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Duality and the Vacuum

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Abstract

We examine the issue of duality both in electrodynamics and in Kalb–Ramond—scalar axion systems. In D spacetime dimensions the dual abelian theories of $p-1$- and $D-p-1$-form potentials have vacua classified by the dimensions of the cohomology spaces $H^{p-1}(\,^{(D-1)}\mathcal{M})$ or $H^{D-p-1}(\,^{(D-1)}\mathcal{M})$ respectively. The vacua are characterized by topological charges which are expectation values for generalized 'Wilson loop' operators around non-trivial cycles. In certain instances the vacua exhibit a theta angle parametrization much as in QCD. The relation of axionic hair and discrete gauge hair is analyzed in the topologically massive Kalb–Ramond theory. If there are no fundamental strings in the theory, axionic charge is replaced by an irrelevant vacuum angle.

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1. Introduction

It is a well-known fact that certain theories, among them gauge theories and abelian (and even non-abelian[1]) scalar theories, admit description in terms of two different sets of potentials. The duality in these theories is the exchange of field equations and Bianchi identities.

What is not well appreciated is that these theories are not identical. The propagating modes written in one of the potentials do have description[2] in terms of the dual potential, although the relation is non-local. It is precisely the vacua of the theories which differ. It is the vacuum properties of the theories which carry such information as Aharonov-Bohm phases and such exotic features as quantum hair. Duality in terms of potentials is, in fact, another way of expressing the lattice duality which exchanges the strong and weak coupling regimes.[3] In the Ising model duality exchanges the spin lattice with its dual lattice. The present work is an attempt to clarify the issues of the vacua of theories dual to each other and to understand the subtle differences between the different kinds of quantum hair which have been proposed.

We find that the vacua of dual abelian theories have surprisingly rich structures, depending on the topology of the spatial slices of spacetime. The main line of attack is the construction of topological currents and, from them, observables and global symmetries.

We first construct the topological currents both in electrodynamics and in the scalar—Kalb–Ramond axion system, and from them their charges and observables. Only the source-free equations are considered, but sources may be put in by excising the regions from the spatial manifold where the sources are located. The source-free equations are considered on the remaining manifold, which may now have non-trivial topology. The same analysis applies to spatial manifolds having intrinsic topology not created by physical currents. As an aside, we put the Dirac quantization of magnetic charge and the quantization of axionic charge on S^3 into the same language, even though these are not strictly issues of the vacuum. Finally, we discuss the issue of quantum hair in the superconducting vacuum of coupled electromagnetic and axion fields.

2. Duality and Topological Currents

2.1. Electrodynamics

Although we ordinarily think of Maxwell's equations as being the theory of the electromagnetic potential A_μ, it is sometimes useful to step back to the field strengths themselves. We examine Maxwell's equations in the absence of sources.

$$\begin{aligned} \mathrm{d}F &= 0, \\ \mathrm{d}{*F} &= 0. \end{aligned} \tag{2.1}$$

The standard procedure is to solve the first equation by defining the potential: $F = \mathrm{d}A$, at least locally. Usually we think of this as being more reasonable than using the dual potential G with ${*F} = \mathrm{d}G$, because of the extreme scarcity of magnetic charge. For now we assume the absence of any point-like defects in the spatial slices: $H^2({}^3\mathcal{M}) = 0$.

Let us take the point of view that either set of potentials is reasonable in the absence of charged matter. It is useful to look upon the Maxwell equations as defining two conserved currents, and to note the action of these currents upon states. It is necessary to apply to the currents an arbitrary vector function in each case to saturate the Lorentz indices.

$$\begin{aligned} j^{\mu}_{(\lambda)} &= \lambda_\alpha F^{\alpha\mu}, \\ j^{*\mu}_{(\xi)} &= \xi_\alpha {*F}^{\alpha\mu}. \end{aligned} \tag{2.2}$$

For each current to be conserved, the one-forms λ and ξ must satisfy $\mathrm{d}\lambda = \mathrm{d}\xi = 0$. It is important to note that although these currents are conserved, they have nothing to do with the movement of electric or magnetic charges. (The true currents would be given formally by the replacement of the one-forms λ and ξ with the exterior derivative operator.) These currents, rather, encode topological information about the spatial manifold ${}^3\mathcal{M}$.

If the dynamical variables of the system are taken to be the vector potentials $A_\mu(x)$, then the charges associated to the currents (2.2) are

$$
\begin{aligned}
Q_{(\lambda)} &= \int\limits_{^3\mathcal{M}} \lambda_\alpha \Pi^\alpha_A, \\
\widetilde{Q}_{(\xi)} &= \int\limits_{^3\mathcal{M}} \xi \wedge \mathrm{d}A,
\end{aligned}
\tag{2.3}
$$

where Π_A is the canonical momentum conjugate to A. When λ is an exact form of compact support,

$$
\lambda = \mathrm{d}\chi, \quad \chi(\mathbf{x}) = 0 \quad \text{when} \quad |\mathbf{x}| > R \quad \text{for some } R, \tag{2.4}
$$

we may integrate by parts and find that the charge $Q_{(\lambda)}$ generates true, or "small," gauge transformations: $A_\mu \to A_\mu + \partial_\mu \chi$. Thus, when λ is an exact form, the charge $Q_{(\lambda)}$ must have a trivial action upon physical states, because physical states are gauge invariant. Another way to put this is to say that $Q_{(\mathrm{d}\chi)}$ is a constraint in the sense of Dirac, and must annihilate all physical states. Written in terms their respective dual potentials, the charges Q and $\widetilde{Q}$ are the generators of the global symmetry group of "large" gauge transformations when the forms λ and ξ are closed but not exact. These large gauge transformations are transformations on the fields which are locally gauge transformations but which cannot be patched together globally into a single-valued gauge transformation. The two charges Q and $\widetilde{Q}$ commute,

$$
[Q_{(\lambda)}, \widetilde{Q}_{(\xi)}] = -i \int\limits_{^3\mathcal{M}} \xi \wedge \mathrm{d}\lambda = 0, \tag{2.5}
$$

at least formally.[4] For certain spatial topologies the commutator (2.5), which is related to the linking number of two curves, will not vanish and provides us with information about the vacuum. The algebra obeyed by the charges is the same regardless of which potentials are used.

2.2. Axions: Scalar and Two-Form

In four dimensions a system similar to electrodynamics is the free system of equations for a three index antisymmetric tensor field strength[5] $H = \frac{1}{3!} H_{\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma$.

$$\begin{aligned} dH &= 0, \\ d{*}H &= 0. \end{aligned} \tag{2.6}$$

These may be viewed as defining a free theory of either a two-form tensor $B = \frac{1}{2} B_{\alpha\beta} dx^\alpha \wedge dx^\beta$ or a scalar field ϕ.

$$H = dB \quad \text{or} \quad {*}H = d\phi. \tag{2.7}$$

This expresses the same duality as Maxwell's equations above, the duality exchanging the field equations for the Bianchi identity and *vice versa*, but the two potentials here look very different; so different that one has a large set of gauge invariances while the other does not. We also find two very different looking conserved currents

$$\begin{aligned} j^\mu &= {*}H^\mu, \\ j^{\mu\nu\rho} &= H^{\mu\nu\rho}. \end{aligned} \tag{2.8}$$

The current j^μ is the Noether current for the global Peccei-Quinn symmetry[6] $\phi \to \phi + c$, generated by

$$\widetilde{Q}_{(c)} = \int\limits_{{}^3\mathcal{M}} c\, {*}H, \tag{2.9}$$

while the current $j^{\mu\nu\rho}$, after saturating the Lorentz indices, leads to the Noether charge,

$$Q^\Omega = \frac{1}{2} \int\limits_{{}^3\mathcal{M}} \Omega_{\mu\nu} j^{\mu\nu 0} = \frac{1}{2} \int\limits_{{}^3\mathcal{M}} \Omega_{\mu\nu} \Pi_B^{\mu\nu}, \tag{2.10}$$

generating Kalb-Ramond gauge transformations $B \to B + \Omega$. In order that the current $j^\mu_{(\Omega)}$ be conserved, it is necessary that the two-form Ω be closed: $d\Omega = 0$. This is in perfect analogy with the Maxwell case. When Ω is exact with compact support, $\Omega = \mathrm{d}\Lambda$, Q^{Ω} generates true gauge transformations $B \to B + \mathrm{d}\Lambda$, under which all physical states must be invariant. Again, as in electromagnetism, this is to say that Q^{Ω} is a constraint in the sense of Dirac and must annihilate all physical states. When Ω is closed but not exact of compact support, Q^{Ω} generates a symmetry of the action exactly analogous to the "large" gauge transformations of the Maxwell system.

3. Characterization of the Vacuum Manifold

3.1. Observables: Wilson Surfaces, Loops and Points

In the A_μ-representation, we can show that the charge $\widetilde{Q}_{(\xi)}$, dual to the generator of gauge transformations $Q_{(\lambda)}$, leads to the familiar Wilson[7] loop observable $W[\Gamma]$ for certain spatial topologies. We will suppose that the boundary at infinity of the spatial slice is identified to a point; the slice is a three-sphere. Now we excise a thickened loop, Γ, from the interior; ${}^3\mathcal{M} = S^3 \backslash \Gamma$. The dual charge may be written

$$\widetilde{Q}_{(\xi)} = \int\limits_{{}^3\mathcal{M}} \xi \wedge \mathrm{d}A = \int\limits_{\partial\, {}^3\mathcal{M}} A \wedge \xi. \tag{3.1}$$

If ξ is chosen to be a one-form which is locally $e\, \mathrm{d}\theta/2\pi$, where $\mathrm{d}\theta/2\pi$ links the loop Γ once, then in the limit of vanishing loop thickness the charge becomes

$$\widetilde{Q}_{\left(\frac{e\mathrm{d}\theta}{2\pi}\right)} = e \oint\limits_{\Gamma} A. \tag{3.2}$$

The "current" flowing in the Wilson loop can be defined as $j = {*\mathrm{d}\xi}$, where the dual is now taken in the three-dimensional space. The form ξ is the only closed, non-exact one-form available in the spatial slice, so no other objects can be constructed from $\widetilde{Q}$. One can also immediately see that excising a point, for instance, will not lead to any gauge invariant objects, for there will not be any closed, non-exact one-forms from which to build such an object. This construction is a way of motivating the Wilson loop operator by removing from the spatial manifold the points where charges are located. If the manifold has noncontractible loops, the Wilson operators are more easily looked at as the left hand side of (3.2).

In general, one can define observables for the Maxwell system as the Wilson and 't Hooft[8] loops

$$\begin{aligned} W[\Gamma] &= \exp\Big\{ i\, e \int_\Gamma A \Big\} = \exp\Big\{ i\, \widetilde{Q}_{(\frac{e d\theta}{2\pi})} \Big\}, \\ \widetilde{W}[\Gamma] &= \exp\Big\{ i\, g \int_\Gamma G \Big\} = \exp\Big\{ i\, Q_{(\frac{e d\theta}{2\pi})} \Big\}, \end{aligned} \tag{3.3}$$

upon which the charges (2.3) may act.

The actions of the global generators on the loop observables are

$$\begin{aligned} e^{iQ_{(\lambda)}}\, W[\Gamma]\, e^{-iQ_{(\lambda)}} &= e^{i\, e \int_\Gamma \lambda}\, W[\Gamma], \\ e^{i\widetilde{Q}_{(\xi)}}\, \widetilde{W}[\Gamma]\, e^{-i\widetilde{Q}_{(\xi)}} &= e^{i\, g \int_\Gamma \xi}\, \widetilde{W}[\Gamma], \\ e^{iQ_{(\lambda)}}\, \widetilde{W}[\Gamma]\, e^{-iQ_{(\lambda)}} &= \widetilde{W}[\Gamma], \\ e^{i\widetilde{Q}_{(\xi)}}\, W[\Gamma]\, e^{-i\widetilde{Q}_{(\xi)}} &= W[\Gamma], \end{aligned} \tag{3.4}$$

As for the Maxwell case, the consideration of one set of potentials for the axion system (2.6) leads to a conjugate set of "observables." For the Kalb-Ramond potentials $B_{\mu\nu}$ we consider the Peccei-Quinn, or axion,[9] charge

$$Q_{(a)} = \int_{{}^3\mathcal{M}} a(x) H, \tag{3.5}$$

where $a(x)$ must be a locally constant function. Of course, if the spatial slice ${}^3\mathcal{M}$ is S^3, then the charge is fairly trivial, but suppose we excise a two-surface Σ from the interior of the slice; ${}^3\mathcal{M} = S^3 \backslash \Sigma$. Then, let $a(x)$ be the function which links the surface Σ once: $a(x) = 1$ if x is inside Σ and $a(x) = 0$ otherwise. The Wilson surface observable is just the axion charge enclosed by the surface:

$$W[\Sigma] = \exp\Big\{ i\kappa \oint_{\Sigma} B \Big\}. \tag{3.6}$$

Generalized Wilson surfaces first appeared in the work of P. Orland,[3] as order-disorder parameters useful for distinguishing the phases in the continuum limit of abelian n-form theories in D dimensions.

Analyzing the axion system with the scalar potential, we need to find a topology with non-trivial two-forms, so let us excise a point p. The 'Wilson point' observable will be

$$\widetilde{W}(p) = \exp\{ i\,\epsilon\,\phi(p) \}. \tag{3.7}$$

This we recognize as the phase of a charge-ϵ Higgs field at a given point p. It is the obvious dual of the Wilson surface observable defined above.

3.2. Charge Quantization Conditions

We have shown how gauge transformations and observables arise from the topology of spatial slices ${}^3\mathcal{M}$. It is interesting to put the Dirac monopole charge quantization and axion charge quantization conditions into the same language.

If a point is excised from the interior of the spatial slice ${}^3\mathcal{M}$, one has the possibility of having charges reside at the point. The Dirac monopole charge quantization condition follows from demanding that any Wilson loop have the limiting value 1 as the loop is shrunk to a point.

$$W[\Gamma] \to 1 \quad \text{as} \quad \Gamma \to p. \tag{3.8}$$

For any continuous family of loops Γ_t one can show that the Wilson loops satisfy

$$W[\Gamma_1] = \exp\Big\{ i\,e \int_{\Sigma} F \Big\}\, W[\Gamma_0], \tag{3.9}$$

where $\Sigma = \bigcup_{t\in[0,1]} \Gamma_t$ is the surface swept out by the family of curves Γ_t. If the end curves $\Gamma_{0,1}$ are both points and the surface Σ spanned by the curves encloses a monopole, then the Dirac quantization condition is the immediate consequence.

$$2\pi n = e \oint F = 4\pi\, e\, g. \tag{3.10}$$

Exactly analogous to the Dirac quantization condition is the quantization of axion charge[10] over a spatial slice ${}^3\mathcal{M} = S^3$. The argument above is repeated with Wilson surfaces instead sweeping out the whole volume, starting from the center, $r = 0$, and ending at the point at infinity. The condition reads

$$W[\Sigma_{r=\infty}] = \exp\Big\{ i\,\kappa \int_V H \Big\}\, W[\Sigma_{r=0}] \quad \Rightarrow \quad 2\pi n = \kappa Q_{\text{axion}}. \tag{3.11}$$

3.3. Vacuum Angles

Until now, we have considered only the structure of the pure gauge theory in the absence of matter. Whether or not a Wilson operator is an observable depends on the matter content of the theory. In electromagnetism with charges quantized in units of some elementary charge e, the Wilson loops $W[\Gamma]$ are observables and can be measured as the Aharonov-Bohm phases of a beam of elementary charges traversing the loops Γ. By inspecting (3.4), one can see that the action of $e^{iQ_{(\lambda)}}$ is no longer a symmetry unless λ satisfies a quantization condition forcing the exponents to be integral multiples of $2\pi i$ for any closed curves in the manifold;

$$\frac{e}{2\pi} \int_\Gamma \lambda \in \mathbf{Z} \quad \forall \Gamma \in \pi_1({}^3\mathcal{M}). \tag{3.12}$$

The physical states of the system, the so-called θ-states, form a representation of each global $\mathbf{Z}$ gauge symmetry

$$\begin{aligned} e^{i\,n\,Q_{(\lambda)}} \Big| w_\Gamma\,, \theta_\lambda \Big\rangle &= e^{i\,n\,\theta_\lambda} \Big| w_\Gamma\,, \theta_\lambda \Big\rangle\,, \\ \Big\langle w_\Gamma\,, \theta_\lambda \Big| W[\Gamma] \Big| w_\Gamma\,, \theta'_\lambda \Big\rangle &= w_\Gamma\; \delta_{\theta_\lambda, \theta'_\lambda}. \end{aligned} \tag{3.13}$$

This an abelian version of the Hosotani flux-breaking mechanism,[11] which breaks the global $U(1)^{b_1}$ ($b_1 = \dim H^1_c({}^3\mathcal{M})$) gauge symmetry down to $H^1_c({}^3\mathcal{M}; \mathbf{Z}) = \mathbf{Z}^{b_1}$. In a pure Maxwell system, none of the Wilson loops is an observable, and the states must form a representation of the global $U(1)^{b_1}$ gauge symmetry. The quantities θ_λ will have an invariant meaning for all $\lambda \in H^1({}^3\mathcal{M})$ and there will be a new parameter describing the vacuum. If we transcribe the $U(1)$ gauge theory analysis above to the Kalb-Ramond axion system, the analogous statements become

$$
\begin{aligned}
e^{i\,n\,Q^\Omega}\left|w_\Sigma\,,\theta_\Omega\right\rangle &= e^{i\,n\,\theta_\Omega}\left|w_\Sigma\,,\theta_\Omega\right\rangle, \\
\left\langle w_\Sigma\,,\theta_\Omega\right| W[\Sigma] \left|w_\Sigma\,,\theta'_\Omega\right\rangle &= w_\Sigma\;\delta_{\theta_\Omega,\theta'_\Omega}.
\end{aligned}
\tag{3.14}
$$

The Wilson surfaces are observables and can be measured by the Aharonov-Bohm phases of fundamental strings propagating along the closed surface Σ, yielding the axionic charge of the state. If a point p has been excised from ${}^3\mathcal{M}$ to create non-trivial two-cohomology, then the vacuum angle θ is the value of the scalar axion at that point p. Here we are assuming that there are instantons which mediate transitions from one n-state to another, such as occurs in QCD.[12] It is known that there are such instantons in the Kalb-Ramond theory in 3+1 dimensions[3,10] and in the $U(1)$ gauge theory on the spacetime $S^1 \times \mathbf{R}$.[13] In the absence of instanton tunneling transitions, the n-vacua are good quantum vacua and the global gauge symmetry is completely broken.

As for QCD, these theta angles can show up in the action. They are implemented in the path integral by adding total derivative terms to the action:

$$
S_{\text{topological}} = \theta_\lambda \int \mathrm{d}{*j_{(\lambda)}}. \tag{3.15}
$$

Actions similar to (3.15), in that forms on the manifold have been used in their construction, have been considered by Nair and Schiff.[14]

4. Axionic Charge and Quantum Hair

Although we have little to say directly about the purely scalar version of axionic hair, which is more like the linking of two loops rather than the linking of a surface and a point, we do point out that there is a parallel in the theta structure of the Kalb-Ramond theory. For rational theta, $\theta = k/N$, the global symmetry group is $H^2_c({}^3\mathcal{M}, \mathbf{Z}_N)$, which is exactly the symmetry left when a charge N scalar gets a VEV and there is a charge 1 object in the theory. In a theory of an uncharged scalar, the string is a singularity and the field around it is not a vacuum. That is to say, the field strength is non-vanishing and the string has a divergent energy. However, when there is a gauge field coupling to a *charged* scalar, there can be a "vacuum" string solution where the field strengths vanish outside the string. The equations of motion for the topologically coupled two- and one-forms describing the charged scalar theory,

$$\begin{aligned} \mathrm{d}{*H} &= mF, \\ \mathrm{d}{*F} &= -mH, \end{aligned} \tag{4.1}$$

are no more than the London equations, and may be cast either as the original equations of motion for the interacting Kalb-Ramond and abelian gauge fields with the action[4]

$$S = \int \tfrac{1}{2} H \wedge {*H} + \tfrac{1}{2} F \wedge {*F} + mB \wedge F, \tag{4.2}$$

or as the equations of motion for the Stückelberg action

$$S = \int \tfrac{1}{2} F \wedge {*F} + \tfrac{1}{2} m^2 (A + \frac{1}{m}\mathrm{d}\phi) \wedge *(A + \frac{1}{m}\mathrm{d}\phi), \tag{4.3}$$

once the identification

$$*H = \mathrm{d}\phi + mA, \tag{4.4}$$

is made. In (4.1) the 'current' is $j_{em} = m{*H}$. If the operator d* is applied to the second equation, one finds that the field strengths satisfy the massive Klein-Gordon equation d*d*$F = -m^2 F$. In keeping with the philosophy of the previous sections, we work at the level of the field equations (4.1), needing the action only for the sake of identifying the momenta conjugate to the potentials we choose to describe the system.

Applying d to both sides of the equations (4.1), we find that the usual Bianchi identities follow, again leading to topological charges

$$\begin{aligned}\widetilde{Q}_{(\xi)} &= \int\limits_{{}^3\mathcal{M}} \xi \wedge \mathrm{d}A, \\ \widetilde{Q}_{(c)} &= \int\limits_{{}^3\mathcal{M}} c\,{*H}.\end{aligned} \tag{4.5}$$

The equations of motion lead to the charges very similar to before, with one crucial difference, an extra term proportional to m, whose origin is the non-invariance of the Lagrangian density in (4.2).

$$\begin{aligned}Q_{(\lambda)} &= \int\limits_{{}^3\mathcal{M}} \lambda_\alpha \Pi_A^\alpha, \\ Q^\Omega &= \tfrac{1}{2} \int\limits_{{}^3\mathcal{M}} \Omega_{ij}(\Pi_B^{ij} + m\epsilon^{ijk} A_k)\end{aligned} \tag{4.6}$$

The charges (4.6), which act independently when $m = 0$, now do not commute. The global generators for the charges (4.6) have a co-cycle in their commutation relations:

$$\exp(iQ_{(\lambda)}) \exp(iQ^\Omega) = e^{i\,m \int_{{}^3\mathcal{M}} \lambda\wedge\Omega} \exp(iQ^\Omega) \exp(iQ_{(\lambda)}). \tag{4.7}$$

When $m \neq 0$ the two large gauge transformations are not independent. Let us consider the black hole geometry which can carry axionic charge by virtue of its non-trivial second cohomology. If the $U(1)$ gauge group is compact, then all the charges are quantized in multiples of some smallest unit e. Under a large $U(1)$ gauge transformation using $\lambda = d\chi$, with $\chi \to 2\pi/e$ as $|\mathbf{x}| \to \infty$, all states must be invariant. This has the effect of quantizing the allowable Ω in (4.6). We find

$$\frac{m}{e} \oint_{r=\infty} \Omega \in \mathbf{Z}. \tag{4.8}$$

If there are fundamental strings in the theory, the large Kalb-Ramond gauge transformations must leave the Wilson surfaces invariant as well:

$$e^{iQ^\Omega} W[\Sigma] e^{-iQ^\Omega} = e^{i\kappa \oint_\Sigma \Omega} W[\Sigma] \Rightarrow \frac{\kappa}{2\pi} \oint_\Sigma \Omega \in \mathbf{Z}. \tag{4.9}$$

If the string coupling $\frac{\kappa}{2\pi}$ and the mass coefficient $\frac{m}{e}$ are not rationally related, then there will be no allowable large Kalb-Ramond gauge transformations at all.

Let us examine the scalar representation of the algebra of large gauge transformations (4.7). We denote the generator $e^{iQ_{(\lambda)}}$ by $U(\omega)$ when the limiting value of $e\chi$ is ω, where $\omega \in [0, 2\pi)$ and $\lambda = d\chi$. We also denote by $V(n_\Omega)$ the generator e^{iQ^Ω} where the winding number of Ω given in eq. (4.8) is n_Ω. Thus (4.7) becomes

$$U(\omega)V(n_\Omega) = e^{i\omega n_\Omega} V(n_\Omega) U(\omega). \tag{4.10}$$

As usual, the states are labeled by charge Q

$$U(\omega) |Q\rangle = e^{i\omega Q} |Q\rangle, \tag{4.11}$$

which we wish to preserve as a representation of the subgroup of the full group of transformations (4.10). If there were no cocycle in the relations (4.10) then the irreps would be labeled by an angle as well. In the presence of the cocycle, the states may be labeled by an angle, just as in the massless case, but now the large Kalb-Ramond gauge transformation also changes the charge sector:

$$\begin{aligned} V(n_\Omega)\,|Q,\,\theta\rangle &= e^{i\theta n_\Omega}\,|Q+n_\Omega,\,\theta\rangle\,, \\ U(\omega)\,|Q,\,\theta\rangle &= e^{i\omega Q}\,|Q,\,\theta\rangle\,. \end{aligned} \tag{4.12}$$

This representation is reducible, being the product of an irreducible representation and a circle. That is, the states may be redefined so that everything is independent of the angle θ.

If there is a Wilson surface observable, $W[\Sigma]$, as in (4.9), which breaks the large Kalb-Ramond gauge transformations completely,

$$\begin{aligned} U(-\omega)W[\Sigma]U(\omega) &= W[\Sigma], \\ V(n)W[\Sigma]V(-n) &= e^{i\kappa n}W[\Sigma], \end{aligned} \tag{4.13}$$

then the states are labeled by axionic charge instead of a vacuum angle.

To see this, we let the phase of the Wilson surface expectation value be given by the axionic charge

$$\frac{\langle Q_{\rm ax}|W[\Sigma]|Q_{\rm ax}\rangle}{|\,\langle Q_{\rm ax}|W[\Sigma]|Q_{\rm ax}\rangle\,|} = e^{i\kappa Q_{\rm ax}}, \tag{4.14}$$

and find, using (4.10) and (4.12), that the states of axionic charge are

$$|Q,\,Q_{\rm ax}\rangle = \int \frac{d\theta}{\sqrt{2\pi}}\, e^{i\frac{m}{e}\theta Q_{\rm ax}}\,|Q,\,\theta\rangle\,. \tag{4.15}$$

Thus there are both types of dual topological charges corresponding to Wilson surfaces (2.9) and points (2.10) and Wilson and 't Hooft lines (2.3). The $\mathbf{Z}_N$ quantum hair examined by Krauss and Preskill *et al.*[15] is described in this language as the detection of 't Hooft lines by Wilson lines, and not the detection of a Wilson point by a Wilson surface. That is to say, a charge e object, when transported around a string carrying flux $2\pi/Ne$, will pick up a phase $e^{2\pi i/N}$. This happens regardless of the mass of the gauge field, and depends only upon the total flux, which is confined to the string by the superconductivity of the vacuum. The flux in the string singularity is quantized in the vacuum by virtue of the 'charge' of the Stückelberg scalar. The Kalb-Ramond field is assumed to be given by (4.4). If the 'mass' in eq. (4.4) is quantized, $m = Ne$, the flux in the string becomes quantized in units of $2\pi/N$ if the scalar field ϕ is to be single-valued around the string. A charge e Wilson loop can still detect the flux in the string, however. This is the origin of $\mathbf{Z}_N$ quantum hair. This hair is different from axionic hair, which is the detection of a Wilson point by a Wilson surface. The existence of axionic hair is similarly unaffected by the mass of the the Kalb-Ramond gauge field.[16]

5. Discussion

The relationship between a theory and its dual is more complicated than it may seem at first sight. For a theory of an abelian p-form field strength in D spacetime dimensions, there are descriptions either in terms of a $p-1$-form potential or a $D-p-1$-form potential. If there are observables corresponding to the Wilson loops in both cases, then the vacua of the theory are generically inequivalent in the sense that they are classified by the dimensions of the cohomology spaces $H^{p-1}(\,^{(D-1)}\mathcal{M})$ or $H^{D-p-1}(\,^{(D-1)}\mathcal{M})$ depending on which potential is used. Even though it is the vacua which are different, there can be dynamical consequences[17] of this inequivalence. Once there are charges coupling to the potentials in the theory, the issue of equivalence becomes more complicated. We avoided the issue of sources by excising regions from the spatial manifold and considering only the source-free equations on the remaining part. In the absence of sources, the dual theories are equivalent, except for their vacua. Depending on the theory, there may be θ-vacua even when there are no observables and the θ structure generically will be different for different potentials.

It should be mentioned that topological currents can be constructed for the field equations of $U(1)$ Chern-Simons theory in 2+1 dimensions as well, and lead directly to the vacuum structure there. The topological current and its charge are

$$\begin{aligned} j^{\mu}_{(\lambda)} &= \lambda_\alpha {*A}^{\alpha\mu}, \\ Q_{(\lambda)} &= \int j^0_{(\lambda)} = \int \lambda_\alpha \Pi^\alpha_A. \end{aligned} \tag{5.1}$$

In this respect, the only real difference between the Chern-Simons theory and the Maxwell theory is that there are *only* topological vacuum states in the Chern-Simons theory while the full Maxwell theory has photon states as well.

The issue of duality is interesting not only in four dimensions. Some time ago, Orland noted the equivalence of Hodge duality and duality in statistical mechanics[3] and was able to show that n th rank theories are disordered in $D = n + 2$ but not in $D = n + 3$ and higher. Recently it has been argued that ten-dimensional strings are solitons in a theory of ten-dimensional five-branes and *vice versa*.[18] This duality is interesting for the reason that while string theory is believed to be strongly coupled, the five-brane theory, in agreement with ref. [3], is weakly coupled. Thus, if a perturbation theory for five-branes can be constructed, it should be valid when string perturbation theory is not. It might be interesting to investigate one aspect of this duality; the “axion” in $D = 10$. The two dual charges are constructed from two-cocycles and from six-cocycles. What was the scalar axion in four dimensions will become a six-form axion. If there are instantons in string theory which can change the axionic charge by integral values, it may be that there is a separate phase of the theory wherein the dual topological charges, thought of as Wilson operators, get VEVs signaling compactification of six dimensions. Of course, if the string—five-brane duality is exact, it should explain how the seemingly different field theoretic vacua are actually connected and somehow the same.

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REFERENCES

1. S. G. Rajeev, *Duality and Gauge Invariance*, MIT preprint CTP-1335 (unpublished).
2. E. S. Fradkin and A. A. Tseytlin, *Ann. Phys.* (N.Y.) **162** (1985) 31.
3. P. Orland, *Nucl. Phys.* **B205[FS5]** (1982) 107.
4. T.J. Allen, M.J. Bowick and A. Lahiri, *Mod. Phys. Lett.* **A6** (1991) 559.
5. M. Kalb and P. Ramond, *Phys. Rev.* **D9** (1974) 2273;
   S. Deser, *Phys. Rev.* **187** (1969) 1931.
6. R.D. Peccei and H.R. Quinn, *Phys. Rev. Lett.* **38** (1977) 1440.
7. K. Wilson, *Phys. Rev.* **D10** (1974) 2445.
8. G. 't Hooft, *Nucl. Phys.* **B138** (1978) 1; **B153** (1979) 141.
9. M.J. Bowick, S.B. Giddings, J.A. Harvey, G.T. Horowitz and A. Strominger, *Phys. Rev. Lett.* **61** (1988) 2823.
10. R. Rohm and E. Witten, *Ann. Phys.* (N.Y.) **170** (1986) 454.
11. Y. Hosotani, *Phys. Lett.* **126B** (1983) 309.
12. A.A. Belavin, A.M. Polyakov, A.S. Schwartz, and Yu.S. Tyupkin, *Phys. Lett.* **59B** (1975) 85.
13. C. Callan, R. Dashen, and D. Gross, *Phys. Lett.* **66B** (1977) 375.
14. V.P. Nair and J. Schiff, *Phys. Lett.* **B246** (1990) 423;
    Columbia University preprint CU-TP-521.
15. J. Preskill and L.M. Krauss, *Nucl. Phys.* **B341** (1990) 50;
    J. Preskill, *"Quantum Hair," Phys. Scr.* **T36** (1991) 258;
    S. Coleman, J. Preskill and F. Wilczek, *Mod. Phys. Lett.* **A6** (1991) 1631;
    J.P. Preskill, P. Schwarz, A. Shapere, S. Trivedi, and F. Wilczek, *Mod. Phys. Lett.* **A6** (1991) 2353;
    S. Coleman, J. Preskill and F. Wilczek, *Phys. Rev. Lett.* **67** (1991) 1975.

16. T.J. Allen, M. Bowick and A. Lahiri, *Phys. Lett.* **B237** (1990) 47.

17. M.J. Duff and P. van Nieuwenhuizen, *Phys. Lett.* **94B** (1980) 179;
C.P. Burgess and A. Kshirsagar, *Nucl. Phys.* **B324** (1989) 157.

18. M.J. Duff and J.X. Lu, *Nucl. Phys.* **B354** (1991) 129, 141; **B357** (1991) 534.