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Particle Statistics in Topologically Nontrivial Two-Dimensional Magnetic Systems

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Abstract

It is demonstrated that charged particles may acquire unusual statistics in topologically nontrivial two-dimensional samples placed in strong magnetic fields. These novel statistics follow from an analysis of the self-adjoint extensions of the Landau Hamiltonian which are partially classified by the UIR's of the fundamental group. Superselection rules corresponding to different quantum theories are found. Supersymmetry arguments are used to construct exact ground states.

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1. Introduction

Electrons, the most prosaic of particles, are fermions as chemistry amply teaches us. In dimensions higher than two, only the usual Fermi and Bose and the exotic parastatistics, which seem to be unrealized in nature, are possible statistics for particles, unless the fundamental group of the domain is nontrivial. In two dimensions the possibilities are much richer. Indeed, in two dimensions it is the braid group which is relevant to statistics, not the permutation group as in higher dimensions.[1] Until recently, there was no reason to take these extra possibilities seriously, since the physical dimension of space seems to exclude these exotic possibilities as mere theoretical curiosities. However, the discovery of the fractional quantum Hall effect[2] and its most accepted theoretical explanation[3] have turned this prejudice on its head. Now it is even fashionable to consider that exotic statistics have relevance for high-T_c superconductivity.[4]

In the fractional quantum Hall effect it is the quasiparticle excitations which have the exotic statistics while the electrons themselves remain fermions. In general, whenever the configuration space, Q, is multiply connected, there exist inequivalent quantizations corresponding to different unitary, irreducible representations (UIR's) of $\pi_1(Q)$.[5] The most complete treatment of this is due to R. Sorkin, in ref. 5. When the homotopy group $\pi_1(Q)$ is non-abelian, some of these quantizations will require the use of vector-valued wave functions. Such wave functions are used, for example, in the quantization of the collective rotational motion of odd-A nuclei with three distinct moments of inertia.[6] In this spirit, we investigate the quantization of planar motion in a uniform magnetic field. The magnetic field makes the problem tractable by preventing the electrons from wandering off to infinity, but is interesting because of its connections to studies of the quantum Hall effect. In these quantizations the electrons themselves may acquire statistics more general than Fermi-Dirac by virtue of their monodromies forming a non-trivial representation of the fundamental group. In this more general case one must be careful to distinguish the paths through which the electrons are exchanged. If the path does not encircle a puncture, the many-body wave function must simply change sign. If, on the other hand, the path does encircle a puncture, the many-body wave function will gain an additional factor corresponding to the monodromy of the path. For definiteness, we consider the simplest case, the twice punctured plane, whose homotopy group $\pi_1(\mathbf{R}^2 \backslash \{p_1, p_2\})$ is the free group on two letters.

2. Landau Levels

The electron states in a two dimensional magnetic system are highly degenerate. These degenerate states, the Landau levels, are easily found when a non-symmetric gauge is chosen for the vector potential.

$$\mathbf{A}(x, y) = B_0 x\,\hat{\mathbf{y}} \tag{2.1}$$

The momentum p_y is a conserved quantity of the Landau Hamiltonian

$$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 = \frac{1}{2m}[(-i\partial_x)^2 + (-i\partial_y - eB_0 x)^2]. \tag{2.2}$$

The energy eigenstates are written with harmonic oscillator wave functions, ϕ_n, centered at $x_0 = -\frac{k}{eB_0}$ with frequency $\frac{eB_0}{m}$;

$$\psi_n = e^{iky}\phi_n(x - x_0). \tag{2.3}$$

Another way to view these states is to use the symmetric gauge

$$\mathbf{A}(\mathbf{r}) = \tfrac{1}{2}\mathbf{B} \times \mathbf{r}. \tag{2.4}$$

In this gauge the Hamiltonian reads

$$\begin{aligned} H &= \frac{1}{2m}\{\Pi_x^2 + \Pi_y^2\} = \frac{1}{4m}\{\Pi_z\Pi_{\bar{z}} + \Pi_{\bar{z}}\Pi_z\} \\ &= \frac{1}{2m}\left\{-4\partial\bar{\partial} + (\frac{eB_0}{2})^2|z|^2 - eB_0(z\partial - \bar{z}\bar{\partial})\right\}. \end{aligned} \tag{2.5}$$

Again $\mathbf{\Pi} = -i\nabla - e\mathbf{A}$. The problem has been formulated in complex coordinates

$$\begin{aligned} z &= x + iy, \\ \partial &= \tfrac{1}{2}(\partial_x - i\partial_y), \\ \bar{\partial} &= \tfrac{1}{2}(\partial_x + i\partial_y). \end{aligned} \tag{2.6}$$

Clearly the Π_z and $\Pi_{\bar{z}}$ should be treated as creation and annihilation operators. They satisfy the commutation relations

$$[\Pi_z, \Pi_{\bar{z}}] = 2m\omega. \tag{2.7}$$

Here ω denotes the signed frequency $\frac{eB_0}{m}$. The sign of ω determines which of Π_z and $\Pi_{\bar{z}}$ is the creation operator. In the following $\omega > 0$. The notation is simpler if the complex coordinates are scaled by $(m|\omega|)^{-1/2}$. The spectrum generating operators can then be identified as

$$
\begin{aligned}
\hat{a}^\dagger &= -\frac{i}{\sqrt{2m|\omega|}}\Pi_{\bar{z}} = \sqrt{2}(-\partial + \tfrac{1}{4}\bar{z}), \\
\hat{a} &= \frac{i}{\sqrt{2m|\omega|}}\Pi_z = \sqrt{2}(\bar{\partial} + \tfrac{1}{4}z).
\end{aligned}
\tag{2.8}
$$

Just as for the one dimensional harmonic oscillator, the ground state is annihilated by $\hat{a}$, otherwise it could be lowered to a state of lower energy and the Hamiltonian must be bounded below. It follows that the ground states of the Hamiltonian are of the form

$$
\begin{aligned}
\psi_f(z,\bar{z}) &= \exp(-|z|^2/4)f(z), \\
H\psi_f &= \tfrac{1}{2}|\omega|\psi_f.
\end{aligned}
\tag{2.9}
$$

In the plane it is possible to use angular momentum to label the possible ground states

$$
|0_n\rangle = \exp(-|z|^2/4)z^n. \tag{2.10}
$$

Excited states are built upon any ground state (2.9), by applying creation operators.

$$
|n,f\rangle = \frac{1}{\sqrt{n!}}a^{\dagger n}\exp(-|z|^2/4)f(z). \tag{2.11}
$$

One may also solve the problem by means of separation of variables. The operator $\partial + \frac{1}{4}\bar{z}$ commutes with the Landau Hamiltonian and may be simultaneously diagonalized. Setting $\psi(z,\bar{z}) = \phi(z)\widetilde{\phi}(\bar{z})e^{-|z|^2/4}$ and

$$
\begin{aligned}
(\partial + \frac{1}{4}\bar{z})\psi &= \frac{\beta}{2}\psi, \\
\tfrac{1}{2}|\omega|(-4\partial\bar{\partial} + (\frac{1}{4})|z|^2 - (z\partial - \bar{z}\bar{\partial}))\psi &= E\psi,
\end{aligned}
\tag{2.12}
$$

we find

$$\psi(z,\bar{z}) = (\bar{z}-\beta)^{\frac{2E-\omega}{2\omega}} \exp(-z(\bar{z}-2\beta)/4). \tag{2.13}$$

It is immediate from this that the spectrum is $E = (n+\frac{1}{2})\omega, \quad n = 0, 1, \ldots,$ in order that the solution (2.13) be normalizable and single-valued. Any of the states (2.11) can be expressed as

$$|n, f\rangle = f(\frac{d}{d\beta}\bigg|_{\beta=0})\big[(\bar{z}-\beta)^n \exp(-z(\bar{z}-2\beta)/4)\big]. \tag{2.14}$$

3. Self-adjoint extensions of the Landau Hamiltonian

A fundamental requirement of quantum mechanics is that probability should be conserved. That is, the time evolution of the system ought to be unitary. Ordinarily, we think of this in terms of the Hamiltonian being hermitian. Technically, however, this is not enough. The proper requirement is that the Hamiltonian be self-adjoint. A self-adjoint operator has real eigenvalues while a merely hermitian operator may have complex eigenvalues. The classic example[7] of the latter situation is the momentum operator $\hat{p} = i\frac{d}{dx}$ acting on functions of a real variable. The momentum operator is formally hermitian, yet any function $\exp(\alpha x)$ is an eigenfunction with eigenvalue $i\alpha$. Thus, one must be very careful to specify the allowed functions on which the operator may act. In the present instance, if a domain too small $D_0(\hat{p}) = \{\phi : [0,1] \to \mathbf{C} \,|\, \phi(0) = \phi(1) = 0\}$ is set, then the adjoint operator $\hat{p}^*$ has a domain too large to exclude the above exponential eigenfunction. The proper way to find a self-adjoint extension of the operator $\hat{p}$ acting on a domain at least as large as $D_0(\hat{p})$ is to set the domain just weak enough that the domain of the adjoint, $D(\hat{p}^*)$, is no larger than $D(\hat{p})$. Thus $v \in D_\theta(\hat{p}) = \{\phi : [0,1] \to \mathbf{C} \,|\, \phi(0) = e^{i\theta}\phi(1)\}$ and

$$\langle u, \hat{p}v\rangle - \langle \hat{p}u, v\rangle = \bar{u}(0)v(0) - \bar{u}(1)v(1) = 0 \tag{3.1}$$

together imply that $u \in D_\theta(\hat{p})$, or that $\hat{p}$ acting on $D_\theta(\hat{p})$ is self-adjoint. The implication of this is that the states actually live on the circle and that the transport of the wave function around the circle preserves the norm.

Keeping this in mind, we next imagine studying the motion of electrons under the influence of a constant magnetic field, but in a punctured plane. All of the important features of the problem are displayed by the twice-punctured plane. For simplicity we choose to put the punctures at $z = 0$ and at $z = 1$. Furthermore, we assume that the electrons may be described by vector-valued wave functions, and in this case we will consider them to be valued in $\mathbf{C}^N$, but will give an example for which $N = 2$. For now, the problem is to find the conditions on the states which guarantee that the Landau Hamiltonian (2.5) is self-adjoint.

In the present case, we shall observe that there are two conditions for self-adjointness; unitary monodromies around each of the punctures and conditions on the relative behavior of the functions and their derivatives at the punctures.

Besides the punctures at $z = 0$ and $z = 1$, we choose a boundary for the sample to ensure that the remaining parts of the sample are simply connected. We consider the location of this boundary to be unobservable and in the end the analysis ought to be independent of its position. The extensions we are seeking are extensions of the Landau Hamiltonian acting on smooth functions having compact support in a simply connected region of the punctured plane. It is sufficient to choose the boundary to be a branch cut along the positive real axis and the point at infinity. Figure 1 shows the boundary and punctures explicitly.

The inner product on the states is the standard one:

$$\langle u, v\rangle = \frac{i}{2}\int_{\mathcal{C}} dz\wedge d\bar{z}\,\bar{u}^j(z,\bar{z})v_j(z,\bar{z}). \tag{3.2}$$

Thus, the condition $\langle u, Hv\rangle = \langle Hu, v\rangle$ becomes

$$\oint_{\partial\mathcal{C}}\left\{dz\,\bar{u}^j(\partial+\tfrac{1}{4}\bar{z})v_j + d\bar{z}\,v_j(\bar{\partial}+\tfrac{1}{4}z)\bar{u}^j\right\} = 0, \tag{3.3}$$

where the contour $\partial\mathcal{C}$ is shown in figure 2. We examine the pieces of the boundary separately, and break them up as follows.

$$\partial\mathcal{C} = \mathcal{C}_0\cup\mathcal{C}_1\cup\mathcal{C}_\infty\cup\mathcal{L}_0\cup\mathcal{L}_1, \tag{3.4}$$

where

$$\begin{aligned}
\mathcal{C}_0 &= \{z|\ |z| = 0^+\},\\
\mathcal{C}_1 &= \{z|\ |z-1| = 0^+\},\\
\mathcal{C}_\infty &= \{z|\ |z| = \infty\},\\
\mathcal{L}_{0\pm} &= \{z|\ z = x\pm i0^+,\ x\in\mathbf{R},\ 0<x<1\},\\
\mathcal{L}_{1\pm} &= \{z|\ z = x\pm i0^+,\ x\in\mathbf{R},\ 1<x<\infty\}.
\end{aligned} \tag{3.5}$$

The condition for hermiticity (3.3) becomes the separate conditions

$$\begin{aligned}
\oint_{\mathcal{C}_\mathcal{A}}\left\{dz\,\bar{u}^j(\partial+\tfrac{1}{4}\bar{z})v_j + d\bar{z}\,v_j(\bar{\partial}+\tfrac{1}{4}z)\bar{u}^j\right\} = 0, \quad \mathcal{A} = 0,1,\infty\\
\left(\int_{\mathcal{L}_{\mathcal{A}+}} - \int_{\mathcal{L}_{\mathcal{A}-}}\right)\left\{dz\,\bar{u}^j(\partial+\tfrac{1}{4}\bar{z})v_j + d\bar{z}\,v_j(\bar{\partial}+\tfrac{1}{4}z)\bar{u}^j\right\} = 0, \quad \mathcal{A} = 0,1.
\end{aligned} \tag{3.6}$$

If, for the first condition, we suppose that the function v_i on the upper contour $\mathcal{L}_{\mathcal{A}+}$, denoted $v_i^{(+)}$, is a given constant matrix $\mathcal{M}_{\mathcal{A}}$ times its value, $v_i^{(-)}$, on the lower contour $\mathcal{L}_{\mathcal{A}-}$,

$$v_i^{(+)} = \mathcal{M}_{\mathcal{A}\,ij} v_j^{(-)}, \tag{3.7}$$

then we may recast the first part of (3.6) as

$$\begin{aligned} &\int_{\mathcal{L}_{\mathcal{A}}} \left\{ \bar{u}^{(+)} \mathcal{D} v^{(+)} + \bar{\mathcal{D}} \bar{u}^{(+)} v^{(+)} - \bar{u}^{(-)} \mathcal{D} v^{(-)} - \bar{\mathcal{D}} \bar{u}^{(-)} v^{(-)} \right\} = \\ &\int_{\mathcal{L}_{\mathcal{A}}} \left\{ [\bar{\mathcal{D}} \bar{u}^{(+)} \mathcal{M}_{\mathcal{A}} - \bar{\mathcal{D}} \bar{u}^{(-)}] v^{(-)} + [\bar{u}^{(+)} \mathcal{M}_{\mathcal{A}} - \bar{u}^{(-)}] \mathcal{D} v^{(-)} \right\} = 0, \end{aligned} \tag{3.8}$$

where $\mathcal{D} = (\partial + \frac{1}{4}\bar{z})\, dz$. This implies that

$$\bar{u}^{(+)} \mathcal{M}_{\mathcal{A}} = \bar{u}^{(-)}, \tag{3.9}$$

or, equivalently,

$$u^{(+)} = (\mathcal{M}_{\mathcal{A}}^{\dagger})^{-1} u^{(-)}. \tag{3.10}$$

This is the same condition on u as (3.7) is on v, if

$$\mathcal{M}_{\mathcal{A}}^{\dagger} \mathcal{M}_{\mathcal{A}} = \mathbf{1}, \tag{3.11}$$

which is the first condition for self-adjointness of the Landau Hamiltonian, the unitarity of the monodromies around the punctures of the plane.

The second conditions of (3.6) are the relative behavior at the punctures and infinity of the wave functions and their derivatives.

$$\oint_{\mathcal{C}_{\mathcal{A}}} \left(\bar{\mathcal{D}}\bar{u}^i v_i + \bar{u}^i \mathcal{D} v_i\right) = 0, \quad \mathcal{A} = 0, 1, \infty. \tag{3.12}$$

In coordinates (ϱ, θ) near each puncture,

$$\zeta = z - z_c = \varrho e^{i\theta}, \tag{3.13}$$

the derivative part of (3.12) at the puncture z_c becomes

$$\oint \left(d\bar{\zeta}\, v\bar{\partial}\bar{u} + d\zeta\, \bar{u}\partial v\right) = \tfrac{1}{2}\oint d(\bar{u}v) + \tfrac{i}{2}\oint \varrho\, d\theta\, \left(\bar{u}\partial_\varrho v - v\partial_\varrho \bar{u}\right). \tag{3.14}$$

The first term on the right hand side vanishes because $\bar{u}v$ is single valued in the whole plane. The constant piece of (3.12) becomes

$$\tfrac{1}{4}\oint \left\{\bar{u}(i\bar{z}_c\zeta)v - \bar{u}(iz_c\bar{\zeta})v\right\} d\theta, \tag{3.15}$$

which we will write as

$$\frac{i}{2}\oint \begin{pmatrix} \bar{u} \\ \varrho\bar{\gamma}\bar{u} \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v \\ \varrho\gamma v \end{pmatrix} d\theta, \tag{3.16}$$

where $\gamma = 2\bar{z}_c e^{i\theta}$ is an angle dependent factor. Putting these together, using $D_\varrho = \partial_\varrho + \gamma$, we have

$$\lim_{\varrho\to 0} \tfrac{1}{2}\oint d\theta \left\{ \begin{pmatrix} u \\ \varrho D_\varrho u \end{pmatrix}^\dagger \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} v \\ \varrho D_\varrho v \end{pmatrix} \right\} = 0. \tag{3.17}$$

It is convenient to expand the vector $(\, v_i \quad \varrho D_\varrho v_i \,)$ in terms of eigenvectors of $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. These eigenvectors, $\mathbf{e} = \frac{1}{\sqrt{2}}(\, 1 \quad -i \,)$ and $\bar{\mathbf{e}} = \frac{1}{\sqrt{2}}(\, 1 \quad i \,)$ have eigenvalues 1 and -1 respectively. In general the functions near the puncture may be expanded out in harmonics, $\Theta_{Ii}(\theta), \quad I \in \mathbf{Z}^N$, which are orthogonal and have the proper monodromies.

$$\begin{aligned} \Theta_{Ii}(\theta + 2\pi) &= M_{\mathcal{A}ij} \Theta_{Ij}(\theta), \\ \oint d\theta \, \bar{\Theta}_{Ii}(\theta) \Theta_{Ji}(\theta) &= \delta_{IJ}. \end{aligned} \tag{3.18}$$

We posit a specific expansion for v,

$$\begin{pmatrix} v \\ \varrho D_\varrho v \end{pmatrix}_i = \sum_{I,J \in \mathbf{Z}^N} A_I(\varrho) [\delta_{IJ} \mathbf{e} + U_{IJ} \bar{\mathbf{e}}] \Theta_{Ji}(\theta), \tag{3.19}$$

take the general expansion for u,

$$\begin{pmatrix} u \\ \varrho D_\varrho u \end{pmatrix}_i = \sum_{I \in \mathbf{Z}^N} [B_I(\varrho) \mathbf{e} + C_I(\varrho) \bar{\mathbf{e}}] \Theta(\theta)_{Ii}, \tag{3.20}$$

and use (3.17). For finite ϱ, the function whose limit we wish to evaluate is

$$\sum_{I,J \in \mathbf{Z}^N} [\bar{B}_I - \bar{C}_J U_{IJ}] A_I. \tag{3.21}$$

The vanishing of the above expression (3.21) for finite ϱ would imply that $(\, u \quad \varrho D_\varrho u \,)_i$ has the same form as (3.19)

$$\begin{pmatrix} u \\ \varrho D_\varrho u \end{pmatrix}_i = \sum_{I,J \in \mathbf{Z}^N} B_I(\varrho) [\delta_{IJ} \mathbf{e} + U_{IJ} \bar{\mathbf{e}}] \Theta(\theta)_{Ji}, \tag{3.22}$$

but in the limit, the precise condition is slightly more complicated. If v_i describes a normalizable state, the coefficient A_I in (3.19) must be less singular than ϱ^{-1},

($\varrho A_I(\varrho,\theta) \to 0$ for $\varrho \to 0$). This implies that in the limit the correct condition is

$$\begin{pmatrix} v \\ \varrho D_\varrho v \end{pmatrix}_i = \sum_{I,J\in\mathbf{Z}^N} A_I(\varrho)[\delta_{IJ}\mathbf{e} + U_{IJ}\bar{\mathbf{e}}]\Theta_{Ji}(\theta) + O(\varrho), \tag{3.23}$$

Now, for the vanishing of (3.17). If we posit (3.23) as above, a condition of the exact same form must hold for the vector $(\, u \quad \varrho D_\varrho u\,)_i$, where the coefficients, denoted A_I in the condition (3.23) above need not be the same for the two vectors.

The full set of conditions for the existence of a self-adjoint extension of the Landau Hamiltonian (2.5) in the many-punctured plane are that all of the states in the Hilbert space have the same unitary monodromies around the punctures and that all of the functions have limiting behavior (3.23) with the same values for the unitary matrices U_{IJ} at a given puncture. There are no conditions on the boundary values of the wave functions at spatial infinity because the normalizability requirement makes them vanish sufficiently rapidly there. Each of the parameters $\mathcal{M}_\mathcal{A}$ and U_{IJ} is essentially superselected. That is, in a space of states which contains elements with differing values of these parameters, the Hamiltonian will not be self-adjoint. We have assumed that the parameters $\mathcal{M}_\mathcal{A}$ are constants along the cuts from the punctures to infinity in order that the placement of the cuts be arbitrary as long as the remaining domain of the sample be simply connected.

Not every unitary matrix U_{IJ} defines a self-adjoint extension. It usually happens that the functions realizing the general U_{IJ} are not normalizable. To determine the realizable U_{IJ} is a tedious exercise. To find the number of independent parameters describing the extension, one might turn to the von Neumann theory of self-adjoint extensions. In the von Neumann theory, the kernels

$$\mathcal{K}_{\pm} = \{\psi \in L^2 \,|\, H^*\psi = \pm i\psi\}. \tag{3.24}$$

are used to construct the self-adjoint extensions. The domain of the adjoint of H is given by

$$D(H^*) = D(H) \oplus \mathcal{K}_+ \oplus \mathcal{K}_-. \tag{3.25}$$

The self-adjoint extensions are characterized by the unitary maps $U:\ \mathcal{K}_- \to \mathcal{K}_+$. For each such unitary map U, one finds a domain on which H is self-adjoint:

$$D_{\mathrm{sa}}(H) = \{\phi + \beta(\psi_+ + U\psi_-)|\, \phi \in D(H),\, \beta \in \mathbf{C},\, U\,\text{fixed}\}. \tag{3.26}$$

Again, there is the problem of finding which eigenvectors are normalizable, but this usually is not a problem if the eigenvectors can be found in closed form. An analysis of the simple case of one puncture is given in the appendix.

4. An Exact Ground State

4.1 Supersymmetric Quantum Mechanics

Because there are boundary conditions at the punctures on the allowable states, the creation and annihilation operators do not, in general, generate the spectrum of the Landau Hamiltonian in the punctured plane. Therefore, the argument that analytic functions are ground states is no longer valid. A better argument starts from a supersymmetrized Hamiltonian, since the supersymmetry guarantees that the spectrum is non-negative. It is well known that the supersymmetrization of the spectrum generating algebra for the Landau levels describes the same physics with the introduction of electron spin.[8] In addition to the bosonic oscillator $\hat{a}$, $\hat{a}^\dagger$ we introduce a fermionic oscillator $\hat{\xi}$, $\hat{\xi}^\dagger$ from which we construct a supercharge $Q = \sqrt{|\omega|}(\hat{a}\hat{\xi}^\dagger + \hat{a}^\dagger\hat{\xi})$. The supercharge squares to

$$Q^2 = |\omega|(\hat{\xi}^\dagger\hat{\xi} + \hat{a}^\dagger a) = |\omega|(\mathcal{N}_f + \mathcal{N}_b) = |\omega|(\mathcal{N}_f - \tfrac{1}{2}) + H \tag{4.1}$$

With the value of $\omega = \frac{eB_0}{m}$, (4.1) is exactly (in the absence of corrections to $g-2$) the Hamiltonian for planar motion in a uniform magnetic field with the magnetic moment coupling included. In this case we interpret $(\mathcal{N}_f - \frac{1}{2})$ as $\frac{1}{2}\sigma_z$ and the operator $\hat{\xi}^\dagger$ flips the spin from parallel to antiparallel. The states are paired; an antiparallel state in one Landau level is paired with the parallel state one level up.

4.2 Self-adjoint extensions of the Supercharge

It is interesting to consider the question of when the supercharge is self-adjoint. Whenever a self-adjoint domain of the supercharge contains a self-adjoint domain of the Landau Hamiltonian, the lowest possible energy will be $\frac{1}{2}\omega$. We expect that the domain $D(Q)$ on which Q is self-adjoint will be larger than $D(H)$ because Q is a first-order operator. From the expressions

$$\begin{aligned}
\hat{a}^\dagger &= \sqrt{2}(-\partial + \tfrac{1}{4}\bar{z}),\\
\hat{a} &= \sqrt{2}(\bar{\partial} + \tfrac{1}{4}z),\\
\hat{\xi}(f_0 + \xi f_1) &= \xi f_0,\\
\hat{\xi}^\dagger(f_0 + \xi f_1) &= f_1,
\end{aligned} \tag{4.2}$$

we construct the supercharge $Q = \sqrt{|\omega|}(\hat{a}\hat{\xi}^\dagger + \hat{a}^\dagger\hat{\xi})$ and examine the hermiticity condition

$$(QU, V) - (U, QV) = 0. \tag{4.3}$$

Because the supercharge is off-diagonal in the 1, ξ basis, we expect that the self-adjointness conditions on the wave functions will be quite a bit weaker than for the Landau Hamiltonian. The following calculation bears out this expectation. We put $U = u_0 + \xi u_1$ and $V = v_0 + \xi v_1$ and obtain

$$\begin{aligned}
0 &= \left\langle \hat{a}^\dagger u_0, v_1 \right\rangle + \langle \hat{a} u_1, v_0 \rangle - \langle u_0, \hat{a} v_1 \rangle - \left\langle u_1, \hat{a}^\dagger v_0 \right\rangle \\
&= \sqrt{2} \int_{\mathcal{C}} dz \wedge d\bar{z} \left\{ \partial(\bar{u}_0 v_1) - \bar{\partial}(\bar{u}_1 v_0) \right\} \\
&= \sqrt{2} \oint_{\partial\mathcal{C}} d\bar{z}\, (\bar{u}_0 v_1) + dz\, (\bar{u}_1 v_0). \\
\Rightarrow \quad 0 &= \sqrt{2} \oint_{\partial\mathcal{C}} \frac{d\zeta}{\zeta} \left(\bar{u}_0 \zeta v_1 - \bar{\zeta} \bar{u}_1 v_0 \right) \\
&= \sqrt{2} \oint_{\partial\mathcal{C}} \frac{d\zeta}{\zeta} \begin{pmatrix} u_0 \\ \zeta u_1 \end{pmatrix}^\dagger \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ \zeta v_1 \end{pmatrix}, \\
&\text{and} \quad u_0,\, u_1,\, v_0,\, v_1 \quad \text{have identical unitary monodromies.}
\end{aligned} \tag{4.4}$$

Following the analysis after eqn. (3.17), we find that the self-adjoint extensions of the supercharge are characterized by the boundary conditions at the punctures

$$\begin{pmatrix} v_0 \\ \zeta v_1 \end{pmatrix}_i = \sum_{I,J \in \mathbf{Z}^N} B_I(\varrho) [\delta_{IJ} \mathbf{e} + W_{IJ} \bar{\mathbf{e}}] \Theta_{Ji}(\theta) + O(\varrho). \tag{4.5}$$

If $W_{IJ} = \delta_{IJ}$, there are no restrictions on the derivatives of the states at the punctures. This domain is obviously realizable with normalizable states and is larger than the domains found for the Landau Hamiltonian. Thus the smallest possible eigenvalue of the Landau Hamiltonian is $\frac{1}{2}\omega$.

4.3 An Exact Ground State

The simplest case to consider is that of a $\mathbf{C}^2$ valued wave function, v_i. If the sample has no punctures, we know that the ground states are just given by analytic functions. We know that a ground state of the extended Landau Hamiltonian is also given by an analytic function, according to (2.9), if the analytic function obeys the correct boundary conditions at the punctures. For illustrative purposes, we will restrict ourselves to the two component case.

$$\psi_{i,0} = \exp(-|z|^2/4)\phi_i(z) \quad i = 1, 2 \tag{4.6}$$

Because the punctures are at $z = 0$ and $z = 1$, a suitable ground state may be found by using hypergeometric functions ${}_2F_1(a, b; c; z)$. We would like to construct a pair of functions which has unitary monodromies around the two punctures and vanishes at the punctures as well. The pair of functions

$$\mathbf{u} = \begin{pmatrix} F(a, b; c; z) \\ z^{1-c}F(a + 1 - c, b + 1 - c; 2 - c; z) \end{pmatrix} \tag{4.7}$$

has monodromy

$$\mathbf{u}(z\, e^{2\pi i}) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\pi i c} \end{pmatrix} \mathbf{u}(z) \tag{4.8}$$

around the origin and monodromy

$$\mathbf{u}(1 + (z - 1)\, e^{2\pi i}) = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \mathbf{u}(z) \tag{4.9}$$

around $z = 1$. The B_{ij} are given by[9]

$$
\begin{aligned}
B_{11} &= 1 - 2i\, e^{i\pi(c-a-b)} \frac{\sin(\pi a)\sin(\pi b)}{\sin(\pi c)}, \\
B_{12} &= -2\pi i\, e^{i\pi(c-a-b)} \frac{\Gamma(c)\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(b)\Gamma(a)}, \\
B_{21} &= 2\pi i\, e^{i\pi(c-a-b)} \frac{\Gamma(2-c)\Gamma(1-c)}{\Gamma(1-c+a)\Gamma(1-c+b)\Gamma(1-b)\Gamma(1-a)}, \\
B_{22} &= 1 + 2i\, e^{i\pi(c-a-b)} \frac{\sin(\pi(c-a))\sin(\pi(c-b))}{\sin(\pi c)}.
\end{aligned}
\tag{4.10}
$$

Using these relations, one finds that

$$
\phi(z) = \begin{pmatrix} F(\frac{5}{4}, -\frac{9}{4}; \frac{1}{2}; z) \\ \frac{7i\sqrt{3}}{45}\sqrt{z}F(\frac{7}{4}, -\frac{7}{4}; \frac{3}{2}; z) \end{pmatrix}
\tag{4.11}
$$

has monodromies σ_3 and $i\sigma_2$ around $z = 0$ and $z = 1$ respectively. Furthermore, the vector (4.11) is finite at the two punctures, and has pole of low order at infinity. By multiplying by $z^n(z-1)^m$, for positive integers n and m, we find an infinite number of valid two-component ground states which can be built from ϕ, yielding normalizable ground states of a self-adjoint extension of the Landau Hamiltonian (2.5). Given any set of punctures and any desired monodromies around these punctures, there exists a vector-valued analytic function in the plane minus those punctures which has just those monodromies.[10] This guarantees the existence of many such ground states, so that (4.11) is not an isolated example and its monodromies are by no means the only unitary monodromies possible.

5. Conclusion

We have considered quantization of charged particles in a uniform magnetic field in a topologically nontrivial plane. The nontrivial topology was put in by hand, considering the "punctures" to be part of the boundary of the configuration space. Boundary conditions at the punctures are necessary to keep the evolution of the system unitary. It is natural to inquire what physically could account for the punctures. Any defects in the material which have short range interactions certainly could be considered punctures. More interesting would be to see how far one could push the analogy of punctures to solitons or quasiparticles. Punctures with nontrivial monodromy are physically equivalent to infinitesimally thin flux tubes. When the wave functions are scalars the fluxes are abelian. In the vector-valued quantizations the fluxes are in general non-abelian. In the scalar quantizations with monodromies $e^{2\pi i\theta}$ it is tempting to identify the punctures with some type of anyon. In fact, in considering a many electron state in the plane with such a puncture, one is led to the ground state

$$\psi(z_1,\ldots,z_K) = \exp\bigl(-\sum_i |z_i|^2/4\bigr) \prod_{i=1}^{K}(z_i - z_0)^\theta \prod_{i<j}(z_i - z_j). \tag{5.1}$$

This state bears a striking resemblance to the Laughlin wave function. This suggests that there is a relationship between the two and it would be interesting to find such a relationship.

In the system described above, the statistics of the electrons are altered by the puncture. If the two electrons are exchanged through a path which does not encircle the puncture, the wave function simply changes sign. When the exchange path encircles the puncture, the wave function gets a factor $-e^{2\pi i\theta}$. In the more general vector quantizations the wave function would get an additional unitary matrix factor if the exchange path encircled a puncture.

More general vector-valued wave functions are, in general, mysterious and the analysis here has shed no new light on their interpretation. In fact, proceeding to a many particle state is problematical and no guidance is given by considering analogous systems such as the asymmetric rotor, where it is not necessary. However, in analogy with the scalar states, one expects that the punctures, interpreted as quasiparticles, should have more general "braid" statistics than anyons.

APPENDIX

Here we consider the extension of the Landau Hamiltonian in a plane punctured at the origin. To determine the different self-adjoint extensions, we first need to determine the kernels

$$\mathcal{K}_{\pm} = \ker(H^{*} \mp i). \tag{.1}$$

The dimensions of these kernels are called the deficiency indices, denoted (n_{+}, n_{-}). There exist self-adjoint extensions of the operator H if and only if $n_{+} = n_{-}$. The homotopy group is $\pi_1(\mathbf{R}^2 \backslash \{0\}) = \mathbf{Z}$, all of whose UIR's are one dimensional. Thus we need only consider wave functions having a single component. For the trivial representation, the deficiency indices are $(1, 1)$, while for any other representation, the deficiency indices are $(2, 2)$. Our starting domain, $D(H)$, is $\mathcal{C}_0^{\infty}(\mathbf{R}^2 \backslash \{0\})$, such that $\psi(r, \theta) \in D(H)$ satisfies

$$\psi(r, \theta + 2\pi) = e^{2\pi i \nu} \psi(r, \theta), \quad 0 \leq \nu < 1. \tag{.2}$$

We can solve the Schrödinger equation $H^*\psi = E\psi$ for any E, including $\pm i$, by separation of variables. Set $\psi(r,\theta) = R_n(r)e^{i(n+\nu)\theta}$. The general solution regular at infinity is

$$\begin{aligned} R_n(r) &= e^{-r^2/4} r^{n+\nu} \Psi(\tfrac{1}{2}(1-\frac{2E}{\omega}),\, 1+n+\nu,\, \frac{r^2}{2}), \qquad\qquad n \geq 0, \\ R_{-m}(r) &= e^{-r^2/4} r^{\nu-m} \Psi(\tfrac{1}{2}(1-\frac{2E}{\omega}) + m - \nu,\, 1+m-\nu,\, \frac{r^2}{2}), \quad m \geq 1. \end{aligned} \tag{.3}$$

Here $\Psi(a,b,x)$ is a confluent hypergeometric function.[11] There is another set of solutions involving the other confluent hypergeometric function, $\Phi(a,b,x)$. These solutions are not normalizable, however, because they behave as $e^{r^2/4}$ for large r. The solutions involving Ψ are singular at the origin, but for each of the eigenvalues $E = \pm i$ there are two normalizable solutions, $R_0(r)$ and $R_{-1}(r)$ for $\nu \neq 0$ and only one, $R_0(r)$, for $\nu = 0$. The energy spectrum for $m, n = 1, 2, \ldots$, $E = \frac{\omega}{2}(2k+1)$, $\frac{\omega}{2}(2k+1+2(1-\nu))$, $k = 2, 3, \ldots$, may be read off from these solutions since one knows that the small r behavior of the confluent hypergeometric function $\Psi(a,b,r)$ is

$$\Psi(a,b,r) \sim \frac{\Gamma(b-1)}{\Gamma(a)} r^{1-b} \quad \text{for} \quad b > 1. \tag{.4}$$

If the wave functions all must vanish at the origin then the lowest states have energies $E = \frac{\omega}{2}, \frac{\omega}{2}(3-2\nu)$.

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Figure Captions

Fig. 1: Boundary of the simply connected piece of the sample.

Fig. 2: The boundary contour $\partial\mathcal{C}$.

Figure 1.

Figure 2.