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The Dirac Propagator From Pseudoclassical Mechanics*

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Abstract

In this note it is demonstrated that the unitary propagator is obtained from the pseudoclassical system proposed as a first-quantized version of the Dirac particle by Berezin and Marinov. The action for the system is written in a form which has manifest global supersymmetry on the worldline.

In Memoriam Richard P. Feynman

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1. Introduction

Little is known of the point particle versions of quantum field theories while much is known about quantum field theories. On the other hand, string theories are understood only in their first-quantized form. By understanding well the connection between field theories and their point particle versions, one may hope to understand the connection between strings in a first-quantized form and in a form suitable for nonperturbative calculations.

In this note it is demonstrated that the unitary propagator is obtained from a Batalin-Fradkin-Vilkovisky [1] quantization of the action proposed by Berezin and Marinov [2] for the pseudoclassical description of the Dirac electron.

When computing the propagator, other authors [3,4] have found a factor of γ_5. Such a factor ruins the unitarity of the propagator, which must be unitary since the Batalin-Fradkin-Vilkovisky path integral is manifestly unitary. The reconciliation is arrived at when the second-class constraints are treated carefully.

The pseudoclassical version of the Dirac electron is a nice example of a system whose second-class constraints may be treated in canonical quantization in an elegant fashion.

2. The Classical Dirac Electron

The first classical action for the Dirac electron using anticommuting variables was written by Berezin and Marinov [2].

$$S = \int d\tau[-m\sqrt{-\dot{x}^2} + \frac{i}{2}(\xi^\mu\dot{\xi}_\mu + \xi_5\dot{\xi}_5) - (u_\mu\xi^\mu + \xi_5)\lambda]. \tag{1}$$

In this action the variables ξ_μ, ξ_5 and λ are anticommuting while x^μ is commuting.

The notation u_μ is a shorthand for $\dot{x}_\mu/\sqrt{-\dot{x}^2}$. The canonical momenta

$$
\begin{aligned}
&P_\mu = mu_\mu + i\lambda(-\dot{x}^2)^{-\frac{1}{2}}(\xi_\mu + u_\nu \xi^\nu u_\mu),\\
&P_\xi = \frac{i}{2}\xi,\\
&P_5 = \frac{i}{2}\xi_5,\\
&P_\lambda = 0,
\end{aligned}
\tag{2}
$$

lead to the constraints

$$
\begin{aligned}
P_\lambda &\approx 0,\\
P^2 + m^2 &\approx 0,\\
P_\mu \xi^\mu + m\xi_5 &\approx 0,\\
P_{\xi_\mu} - \frac{i}{2}\xi^\nu \eta_{\mu\nu} &\approx 0,\\
P_5 - \frac{i}{2}\xi_5 &\approx 0.
\end{aligned}
\tag{3}
$$

The first three of these are first-class while the last two are second-class. The two second-class constraints may be removed by introducing the Dirac brackets

$$
\begin{aligned}
&\{\xi_\mu, \xi_\nu\}_{DB} = i\eta_{\mu\nu},\\
&\{\xi_5, \xi_5\}_{DB} = i.
\end{aligned}
\tag{4}
$$

The Hamiltonian, which is a linear combination of the constraints

$$H = \lambda_1(P^2 + m^2) + \lambda_2(P_\mu \xi^\mu + m\xi_5),$$

generates the equations of motion

$$
\begin{aligned}
&\dot{x}_\mu = 2\lambda_1 P_\mu + \xi_\mu \lambda_2,\\
&\dot{P}_\mu = 0,\\
&\dot{\xi}_\mu = iP_\mu \lambda_2,\\
&\dot{\xi}_5 = im\lambda_2.
\end{aligned}
\tag{5}
$$

One may fix a gauge by choosing the multipliers to be

$$\begin{aligned} \lambda_1 &= \frac{1}{2m}, \\ \lambda_2 &= 0. \end{aligned} \tag{6}$$

This Hamiltonian yields a simple set of equations of motion.

The constraint $(P^2 + m^2)$ generates τ reparametrizations just as it does for the scalar particle, while the constraint $(P \cdot \xi + m\xi_5)$ generates translations in an anticommuting time, which we might call ϑ. This system is an example of a one-dimensional supersymmetric theory. This can be demonstrated by writing the theory in a manifestly supersymmetric form. We accomplish this by pairing up the fields which are superpartners into single superfields which are functions of both τ and ϑ. We define the following superfields

$$\begin{aligned} \mathbf{X}^\mu(\tau,\vartheta) &= x^\mu(\tau) + i\vartheta\xi^\mu(\tau), \\ \mathbf{X}_5(\tau,\vartheta) &= \xi_5(\tau) + \vartheta\phi_5(\tau), \\ \mathbf{E}(\tau,\vartheta) &= e_0(\tau) + \vartheta e_1(\tau). \end{aligned} \tag{7}$$

The five superfields $\mathbf{X}^\mu$ and $\mathbf{E}$ are commuting superfields while $\mathbf{X}_5$ is anticommuting. The supersymmetric covariant derivative, D, and the supersymmetry generator, Q, defined by

$$\begin{aligned} D &:= \frac{d}{d\vartheta} - i\vartheta\frac{d}{d\tau}, \\ Q &:= \frac{d}{d\vartheta} + i\vartheta\frac{d}{d\tau}, \end{aligned} \tag{8}$$

satisfy the algebra

$$\begin{aligned} Q^2 &= i\frac{d}{d\tau}, \\ D^2 &= -i\frac{d}{d\tau}, \\ QD + DQ &= 0. \end{aligned} \tag{9}$$

Using the properties (9) and the superfields (7), we may construct an action which is manifestly invariant under the global supersymmetry transformations

$$\delta\mathbf{\Phi} = \epsilon Q\mathbf{\Phi}, \tag{10}$$

for any superfield $\mathbf{\Phi}$. The action

$$S = \int d\tau d\vartheta[-\frac{1}{2\mathbf{E}}D\mathbf{X}^{\mu}D^{2}\mathbf{X}_{\mu} + \frac{1}{2\mathbf{E}}\mathbf{X}_{5}D\mathbf{X}_{5} + m\mathbf{X}_{5}] \tag{11}$$

is also invariant under local τ-reparametrizations

$$\begin{aligned} \delta\mathbf{X}^{\mu} &= \epsilon\dot{\mathbf{X}}^{\mu}, \\ \delta\mathbf{X}_{5} &= iD(\epsilon D\mathbf{X}_{5}), \\ \delta\mathbf{E} &= \frac{d}{d\tau}(\epsilon\mathbf{E}) + i(D\epsilon)(D\mathbf{E}), \end{aligned} \tag{12}$$

with ϵ a commuting function of τ and ϑ.

When written out in component form,

$$S = \int d\tau \left[\frac{1}{2e_0}(\dot{x}^2 + i\xi\cdot\dot{\xi} + \frac{e_1}{e_0}\xi\cdot\dot{x} + i\xi_5\dot{\xi}_5 + \phi_5^2 - \frac{e_1}{e_0}\xi_5\phi_5) + m\phi_5\right], \tag{13}$$

the action (11) can easily be seen to be equivalent to the action written by Berezin and Marinov (1). A locally supersymmetric superfield formulation has been written by Brink, Deser, Zumino, Di Vecchia, and Howe [5] for the massless case. To write a locally supersymmetric version for the massive case, presumably one should treat $\mathbf{X_5}$ as a $D = 1$ spinor and introduce a $D = 1$ supermetric.

3. Quantization

Upon quantization, the anticommuting variables ξ^{μ} and ξ_5 will become

operators with anticommutators given by the Dirac brackets (4). Berezin and Marinov have identified the quantum operators corresponding to ξ_{μ} and ξ_5 as the elements of the Dirac gamma algebra $i\sqrt{\frac{\hbar}{2}}\gamma_5\gamma_{\mu}$ and $i\sqrt{\frac{\hbar}{2}}\gamma_5$. In fact, as we shall see, the situation is more subtle than this. The second-class constraints must be handled carefully in order to obtain a consistent quantization. First let us observe that there are an even number of Grassmann variables and an odd number of second-class constraints on those variables. The reduced phase space is thus odd dimensional. The variables ξ_μ are easily separated into pairs conjugate under the Dirac brackets (4):

$$\begin{aligned} \eta_1 &= \frac{\xi_0 + \xi_3}{\sqrt{2}}, \\ \eta_1^* &= \frac{-\xi_0 + \xi_3}{\sqrt{2}}, \\ \eta_2 &= \frac{\xi_1 + i\xi_2}{\sqrt{2}}, \\ \eta_2^* &= \frac{\xi_1 - i\xi_2}{\sqrt{2}}, \end{aligned} \tag{14}$$

satisfying the relations

$$\{\eta_i, \eta_j^*\}_{DB} = i\delta_{ij}. \tag{15}$$

The constraint on ξ_5 is not as easily solved. In fact, because it is second-class, it cannot be imposed on a state directly. The easiest way to take it into account is to follow Bordi and Casalbuoni [3] in imposing the condition that the Hilbert space separate into physical states and unphysical states which are orthogonal to the physical states. Further, we suppose that the constraint maps physical into unphysical states. This is the condition:

$$\int d\xi_5\, \chi^*(\xi_5)[\widehat{P}_5 - i\xi_5/2]\chi(\xi_5) = 0. \tag{16}$$

This implies that the physical states are all proportional to $\chi_\alpha = (\sqrt{2} + e^{i\alpha}\xi_5)$, all with the same value of α. All of the wavefunctions are then of the form

$$\Psi = \chi_\alpha(\xi_5)\phi(\eta_1^*, \eta_2^*, x), \tag{17}$$

with ϕ, as well as χ_α, of mixed Grassmann parity. The wavefunction ϕ is specified by four complex functions $\phi_{ij}(x)$:

$$\phi(\eta_1^*, \eta_2^*, x) = \phi_{00}(x) + \phi_{10}(x)\eta_1^* + \phi_{01}(x)\eta_2^* + \phi_{11}(x)\eta_1^*\eta_2^*. \tag{18}$$

This ϕ carries exactly the same information as a Dirac spinor wavefunction and transforms as a spinor under $SO(3,1)$ rotations.

We may find an operator for ξ_5 which realizes the relation (4) as an anticommutation relation. This operator is

$$\widehat{\xi}_5 = \frac{\partial}{\partial \xi_5} - \frac{\xi_5}{2}, \tag{19}$$

which, strangely enough, is the direct transcription of the last constraint in (3) into operator form. To realize the relations (15) one assigns the operators

$$\begin{aligned} \eta_1 &\to i\frac{\partial}{\partial \eta_1^*}, \\ \eta_2 &\to i\frac{\partial}{\partial \eta_2^*}. \end{aligned} \tag{20}$$

The constraints then become the conditions

$$\begin{aligned} 0 &= (\Box - m^2)\phi_{ij}(x), \\ 0 &= me^{i\alpha}\chi_{-\alpha}(-\xi_5)\phi(\eta_1^*, \eta_2^*, x) + \chi_\alpha(-\xi_5)(i\sqrt{2}\widehat{\xi}^\mu \partial_\mu)\phi(\eta_1^*, \eta_2^*, x). \end{aligned} \tag{21}$$

This last condition cannot be satisfied unless $\alpha = 0, \pi$ or $\phi(\eta_1^*, \eta_2^*, x) \equiv 0$. Thus one is forced to choose $\alpha = 0, \pi$ in order to obtain any quantum theory at all. For these values of α the fermionic constraint is equivalent to the condition

$$(\pm m + \sqrt{2} i \widehat{\xi} \cdot \partial)\phi(\eta_1^*, \eta_2^*, x) = 0. \tag{22}$$

If we require the norm of the state functions

$$\langle \chi_\alpha | \chi_\alpha \rangle = \int d\xi_5\, \chi_\alpha^*(\xi_5)\chi_\alpha(\xi_5) = \sqrt{8}\cos\alpha\,, \tag{23}$$

to be positive, then α must be zero.

The condition (22) is obviously the Dirac equation, and does not contain an extra factor of γ_5. The correct propagator may be obtained by the BFV prescription.

To begin the BFV quantization, one first computes the BRS charge. Because the constraints do not form an abelian algebra, there is a term containing more than one ghost. The ghosts c_1 and b_1 are anticommuting while c_2 and b_2 are commuting. The BRS charge is

$$\Omega = c_1(P^2 + m^2) + c_2(P \cdot \xi + m\xi_5) - \frac{i}{2}c_2 c_2 b_1 + \bar{b}_1\pi_1 + \bar{b}_2\pi_2. \tag{24}$$

It is simplest to choose the gauge-fixing function to be

$$\Psi = -(b_1\lambda_1 + b_2\lambda_2)/\Delta\tau. \tag{25}$$

The propagator is computed from the BFV path integral

$$Z_{BFV} = \int \mathcal{D}P\,\mathcal{D}Q\, e^{i\int d\tau\,(P\dot{Q} - H_0 + \{\Psi,\Omega\}_{DB})}. \tag{26}$$

Here P and Q stand for all of the phase space degrees of freedom, including the ghosts and the Lagrange multiplier variables. The canonical Hamiltonian, H_0, of any reparametrization invariant action (such as the Berezin-Marinov action (1)) vanishes, thus the "Hamiltonian" governing τ evolution is just the "gauge fixing" piece

$$\begin{aligned} H &= -\{\Psi, \Omega\}_{DB} \\ &= \lambda_1(P^2 + m^2) - \lambda_2(\xi \cdot P + m\xi_5) + i\lambda_2 c_2 b_1 + b_i\bar{b}_i. \end{aligned} \tag{26}$$

Imposing the Feynman boundary conditions that the positive (negative) energy particles move forward (backward) in time on the motion of the particle, one finds from the x^μ equations of motion that λ_1 must be restricted to nonnegative values:

$$\dot{x}^\mu = \frac{\partial H}{\partial P_\mu} = 2\lambda_1 P^\mu - \lambda_2\xi^\mu. \tag{27}$$

Unlike the simple case of the bosonic particle, there is a nontrivial ghost integral to evaluate. With the ghost boundary conditions $c_i = \pi_i = 0$, the ghost integrations lead to a factor

$$\delta^2(\bar{b}_f)\delta^2(\bar{b}_i)\,\mathrm{sdet}^{-1}\begin{pmatrix} \frac{\partial^2}{\partial\tau^2} & i\lambda_2\frac{\partial}{\partial\tau} \\ 0 & \frac{\partial^2}{\partial\tau^2} \end{pmatrix} = \delta^2(\bar{b}_f)\delta^2(\bar{b}_i), \tag{28}$$

which will be implicit in the following.

The path integral over the variable ξ_5 can be done as in Bordi and Casalbuoni [3], or directly by discretization of the integral

$$K(\xi_f|\xi_i) := \int_{\xi(0)=\xi_i}^{\xi(\tau)=\xi_f} \mathcal{D}\xi\, e^{i\int_0^\tau dt(\frac{i}{2}\xi\dot{\xi}+\eta\xi)} = (\xi_f - \xi_i + i\eta\tau)e^{-\xi_f\xi_i/2}. \tag{29}$$

This expression is not the propagator, but the symbol of the propagator [6], since it does not preserve the physical states under time evolution. The propagator defined from the symbol

$$\widetilde{K}(\xi_f|\xi_i) = K(\xi_f|\xi_i)e^{\xi_f\xi_i} \tag{30}$$

preserves the orthogonality of the physical states χ_o and the unphysical states χ_π under τ evolution.

$$\int d\xi_f\, d\xi_i\, \chi_\pi^*(\xi_f)\widetilde{K}(\xi_f|-\xi_i)\chi_o(\xi_i) = 0. \tag{31}$$

The rest of the transition amplitude is easy to evaluate. The full propagator is

$$\begin{aligned} K(x_f,\eta_f,\xi_{5f}|x_i,\eta_i^*,\xi_{5i}) = \int d\lambda_2\, d^4p\, &\frac{e^{ip\cdot(x_f-x_i)}}{p^2+m^2-i\epsilon} \\ &\cdot(\xi_{5f}-\xi_{5i}-im\lambda_2)\, e^{\xi_{5f}\xi_{5i}/2}e^{\eta_f\cdot\eta_i^*}e^{i\lambda_2 p\cdot\xi}. \end{aligned} \tag{32}$$

The boundary conditions must be handled carefully. The transition element (32) must be evaluated between physical states. In particular, the ξ_5 dependence is crucial.

To obtain the correct propagator, we take the matrix element of (32) between states whose dependence on ξ_5 is physical. Thus

$$\begin{aligned}\langle x_f, \eta_f | x_i, \eta_i^* \rangle &= \int d\xi_{5i}\, d\xi_{5f}\, \chi_o^*(\xi_{5f}) K(x_f, \eta_f, \xi_{5f} | x_i, \eta_i^*, -\xi_{5i}) \chi_o(\xi_{5i}) \\ &\propto \int d^4p \frac{e^{ip\cdot(x_f - x_i)}}{p^2 + m^2 - i\epsilon}(m - \sqrt{2} p \cdot \xi),\end{aligned} \tag{33}$$

with ξ_μ identified as the expressions in (14), is the the Dirac propagator when the identification

$$\sqrt{2}\xi_\mu \to \gamma_\mu \tag{34}$$

is made. There is no factor of γ_5 to spoil unitarity, which is as it must be since the Hamiltonian path integral is manifestly unitary.

References

[1] E. S. Fradkin and G. A. Vilkovisky, *Phys. Lett.* **55B** (1975)224.

I. A. Batalin and G. A. Vilkovisky, *Phys. Lett.* **69B** (1977) 309.

E. S. Fradkin and T. E. Fradkina, *Phys. Lett.* **72B** (1978) 3 43.

[2] F. A. Berezin and M. S. Marinov, *Ann. Phys.* **104** (1977) 33 6.

[3] F. Bordi and R. Casalbuoni, *Phys. Lett.* **93B** (1980) 308.

[4] S. Monaghan, *Phys. Lett.* **181B** (1986) 101.

[5] L. Brink, S. Deser, B. Zumino, P. Di Vecchia and P. Howe, *Phys. Lett.* **64B** (1976) 435.

[6] L. D. Faddeev and A. A. Slavnov, *Gauge Fields: Introduction to Quantum Theory*, Benjamin Cummings(Reading,1980).