (10) H is the Hubble constant during the inflationary era. The factor of $\Omega_a \equiv \rho_a/\rho_c$ arises because we are not necessarily assuming that the axions comprise the dark matter of the universe. In order for galaxy formation to be possible and for microwave background constraints to be satisfied these fluctuations must be of order 10^{-5}.

$$H\Omega_a/f \sim 10^{-5}. \tag{11}$$

If the axions comprise the dark matter of the universe, then $\Omega_a = 1$ and eq. (11) implies that the Hubble constant during the inflationary era is much smaller than the axion decay constant f. In this case, the connected part of the four-point correlation of the primordial mass density fluctuations [eq. (9)] is negligible compared with the disconnected part. The smallness of the mass density fluctuations has ensured that they are approximately gaussian. A simple way around this is to have the axion fluctuations large so $H/f \sim 1$ but to make the resulting mass density fluctuations small by having the axions make up only a small fraction of the total mass density. Since the main motivation for the axion is the strong CP puzzle there is no reason for us to demand that axions be the dark matter. Some other particle (*e.g.*, the photino) is assumed to dominate the mass density of the universe.

The fraction of critical density that comes from axions is given approximately by[9]

$$\Omega_a \sim (f/4 \times 10^{12}\ GeV). \tag{12}$$

Thus, eq. (11) is satisfied with $H/f \sim 1$ by having $f \sim 10^8$ GeV. Such a small value for the axion decay constant can lead to astrophysical problems due to excessive axion emission from stars. However, the couplings of the axion to quarks, leptons and photons are model-dependent and they can be arranged so that with an axion decay constant of order 10^8 GeV the rate for stellar axion emission is acceptable.[12]

The scenario presented here assumes that the Peccei-Quinn symmetry is broken during and after the inflationary era. If the reheating is efficient (*i.e.*, all the vacuum energy is converted into radiation), then the reheating temperature is

$$T_{\rm rh} \sim \sqrt{M_{\rm Pl} H / N_{\rm eff}^{1/2}} \tag{13}$$

Here $N_{\rm eff}$ is the effective number of radiation species after reheating and $M_{\rm Pl}$ is the Planck mass. For $H/f \sim 1$, and an axion decay constant of order 10^8 GeV, this reheating temperature is significantly greater than the axion decay constant. Although there is no general relation between the temperature at which a symmetry is restored and the magnitude of the vacuum expectation value associated with the symmetry breaking, it seems likely that a bizarre fine tuning of coupling constants would be required for the Peccei-Quinn symmetry to remain broken at a temperature several orders of magnitude greater than the axion decay constant. We prefer to imagine that the reheating is not very efficient so that the universe only reheats to a temperature of order the axion decay constant, thus leaving the Peccei-Quinn symmetry broken. (This is not difficult to achieve. In fact, many inflationary models suffer from problems due to insufficient reheating.[13]) The small reheating temperature may make the generation of a baryon excess through the standard mechanisms difficult.[14][15] However, there are ways to generate an acceptable baryon excess at quite low temperatures.[16] Finally, we note it is *not* the phase transition associated with the breaking of the Peccei-Quinn that is driving the inflation. In fact, the curvature of the potential for the scalar fields whose vacuum expectation values break the Peccei-Quinn symmetry should be greater than the Hubble constant during the inflationary era to ensure that the Peccei-Quinn symmetry is broken during the inflationary phase.

The fluctuations in the mass density presented in eqs. (8) and (9) resulted from linearizing in the axion field. The connected four-point correlation in eq. (9) is important, for H/f of order unity, because then the axion is an interacting field during the de Sitter epoch. With H/f of order unity there will also be non-gaussian fluctuations induced through the non-linear relationship between the axion filed and the mass density. These we have neglected. We have also neglected the fact that a/f is an angular variable. It has been argued in ref. [17] that this provides an infrared cutoff that breaks the scale invariance of the fluctuations. However, as long as H/f is not larger than unity this violation of scale invariance is likely to be negligible on astrophysically relevant length scales. (In ref. [17] it was also noted that acceptable isocurvature mass density fluctuations can arise for large H/f provided axions comprise only a small fraction of the total mass density. However, the main emphasis of ref. [17] was on the breaking of scale invariance, due to the angular nature of a/f, rather than on producing non-gaussian fluctuations.)

The above model has (for λ of order unity) significantly non-gaussian primordial mass density fluctuations. The connected part of the four-point correlation of the mass density fluctuations, averaged over the horizon volume, is as large as the disconnected part. The two-point spatial correlation of biased objects (*e.g.*, rich clusters of galaxies[18]) typically depends on all the connected correlations of the mass density fluctuations. When the number density of the objects is determined by local properties of the primordial mass density fluctuations (*e.g.*, objects arising only at high peaks of the filtered primordial fluctuations[17]), it is easy to characterize how the connected correlations of the mass density fluctuations affect the two-point spatial correlation of the objects.[3,19] The connected n-point correlation of the mass density fluctuations depends on the locations of n points $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Suppose m of these points are kept near each other (*i.e.*, within the distance given by the filtering used to define the objects) but separated a large distance r from the remaining $n-m$ points (which are also near each other). If in this limit the connected n-point correlation of the mass density fluctuations behaves like r^{-p}, then it gives a contribution to the two-point correlation of the objects $\xi_0(r)$ that goes like r^{-p}, for large r. An analogous criterion holds in Fourier space. The Fourier transform of the connected n-point correlation of the primordial mass density fluctuations depends on n wave vectors $\mathbf{k}_1, \ldots, \mathbf{k}_n$. If as any partial sum of wave vectors $\mathbf{k}_s = \mathbf{k}_1 + \cdots + \mathbf{k}_j$, $j < n$ goes to zero (but the individual wave vectors do not), the Fourier transform of the connected n-point correlation of the primordial mass density fluctuations diverges as k_s^{-p}, then the power spectrum for the two-point correlation of the biased objects $P_0(k)$ gets a contribution that also diverges as k^{-p} for small k.

In the model we have constructed the connected correlations of the mass density fluctuations give a contribution to the power spectrum for the two-point correlation of biased objects that goes to a constant at small wave numbers. It is possible to complicate the model somewhat so that the higher correlations have a more dramatic impact on the power spectrum for the two-point correlation of biased objects. For example, imagine that there is another (essentially) massless field χ that is coupled to the axion in a way that is described by the interaction lagrangian density

$$\mathcal{L}_{\text{int}} = (\lambda'/f)\chi g^{\mu\nu}\partial_\mu a \partial_\nu a. \tag{14}$$

Now there is a contribution to the connected part of the axion four-point correlation that comes from tree level χ exchange. The most divergent part of the Fourier transform of this correlation, as $\mathbf{k}_s = \mathbf{k}_1 + \mathbf{k}_2$ goes to zero, is given at small $k\tau$ by

$$\langle \tilde{a}(\mathbf{k}_1, \tau) \ldots \tilde{a}(\mathbf{k}_4, \tau) \rangle_c \simeq H^4 (\lambda' H/f)^2 \frac{1}{2k^6 k_s^3} (2\pi)^3 \delta^3(\mathbf{k}_1 + \ldots + \mathbf{k}_4). \tag{15}$$

Here for simplicity we have set $k = k_1 = k_2 = k_3 = k_4$. When multiplied by the transfer functions, the axion four-point correlation in eq. (15) corresponds to non-gaussian mass density fluctuations that five a contribution to the power spectrum for the two-point correlation of biased objects (*i.e.*, objects that do not trace the mass) which diverges like k^{-3} at small k! This "bad" infrared behavior occurs because the propagator for a massless scalar field in de Sitter space is badly behaved in the infrared. Even for very small $\lambda'(H/f)$ this could have an observable impact on the large-scale structure of the universe. It implies fluctuations in the number of objects in a large volume V tends to infinity. Webster's analysis of the clustering of a sample of about 5000 radio sources provides a limit of about 10^{-3} on the normalization of a k^{-3} term in their power spectrum.[20]

The infrared divergence associated with the k^{-3} spectrum is an artifact of approximations made in the computation of eq. (15). It is cut off by the size of the region that has been in causal contact, the χ mass and the angular nature of a/f. We have assumed that the length scales associated with these cutoffs are greater than those relevant for astrophysics.

In this letter we have constructed a model where fluctuations in an axion field during the inflationary era give rise to significantly non-gaussian primordial mass density fluctuations. The smallness of the primordial mass density fluctuations did not restrict the axion to be weakly coupled because the axions comprised only a small fraction of the total mass density. While a more detailed analysis would be necessary to determine if the model presented here is completely realistic, we believe it serves the purpose of illustrating that (even without cosmic strings) the primordial mass density fluctuations can be highly non-gaussian. We have also seen that the peculiar infrared propertied of the fluctuations of massless fields in de Sitter space can translate itself into unusual behavior for the spatial distribution of objects that do not trace the mass.

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