

Quantization of Pseudoclassical Systems in the Schrödinger Realization

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We examine the quantization of pseudoclassical dynamical systems, models that have classically anticommuting variables, in the Schrödinger picture. We quantize these systems, which can be viewed as classical models of particle spin, using the generalized Gupta-Bleuler method as well as the reduced phase space method in even dimensions. With minimal modifications, the standard constructions of Schrödinger quantum mechanics of constrained systems work for pseudoclassical systems. We generalize the standard Schrödinger norm and implement the correct adjointness properties of observables and constraints. We construct the state space corresponding to spinors as physical wave functions of anticommuting variables, finding that there are superselection sectors in both the physical and ghost subspaces. The physical states are isomorphic to those of the Dirac-Kähler formulation of fermions though the inner product in Dirac-Kähler theory is not equivalent to ours.

I. INTRODUCTION

Anticommuting variables, also called Grassmann numbers, have a long history in theoretical physics,¹⁻⁵ with applications ranging from the path integral formulation of fermions to superspace constructions for supersymmetric theories. Pseudoclassical mechanics, which incorporates anticommuting dynamical variables, arises as the $\hbar \rightarrow 0$ classical limit^{6,7} of quantum mechanical systems with spin. Despite the key role of anticommuting variables in theoretical physics, the Schrödinger picture approach to such systems heretofore has not received full attention. We aim to remedy this oversight.

In their renowned paper on the use of anticommuting variables to describe relativistic and non-relativistic spin degrees of freedom, Berezin and Marinov⁷ posit a three dimensional real vector-valued anticommuting variable ξ_k with the real Grassmann-even action

$$S = \int dt \left[\frac{1}{2} \tilde{\omega}_{kl} \xi_k \dot{\xi}_l - H(\xi) \right], \quad (1.1)$$

with H a real Grassmann-even function and $\tilde{\omega}$ an imaginary symmetric 3×3 matrix, to describe the non-relativistic spin degrees of freedom of a spin-1/2 particle. The matrix $\tilde{\omega}$ can be reduced to

$$\tilde{\omega}_{kl} = i\delta_{kl} \quad (1.2)$$

by a linear transformation of variables. With this choice the kinetic term of the action is $O(3)$ invariant, and the full action will be likewise if the function $H(\xi)$ is $O(3)$ invariant.

Berezin and Marinov note from the form of the action that the variables ξ_k may be taken as phase-space coordinates and then define a Poisson bracket from the (ortho)symplectic form $\tilde{\omega}_{kl}$ that gives the correct equations of motion. After quantization, the operators $\hat{\xi}_k$ corresponding to the pseudoclassical variables ξ_k become the generators of the Clifford algebra with three Euclidean generators and satisfy the Pauli matrix anticommutation relations. Consistent with their abstract approach to mechanics, Berezin and Marinov appeal to the representation theory of Clifford algebras, and take the space of states to be the essentially unique irreducible representation of that algebra, which is the space of two-component spinors.

While this approach is certainly elegant and efficient, as a phase space method it bypasses the quantization on configuration space and provides no insight into the issues of dealing with first-order actions on configuration space. When particle position is considered as an additional configuration space variable, as Berezin and Marinov⁷ also do in their paper, the resulting actions have not only a global rotational (or Lorentz) invariance that acts on both particle position and the Grassmann-odd coordinates, but also a world-line supersymmetry relating the Grassmann-even and -odd coordinates, strongly suggesting their treatment on an equal footing.

Our purpose in this paper is to analyze pseudoclassical systems in D dimensions of the form introduced by Berezin and Marinov through the explicit application of Dirac's methods for constrained systems and the use of the Schrödinger representation for the quantum states and their norms, assuming that all the ξ_i variables are configuration space coordinates. Certainly, the approach of taking the configuration space to be the full set of anticommuting variables is not new, having been explored, for example, by Mankoč Borštnik^{8,9} and Mankoč Boršt-

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nik and Nielsen,¹⁰ but our approach to implementing quantization in the full Schrödinger representation does appear to be new. Our work offers concrete insights into both pseudoclassical mechanics and constrained quantization. We make as close an analogy with the standard Schrödinger picture as we can by using the calculus of anticommuting variables^{5,11,12} in the standard constructions.

We denote left and right derivatives with respect to an anticommuting variable ξ as $\partial^L/\partial\xi$ and $\partial^R/\partial\xi$ respectively.

II. $D = 2$ ANTICOMMUTING VARIABLES

A. Pseudoclassical Hamiltonian System

As a system with one anticommuting coordinate is something of a special case, which we will address later, we start by analyzing in detail the system with two anticommuting coordinates, proceed to the general case of an even number of anticommuting coordinates, and then come back to the one, three, and general odd-dimensional cases later.

Barring for the moment actions with non-dynamical Grassmann-odd parameters, the simplest non-trivial pseudoclassical action for two real Grassmann-odd coordinates ξ_1 and ξ_2 has the essentially unique Lagrangian,

$$L = \frac{i}{2}(\xi_1 \dot{\xi}_1 + \xi_2 \dot{\xi}_2) + i\omega \xi_1 \xi_2, \quad (2.1)$$

which is also the most general action invariant under the rotation group $SO(2)$. Here ω is a real Grassmann-even constant. (In the form of (1.1), $H(\xi) = -\frac{i}{2}\epsilon_{ij}\xi_i\xi_j$ is rotationally invariant and unique up to the addition of a real Grassmann-even constant.) The Euler-Lagrange equations of motion that follow from (2.1),

$$\frac{d}{dt} \left(\frac{\partial^R L}{\partial \dot{\xi}_i} \right) = \frac{\partial^R L}{\partial \xi_i}, \quad (2.2)$$

are

$$\dot{\xi}_i = \omega(\xi_1 \delta_{i2} - \xi_2 \delta_{i1}) = -\omega \epsilon_{ij} \xi_j. \quad (2.3)$$

The Hamiltonian description, with Poisson brackets

$$\{f, g\} = \sum_{i=1,2} \left(\frac{\partial^R f}{\partial \xi_i} \frac{\partial^L g}{\partial \pi_i} + \frac{\partial^R f}{\partial \pi_i} \frac{\partial^L g}{\partial \xi_i} \right) \quad (2.4)$$

and canonical momenta,

$$\pi_i = \frac{\partial^R L}{\partial \dot{\xi}_i} = \frac{i}{2} \xi_i \quad (2.5)$$

that do not depend on velocities $\dot{\xi}_i$, is complicated by the presence of constraint functions on the phase-space

$$\varphi_i = \pi_i - \frac{i}{2} \xi_i \approx 0, \quad (2.6)$$

which are second-class in Dirac's¹³ classification, because they do not have vanishing Poisson brackets with themselves. These constraints reduce the dimension of the phase space from four to two. Dirac's methods for analyzing both classical and quantum constrained systems are well explained in the literature.^{13–18}

On the physical phase space defined by the constraints (2.6), the Hamiltonian is equal to

$$H = \pi_i \dot{\xi}_i - L = -i\omega \xi_1 \xi_2, \quad (2.7)$$

but the evolution of the system on the physical phase space defined by the constraints (2.6) must stay in that phase space. In other words, the constraints must be conserved in time. The most general Hamiltonian that agrees with (2.7) on the physical phase space is

$$H' = -i\omega \xi_1 \xi_2 + \lambda_i \varphi_i, \quad (2.8)$$

where the λ_i are Grassmann-odd phase space functions.

The coefficients λ_i are determined by requiring that the constraints remain zero on the reduced phase space,

$$\dot{\varphi}_i = \{\varphi_i, H'\} = i(\lambda_i - \omega \epsilon_{ij} \xi_j) + \left(\frac{\partial^L \lambda_j}{\partial \xi_i} - \frac{i}{2} \frac{\partial^L \lambda_j}{\partial \pi_i} \right) \varphi_j \approx 0. \quad (2.9)$$

Equation (2.9) can be made to hold identically, not just weakly, when we choose

$$\lambda_i = \frac{3}{4} \omega \epsilon_{ij} \xi_j - \frac{i}{2} \omega \epsilon_{ij} \pi_j, \quad (2.10)$$

from which we have

$$\begin{aligned} H' &= -\frac{i}{4} \omega \xi_1 \xi_2 + \frac{1}{2} \omega (\xi_2 \pi_1 - \xi_1 \pi_2) + i\omega \pi_1 \pi_2 \\ &= i\omega \left(\pi_1 + \frac{i}{2} \xi_1 \right) \left(\pi_2 + \frac{i}{2} \xi_2 \right). \end{aligned} \quad (2.11)$$

B. Wave functions, involution, and operators

Generalizing from the standard quantum mechanics, we take the wave functions for the quantum states to be complex Grassmann-valued functions of the coordinates ξ_i , defined through their power series,

$$\psi = \psi(\xi_1, \xi_2) = \psi_0 + \psi_1 \xi_1 + \psi_2 \xi_2 + \psi_3 \xi_1 \xi_2, \quad (2.12)$$

with complex-valued (Grassmann-even) coefficients ψ_i .

While the Grassmann coordinates ξ_i are taken to be real, $\xi_i^* = \xi_i$, because of the properties of the involution

$$(\xi^*)^* = \xi,$$

$$(\xi\theta)^* = \theta^*\xi^* \quad (2.13)$$

that generalizes complex conjugation to Grassmann variables, the complex conjugate of a product of two odd elements of a Grassmann algebra is $(\xi\theta)^* = \theta^*\xi^* = -\xi^*\theta^*$. This involution property, when combined with anticommutativity, yields the unfamiliar result that the product of two real anticommuting numbers is imaginary, as is the product of two imaginary anticommuting numbers. The properties of classical variables under complex conjugation carry over to the adjointness properties of their corresponding quantum operators.

In the Schrödinger representation the phase space variables are replaced by the operators

$$\begin{aligned} \hat{\xi}_i &= \xi_i, \\ \hat{\pi}_i &= i\hbar \frac{\partial^L}{\partial \xi_i}. \end{aligned} \quad (2.14)$$

In what follows, we use units where $\hbar = 1$.

C. Inner product and physical states

In keeping with the analogy to the commuting case, we take the inner product to be proportional to the $SO(2)$ invariant Grassmann integral over configuration space. In the notation of (2.12),

$$\int \phi^* \psi d\xi_1 d\xi_2 = \phi_3^* \psi_0 + \phi_2^* \psi_1 - \phi_1^* \psi_2 - \phi_0^* \psi_3. \quad (2.15)$$

With an explicit factor of i , the Grassmann integral (2.15) gives an inner product,

$$\langle \phi | \psi \rangle = i \int \phi^* \psi d\xi_1 d\xi_2 = \langle \psi | \phi \rangle^*, \quad (2.16)$$

that yields a manifestly real norm, but one that is not positive definite on the full space of wave functions (2.12). However, as in the case of gauge theories, the inner product need only be positive definite on the space of physical states. States of non-positive norm are unphysical “ghost” states.

D. Generalized Gupta-Bleuler Quantization

In Dirac quantization, physical states are annihilated by all first-class constraints. Second-class constraints cannot be imposed this way as they do not commute amongst themselves. The physical states in the presence of second-class constraints can be found by imposing the generalized Gupta-Bleuler conditions,^{19–21} which are that the physical matrix elements of all second-class

constraints vanish. Thus, the constraint matrix elements between physical states must satisfy

$$\begin{aligned} \langle \phi | \hat{\phi}_1 | \psi \rangle &= -(\phi_3^* \psi_1 - \phi_1^* \psi_3) - \frac{1}{2}(\phi_0^* \psi_2 - \phi_2^* \psi_0) = 0, \\ \langle \phi | \hat{\phi}_2 | \psi \rangle &= -(\phi_3^* \psi_2 - \phi_2^* \psi_3) + \frac{1}{2}(\phi_0^* \psi_1 - \phi_1^* \psi_0) = 0. \end{aligned}$$

Because the matrix elements of a Grassmann-odd operator between two states of the same Grassmann parity vanish automatically, the task of identifying physical states reduces to finding states of definite Grassmann parity such that matrix elements of the constraints between states of opposite parity vanish. We find that

$$\begin{aligned} \langle \phi_{\text{even}} | \hat{\phi}_1 | \psi_{\text{odd}} \rangle &= -(\phi_3^* \psi_1) - \frac{1}{2}(\phi_0^* \psi_2) = 0, \\ \langle \phi_{\text{even}} | \hat{\phi}_2 | \psi_{\text{odd}} \rangle &= -(\phi_3^* \psi_2) + \frac{1}{2}(\phi_0^* \psi_1) = 0, \end{aligned} \quad (2.17)$$

are satisfied when $2(\phi_3/\phi_0)^* = -(\psi_2/\psi_1) = (\psi_1/\psi_2)$.

These conditions yield an orthonormal basis for the wave functions of the full Schrödinger state space,

$$\begin{aligned} |0\rangle &= 1 + \frac{i}{2}\xi_1\xi_2, \\ |1\rangle &= \frac{1}{\sqrt{2}}(\xi_1 + i\xi_2), \\ |\bar{0}\rangle &= 1 - \frac{i}{2}\xi_1\xi_2, \\ |\bar{1}\rangle &= \frac{1}{\sqrt{2}}(\xi_1 - i\xi_2). \end{aligned} \quad (2.18)$$

The inner product (2.16) gives

$$\langle \alpha | \beta \rangle = \delta_{\alpha\beta} = -\langle \bar{\alpha} | \bar{\beta} \rangle, \quad (2.19)$$

for $\alpha = 0, 1$ and $\beta = 0, 1$, and the matrix elements of the constraints vanish in the physical basis $|\alpha\rangle$,

$$\langle \alpha | \hat{\phi}_k | \beta \rangle = 0, \quad (2.20)$$

decomposing the full Schrödinger Hilbert space into a physical space and an orthogonal negative-norm “ghost” space,

$$\mathfrak{H}_{\text{Schrödinger}} = \mathfrak{H}_{\text{physical}} \oplus \mathfrak{H}_{\text{ghost}}. \quad (2.21)$$

The constraint operators each map the physical state space to the ghost state space and vice versa:

$$\begin{aligned} \hat{\phi}_k | \alpha \rangle &= -\frac{i^k}{\sqrt{2}} \epsilon_{\alpha\beta} | \bar{\beta} \rangle, \\ \hat{\phi}_k | \bar{\alpha} \rangle &= \frac{(-i)^k}{\sqrt{2}} \epsilon_{\alpha\beta} | \beta \rangle, \end{aligned} \quad (2.22)$$

where $\epsilon_{01} = -\epsilon_{10} = 1$ and $\epsilon_{00} = \epsilon_{11} = 0$.

The remaining condition on an inner product is the self-adjointness of all observables. Anticommuting

variables, being nilpotent, cannot be observables, but nonetheless $\hat{\xi}_i$ is self-adjoint,

$$(\xi_i \psi(\xi_1, \xi_2))^* = \psi(\xi_1, \xi_2)^* \xi_i, \quad (2.23)$$

and we have $\langle \phi | \hat{\xi}_i \psi \rangle = \langle \hat{\xi}_i \phi | \psi \rangle$, or $\hat{\xi}_i^\dagger = \hat{\xi}_i$. Although ξ_i is real, its conjugate momentum π_i is purely imaginary. The momentum operator $\hat{\pi}_i$ should therefore be anti-self-adjoint, which one can check by direct calculation:

$$\int \phi^* \left(i \frac{\partial^L}{\partial \xi_i} \psi \right) d\xi_1 d\xi_2 = - \int \left(i \frac{\partial^L}{\partial \xi_i} \phi \right)^* \psi d\xi_1 d\xi_2, \quad (2.24)$$

and so $\hat{\pi}_i^\dagger = -\hat{\pi}_i$.

Up to a constant factor, the only observable in this system is the Hamiltonian corresponding to Eq. (2.11),

$$\hat{H}' = i\omega \left(\hat{\pi}_1 + \frac{i}{2} \hat{\xi}_1 \right) \left(\hat{\pi}_2 + \frac{i}{2} \hat{\xi}_2 \right), \quad (2.25)$$

which is manifestly self-adjoint.

Because the Hamiltonian (2.25) comes from the Grassmann-even Hamiltonian (2.11), the physical eigenstates can be taken to have definite Grassmann parity. We find

$$\begin{aligned} \hat{H}'|0\rangle &= -\frac{\omega}{2}|0\rangle, \\ \hat{H}'|1\rangle &= \frac{\omega}{2}|1\rangle. \end{aligned} \quad (2.26)$$

The ghost states, though unphysical, are also eigenstates of the Hamiltonian (2.25).

E. Comparison with Reduced Phase Space Quantization

1. Dirac brackets and the reduced phase space

Since the physical evolution of a classical constrained system must remain on the “constraint surface” where the second-class constraints vanish, it is possible to set the constraints identically to zero both inside and outside of Poisson brackets and work purely with functions on the constraint surface, the reduced phase space. We examine this approach to quantization here. Consistency requires that the Poisson bracket on the full phase space be replaced by the Dirac bracket¹³ on the reduced phase space,

$$\{f, g\}_{DB} = \{f, g\} - \{f, \varphi_n\} \Delta^{nm} \{\varphi_m, g\}, \quad (2.27)$$

where Δ^{nm} is the inverse matrix to $\{\varphi_n, \varphi_m\}$. The constraints $\varphi_m \approx 0$ can be taken to be strongly zero because the Dirac bracket of anything with a constraint vanishes identically,

$$\{f, \varphi_k\}_{DB} = \{f, \varphi_k\} - \{f, \varphi_n\} \Delta^{nm} \{\varphi_m, \varphi_k\}$$

$$= \{f, \varphi_k\} - \{f, \varphi_n\} \delta_k^n \equiv 0. \quad (2.28)$$

The Dirac bracket has the same symmetry properties as the Poisson bracket and satisfies the Jacobi identity.

In our case, the matrix of Poisson brackets of the constraints is

$$\{\varphi_k, \varphi_\ell\} = -i\delta_{k\ell}, \quad (2.29)$$

so the Dirac bracket becomes

$$\{f, g\}_{DB} = \{f, g\} - i\{f, \varphi_k\}\{\varphi_k, g\}. \quad (2.30)$$

The full phase space is four-dimensional while the constraint surface is two-dimensional. The two coordinates ξ_1 and ξ_2 can be used as phase space coordinates on the constraint surface. Their Dirac brackets are

$$\begin{aligned} \{\xi_i, \xi_j\}_{DB} &= \{\xi_i, \xi_j\} - i\{\xi_i, \varphi_k\}\{\varphi_k, \xi_j\} \\ &= 0 - i\delta_{ik}\delta_{kj} = -i\delta_{ij}, \end{aligned} \quad (2.31)$$

so that the Dirac bracket of functions $f(\xi_1, \xi_2)$ and $g(\xi_1, \xi_2)$ on the constraint surface is

$$\{f, g\}_{DB} = -i \sum_{k=1,2} \left(\frac{\partial^R f}{\partial \xi_k} \frac{\partial^L g}{\partial \xi_k} \right). \quad (2.32)$$

2. Operators and states

As ξ_1 and ξ_2 are coordinates of the two-dimensional reduced phase space, for quantization in the Schrödinger picture one must choose one position coordinate and one canonical momentum to proceed. Neither ξ_1 nor ξ_2 can fulfill either role as each has non-vanishing Dirac bracket with itself.

Instead, following the holomorphic representation,^{4,11} we consider the complex phase space coordinates,

$$\begin{aligned} \eta &= \frac{1}{\sqrt{2}}(\xi_1 + i\xi_2), \\ \bar{\eta} &= \frac{1}{\sqrt{2}}(\xi_1 - i\xi_2), \end{aligned} \quad (2.33)$$

which satisfy

$$\{\eta, \eta\}_{DB} = \{\bar{\eta}, \bar{\eta}\}_{DB} = 0, \quad \{\eta, \bar{\eta}\}_{DB} = -i. \quad (2.34)$$

For quantization we need operators that satisfy the Dirac anticommutation relations,

$$\hat{\eta}\hat{\eta} + \hat{\eta}\hat{\eta} = i\hbar\{\bar{\eta}, \eta\}_{DB} = \hbar, \quad (2.35)$$

and can proceed with quantization in the Schrödinger picture by taking states to be functions of η alone,

$$\psi = \psi(\eta) = \psi_0 + \psi_1\eta, \quad (2.36)$$

and the operators $\hat{\eta}$ and $\hat{\bar{\eta}}$ acting upon them to be

$$\begin{aligned}\hat{\eta} &= \eta, \\ \hat{\bar{\eta}} &= \hbar \frac{\partial^L}{\partial \eta}.\end{aligned}\quad (2.37)$$

Again, we use units in which $\hbar = 1$ in what follows. As we are working in the reduced phase space, the constraints were eliminated before quantization, so all we need do now is construct the inner product and find the spectrum.

3. Inner product

While the wave function is a function of η , its complex conjugate is a function of $\bar{\eta}$. Thus it is necessary to consider inner products of holomorphic form,^{4,11}

$$\langle \phi | \psi \rangle = \int \phi^*(\bar{\eta}) \psi(\eta) \mathcal{M}(\bar{\eta}, \eta) d\eta d\bar{\eta}, \quad (2.38)$$

where $\mathcal{M}(\bar{\eta}, \eta)$ is a measure factor needed to enforce the adjointness relations coming from the complex conjugate nature of the variables $\bar{\eta}$ and η , $\eta^* = \bar{\eta}$. We need to have

$$\hat{\eta}^\dagger = \hat{\bar{\eta}} = \frac{\partial^L}{\partial \eta}, \quad (2.39)$$

or, for any two states ϕ and ψ ,

$$\begin{aligned}\langle \hat{\bar{\eta}} \phi | \psi \rangle &= \int \left(\frac{\partial^L \phi}{\partial \eta} \right)^* \psi(\eta) \mathcal{M}(\bar{\eta}, \eta) d\eta d\bar{\eta} \\ &= \int \phi^*(\bar{\eta}) \eta \psi(\eta) \mathcal{M}(\bar{\eta}, \eta) d\eta d\bar{\eta} \\ &= \langle \phi | \hat{\eta} \psi \rangle.\end{aligned}\quad (2.40)$$

Similarly, it is necessary that $\hat{\eta} = \hat{\bar{\eta}}^\dagger$, or

$$\begin{aligned}\langle \hat{\eta} \phi | \psi \rangle &= \int (\eta \phi)^* \psi(\eta) \mathcal{M}(\bar{\eta}, \eta) d\eta d\bar{\eta} \\ &= \int \phi^*(\bar{\eta}) \bar{\eta} \psi(\eta) \mathcal{M}(\bar{\eta}, \eta) d\eta d\bar{\eta} \\ &= \int \phi^*(\bar{\eta}) \frac{\partial^L \psi}{\partial \eta} \mathcal{M}(\bar{\eta}, \eta) d\eta d\bar{\eta} \\ &= \langle \phi | \hat{\eta} \psi \rangle.\end{aligned}\quad (2.41)$$

For the adjointness conditions Eqs. (2.40) and (2.41) to hold, it is necessary and sufficient that up to an overall factor,

$$\mathcal{M}(\bar{\eta}, \eta) = 1 + \bar{\eta} \eta = \exp(\bar{\eta} \eta). \quad (2.42)$$

With the measure Eq. (2.42), the inner product on states $\psi(\eta) = \psi_0 + \psi_1 \eta$ and $\phi(\eta) = \phi_0 + \phi_1 \eta$ is

$$\langle \phi | \psi \rangle = \int \phi^* \psi \exp(\bar{\eta} \eta) d\eta d\bar{\eta} = \psi_0^* \phi_0 + \psi_1^* \phi_1. \quad (2.43)$$

This inner product leads to positive definite norms for states. Because the constraints have been implemented prior to quantization, there is no ghost sector.

Note that the basis eigenstates in this system, $|0\rangle$ and $|1\rangle$, have wave functions, 1 and η respectively, with definite Grassmann parity that correspond to the Grassmann parities of the equivalent states found under Gupta-Bleuler quantization. The similarity between the states of the two different quantizations is stronger than just their Grassmann parities, however.

4. Gupta-Bleuler and Reduced Phase Space Wave Function Correspondence

The Gupta-Bleuler configuration space, rather than the reduced phase space, can also be parametrized by the η and $\bar{\eta}$ coordinates of Eq. (2.33), allowing us to rewrite the physical Gupta-Bleuler wave functions given in Eq. (2.18) as the reduced phase space ones times the square root of the reduced phase space measure factor,

$$\begin{aligned}|0\rangle &= 1 + \frac{i}{2} \xi_1 \xi_2 = 1 + \frac{1}{2} \bar{\eta} \eta = \sqrt{e^{\bar{\eta} \eta}} = \sqrt{\mathcal{M}}, \\ |1\rangle &= \frac{1}{\sqrt{2}} (\xi_1 + i \xi_2) = \eta = \eta \sqrt{e^{\bar{\eta} \eta}} = \eta \sqrt{\mathcal{M}},\end{aligned}\quad (2.44)$$

which is to say

$$(\psi_n(\xi_1, \xi_2))_{\text{GB}} = (\psi_n(\eta) \sqrt{\mathcal{M}})_{\text{RPS}}. \quad (2.45)$$

The inner product on the physical Gupta-Bleuler space of states is the integral over the ξ_1, ξ_2 configuration space, which can be reparametrized as an integral over the $\eta, \bar{\eta}$ reduced phase space, making the orthonormality of the one set understandable in terms of the other.

F. The Primed Variables of Hanson, Regge & Teitelboim

Instead of replacing the Poisson brackets with Dirac brackets in a system with second-class constraints, Hanson, Regge, and Teitelboim¹⁴ show that one can replace the canonical variables, or indeed any dynamical variables, by primed versions that agree on the constraint surface and whose Poisson brackets with any other quantity also agree with the Dirac brackets of those quantities on the constraint surface. These are the so-called primed variables,

$$A' = A - \{A, \varphi_n\} \Delta^{nm} \varphi_m \approx A, \quad (2.46)$$

where again Δ^{nm} is the inverse matrix to $\{\varphi_n, \varphi_m\}$. The Dirac bracket satisfies the weak equalities¹⁴

$$\{A, B\}_{\text{DB}} \approx \{A', B'\} \approx \{A', B\} \approx \{A, B'\}. \quad (2.47)$$

Because the matrix of Poisson brackets of the constraints is constant, the primed versions of the ξ_i variables with the constraints (2.6),

$$\xi'_k = \frac{1}{2}\xi_k - i\pi_k, \quad (2.48)$$

have strongly vanishing Poisson brackets with the constraints

$$\{\varphi_j, \xi'_k\} = 0. \quad (2.49)$$

These primed variables will be important for extending this system to general Hamiltonians.

III. $D = 2N$ ANTICOMMUTING VARIABLES

To handle the $D = 2N$ case, we generalize the measure for computing inner products of wave functions analogously to the construction for commuting coordinates,

$$d\mu_{2N} = d\mu_{2N-2} (i d\xi_{2N-1} d\xi_{2N}) = i^N d\xi_1 \dots d\xi_{2N}. \quad (3.1)$$

As we already have the quantization for $D = 2$ case, we here show how to go from $D = 2N - 2$ to $D = 2N$. Suppose that $\Psi_{2N-2}^+(\xi_1, \xi_2, \dots, \xi_{2N-2})$ is a positive norm physical state for $D = 2N - 2$ variables with definite Grassmann parity. Then if $\psi_{2N}^+(\xi_{2N-1}, \xi_{2N})$ is also a positive norm physical state of definite Grassmann parity for a system consisting of just the two variables ξ_{2N-1} and ξ_{2N} , then

$$|++\rangle_{2N} = \Psi_{2N-2}^+ \psi_{2N}^+ \quad (3.2)$$

is a positive norm physical state of definite Grassmann parity for the system consisting of $D = 2N$ anticommuting variables.

Similarly, suppose that $\Psi_{2N-2}^-(\xi_1, \xi_2, \dots, \xi_{2N-2})$ is a negative norm ghost state for $D = 2N - 2$ variables with definite Grassmann parity. Then if $\psi_{2N}^-(\xi_{2N-1}, \xi_{2N})$ is a negative norm ghost state of definite Grassmann parity for a system consisting of just the two variables ξ_{2N-1} and ξ_{2N} , then

$$|--\rangle_{2N} = \Psi_{2N-2}^- \psi_{2N}^- \quad (3.3)$$

is also positive norm physical state of definite Grassmann parity for the system consisting of $D = 2N$ anticommuting variables. One has to check that the norms work as stated and that the Gupta-Bleuler conditions hold. This is straightforward, if tedious. Similarly, the states

$$\begin{aligned} |+-\rangle_{2N} &= \Psi_{2N-2}^+ \psi_{2N}^-, \\ |-+\rangle_{2N} &= \Psi_{2N-2}^- \psi_{2N}^+ \end{aligned} \quad (3.4)$$

are negative norm ghost states. Thus we have $2^{2N-1} = 2^{D-1}$ physical states and an equal number of ghost states. The total number of physical and ghost states is $2^{D-1} + 2^{D-1} = 2^D$, the total number of terms in a function of D anticommuting variables.

IV. $D = 2N + 1$ ANTICOMMUTING VARIABLES

A. One anticommuting variable, $N = 0$

We begin by considering the special case of a single real anticommuting variable, with an eye to the general case. The case of a single anticommuting variable has also been examined by Bordi and Casalbuoni,²² and Bordi, Casalbuoni, and Barducci.²³

With only one anticommuting variable, and absent anticommuting constant parameters, the only possible term in the Lagrangian is the kinetic term,

$$L = \frac{i}{2} \dot{\xi} \dot{\xi}. \quad (4.1)$$

The equation of motion for ξ is that it is a constant. The momentum of the system does not depend on the velocity,

$$\pi = \frac{\partial^R L}{\partial \dot{\xi}} = \frac{i}{2} \xi, \quad (4.2)$$

so there is a constraint

$$\varphi = \pi - \frac{i}{2} \xi \approx 0, \quad (4.3)$$

and the only dynamics are that the system obeys the constraint, because the Hamiltonian vanishes identically. The phase space consists of the single variable ξ . Effectively there is just “half a degree of freedom.” The only way to quantize this system is to use the Gupta-Bleuler quantization to impose the constraint. In the Schrödinger representation, the wave function is a linear function of ξ ,

$$\psi(\xi) = \psi_0 + \psi_1 \xi, \quad (4.4)$$

with ψ_0 and ψ_1 being complex numbers. Because the Poisson brackets, as we will see, are $\{\pi, \xi\} = \{\xi, \pi\} = 1$, the Dirac quantization rule gives the momentum operator (in $\hbar = 1$ units)

$$\hat{\pi} = i \frac{\partial^L}{\partial \xi}. \quad (4.5)$$

In analogy to the quantum mechanics of one commuting variable, we set the inner product to be the integral

$$\langle \phi | \psi \rangle = \int \phi^*(\xi) \psi(\xi) d\xi = \phi_1^* \psi_0 + \phi_0^* \psi_1. \quad (4.6)$$

Since the variable ξ is real, $(\phi_0 + \phi_1 \xi)^* = \phi_0^* + \phi_1^* \xi$. Gupta-Bleuler quantization requires the constraint to have vanishing matrix elements between any two physical states, namely

$$\langle \phi | \hat{\pi} | \psi \rangle = i(\phi_1^* \psi_1 - \frac{1}{2} \phi_0^* \psi_0) = 0, \quad (4.7)$$

which implies that up to an overall phase, there is just a single normalized physical state of positive norm,

$$\psi_{\text{phys}}(\xi) = \frac{1}{\sqrt[4]{2}} \left(1 + \frac{1}{\sqrt{2}} \xi \right), \quad (4.8)$$

and a single orthogonal ghost state of negative norm,

$$\psi_{\text{ghost}}(\xi) = \frac{1}{\sqrt[4]{2}} \left(1 - \frac{1}{\sqrt{2}} \xi \right). \quad (4.9)$$

The states are eigenstates of $\sqrt{2}\hat{\xi}'$ with eigenvalues ± 1 .

$$\begin{aligned} \sqrt{2}\hat{\xi}'\psi_{\text{phys}}(\xi) &= +\psi_{\text{phys}}(\xi), \\ \sqrt{2}\hat{\xi}'\psi_{\text{ghost}}(\xi) &= -\psi_{\text{ghost}}(\xi). \end{aligned} \quad (4.10)$$

Note that if we were to take our single anticommuting variable to be imaginary, $\zeta = i\xi$, we would change the sign of the kinetic term. This will be important in considering the Lorentzian case.

It is important to mention that keeping the Lagrangian as (4.1) with a positive overall sign but positing an imaginary ξ , or equivalently, having a negative sign for the Lagrangian and a real ξ , makes it impossible to impose the constraint (4.3) through the integral (4.7) because that expression becomes proportional to a positive definite expression, $\langle \psi | \hat{\phi} | \psi \rangle \propto \psi_1^* \psi_1 + \psi_0^* \psi_0 / 2$.

B. Three anticommuting variables

We now generalize to the case of three anticommuting variables, the first case treated by Berezin and Marinov,⁷ handled here in our Schrödinger formalism. After diagonalization of the kinetic terms, the most general Lagrangian is

$$L = \frac{i}{2} \xi_k \dot{\xi}_k + i\omega_k \epsilon_{ijk} \xi_i \xi_j, \quad (4.11)$$

which contains three arbitrary commuting constants, ω_k . A further rotation of the ξ_i and ω_k allows the reduction of the Lagrangian to

$$L = \frac{i}{2} \xi_k \dot{\xi}_k + i\omega \xi_1 \xi_2, \quad (4.12)$$

which has the same form as the Lagrangian (2.1), except now the kinetic term contains the additional piece $\frac{i}{2} \xi_3 \dot{\xi}_3$. As a consequence, we might try to anticipate the result of the explicit quantization. Since the Lagrangian (4.12) separates into two non-interacting parts, one involving ξ_1 and ξ_2 and having the form of the two-variable system analyzed earlier, and the other involving ξ_3 and having the form analyzed in the preceding section, the basis states of the three-variable system can be written

in terms of products of the basis states of those two simpler systems. The Hamiltonian that commutes with the constraints will be identical to (2.11).

Note that when we compare the three-variable system to the two-variable system, two of the constraints and two of the equations of motion are the same but there is one additional constraint, which has the same form as the other two constraints,

$$\varphi_3 = \pi_3 - \frac{i}{2} \xi_3 \approx 0, \quad (4.13)$$

and one additional equation of motion,

$$\dot{\xi}_3 = 0. \quad (4.14)$$

As we know, the one-variable system has a Hamiltonian that vanishes identically, and so the Hamiltonian for the three-variable system has the same form as Eq. (2.11), although the wave functions can have ξ_3 dependence.

We now give the results of explicit quantization.

1. States

With three Grassmann coordinates, the “measure” will now be the Grassmann-odd product $i d\xi_1 d\xi_2 d\xi_3$. With this measure, normalizable states cannot have a definite Grassmann parity. For the system described by (4.12), the wave functions of the system can be factorized as

$$\Psi(\xi_1, \xi_2, \xi_3) = \psi(\xi_1, \xi_2) u(\xi_3). \quad (4.15)$$

If the two-dimensional wave functions $\psi(\xi_1, \xi_2)$ have definite Grassmann parity, then it is easy to see that the matrix elements of the first two second-class constraints will vanish if the two-dimensional factors of the wave functions are either both in $\mathfrak{H}_{\text{physical}}$ or both in $\mathfrak{H}_{\text{ghost}}$ of the two-variable Hilbert space (2.21);

$$\begin{aligned} \langle \Phi | \hat{\phi}_{1,2} | \Psi \rangle &= i \int (\phi v)^* \hat{\phi}_{1,2} \psi u d\xi_1 d\xi_2 d\xi_3 \\ &= i \int (v^* \tilde{u}) (\phi^* \hat{\phi}_{1,2} \psi) d\xi_1 d\xi_2 d\xi_3 \\ &= 0, \end{aligned} \quad (4.16)$$

where \tilde{u} is either $u(-\xi_3)$ or $u(\xi_3)$, depending on whether the Grassmann parities of $\phi(\xi_1, \xi_2)$ and $\psi(\xi_1, \xi_2)$ are the same or different respectively. The matrix elements of the third constraint are

$$\begin{aligned} \langle \Phi | \hat{\phi}_3 | \Psi \rangle &= i \int (\phi v)^* \hat{\phi}_3 \psi u d\xi_1 d\xi_2 d\xi_3 \\ &= i \int v^* \phi^* \hat{\phi}_3 \psi u d\xi_1 d\xi_2 d\xi_3 \\ &= i \int (v^* \hat{\phi}_3 \tilde{u}) (\phi^* \psi) d\xi_1 d\xi_2 d\xi_3 \end{aligned}$$

$$= i \int (v^* \hat{\phi}_3 \tilde{u}) d\xi_3 \int \phi^* \psi d\xi_1 d\xi_2, \quad (4.17)$$

where $\tilde{u}(\xi_3)$ is $(-1)^{g_\phi u}((-1)^{g_\phi + g_\psi} \xi_3)$, where g_ϕ and g_ψ denote the Grassmann parities of ϕ and ψ , respectively. The second factor, $\int \phi^* \psi d\xi_1 d\xi_2$, vanishes unless $g_\phi = g_\psi$. In that case, $\tilde{u} = (-1)^{g_\phi} u$ and there are two solutions that make the matrix elements (4.17) vanish, both of which have $u = v$, namely

$$u(\xi_3) = v(\xi_3) = \frac{1}{\sqrt{2}} \left(1 \pm \frac{\xi_3}{\sqrt{2}} \right). \quad (4.18)$$

The norm of a product wave function $\psi(\xi_1, \xi_2) u(\xi_3)$ is the product of the norms of its factors,

$$\begin{aligned} \langle \psi u | \psi u \rangle &= i \int u^* \psi^* \psi u d\xi_1 d\xi_2 d\xi_3 \\ &= \left(\int u^* u d\xi_3 \right) \left(i \int \psi^* \psi d\xi_1 d\xi_2 \right). \end{aligned} \quad (4.19)$$

Consequently, the positive norm physical states are spanned by the orthonormal basis

$$\begin{aligned} |0\rangle &= \frac{1}{\sqrt{2}} \left(1 + \frac{i}{2} \xi_1 \xi_2 \right) \left(1 + \frac{\xi_3}{\sqrt{2}} \right), \\ |1\rangle &= \frac{1}{\sqrt{8}} (\xi_1 + i \xi_2) \left(1 + \frac{\xi_3}{\sqrt{2}} \right), \\ |0'\rangle &= \frac{1}{\sqrt{2}} \left(1 - \frac{i}{2} \xi_1 \xi_2 \right) \left(1 - \frac{\xi_3}{\sqrt{2}} \right), \\ |1'\rangle &= \frac{1}{\sqrt{8}} (\xi_1 - i \xi_2) \left(1 - \frac{\xi_3}{\sqrt{2}} \right). \end{aligned} \quad (4.20)$$

The negative norm ghost states are spanned by the orthogonal anti-normal basis

$$\begin{aligned} |\bar{0}\rangle &= \frac{1}{\sqrt{2}} \left(1 - \frac{i}{2} \xi_1 \xi_2 \right) \left(1 + \frac{\xi_3}{\sqrt{2}} \right), \\ |\bar{1}\rangle &= \frac{1}{\sqrt{8}} (\xi_1 - i \xi_2) \left(1 + \frac{\xi_3}{\sqrt{2}} \right), \\ |\bar{0}'\rangle &= \frac{1}{\sqrt{2}} \left(1 + \frac{i}{2} \xi_1 \xi_2 \right) \left(1 - \frac{\xi_3}{\sqrt{2}} \right), \\ |\bar{1}'\rangle &= \frac{1}{\sqrt{8}} (\xi_1 + i \xi_2) \left(1 - \frac{\xi_3}{\sqrt{2}} \right). \end{aligned} \quad (4.21)$$

The large Schrödinger Hilbert space once again splits as in Eq. (2.21), but this time both the physical and ghost Hilbert spaces have dimension four. Each of these Hilbert spaces is the reducible $2 \oplus 2$ representation of the three-dimensional Clifford algebra.

2. Physical spectrum

The states (4.20) are eigenstates of the Hamiltonian:

$$\hat{H}_{\text{phys}}|0\rangle = -\frac{\omega}{2}|0\rangle, \quad \hat{H}_{\text{phys}}|0'\rangle = -\frac{\omega}{2}|0'\rangle,$$

$$\hat{H}_{\text{phys}}|1\rangle = +\frac{\omega}{2}|1\rangle, \quad \hat{H}_{\text{phys}}|1'\rangle = +\frac{\omega}{2}|1'\rangle. \quad (4.22)$$

3. Matrix elements of $\hat{\xi}_i$ and Pauli algebra

We find the matrix elements of the position operators $\hat{\xi}_i$ in the physical basis $\{|0\rangle, |1\rangle\}$ to be

$$\langle \alpha | \hat{\xi}_k | \beta \rangle = \frac{1}{\sqrt{2}} (\sigma_k)_{\beta\alpha}, \quad (4.23)$$

where σ_k are the standard Pauli matrices. It is instructive to note that the diagonal entries in σ_3 in Eq. (4.23) result from the even or odd definite Grassmann parities of the $\psi(\xi_1, \xi_2)$ pieces (4.15) of the basis states $|0\rangle$ and $|1\rangle$ of (4.20).

While the matrix elements of the $\hat{\xi}_k$ yield the Pauli matrices, the $\hat{\xi}_k$ operators themselves do not form a Clifford algebra; they are still nilpotent generators of a Grassmann algebra. However, the $\hat{\xi}'_k$ operators corresponding to Eq. (2.48) do form a Clifford algebra, although one not obeying a definite Pauli algebra, either $\sqrt{2}\hat{\xi}'_i\hat{\xi}'_j = +i\epsilon_{ijk}\hat{\xi}'_k$ or $\sqrt{2}\hat{\xi}'_i\hat{\xi}'_j = -i\epsilon_{ijk}\hat{\xi}'_k$, unless one restricts to a superselection sector. In the physical sectors that Pauli algebra is left-handed, while in the ghost sectors it is right-handed.

C. General $D = 2N + 1$

As with even dimensions, for the general odd dimensional case, we generalize the measure for computing inner products of wave functions analogously to the construction for commuting coordinates, setting

$$d\mu_{2N+1} = d\mu_{2N} (d\xi_{2N+1}) = i^N d\xi_1 d\xi_2 \dots d\xi_{2N+1}. \quad (4.24)$$

As we already have the quantization for $D = 2N$ case, we show here how to go from $D = 2N$ to $D = 2N + 1$. Suppose that $\Psi_{2N}^+(\xi_1, \xi_2, \dots, \xi_{2N})$ is a positive norm physical state for $D = 2N$ variables with definite Grassmann parity. Then if $\psi_{2N+1}^+(\xi_{2N+1})$ is the positive norm physical state (which, as we saw, must have mixed Grassmann parity) for a system consisting of just the single variables ξ_{2N+1} , then

$$|++\rangle_{2N+1} = \Psi_{2N}^+ \psi_{2N+1}^+ \quad (4.25)$$

is a positive norm physical state of mixed Grassmann parity for the system consisting of $D = 2N + 1$ anticommuting variables.

Similarly, suppose that $\Psi_{2N}^-(\xi_1, \xi_2, \dots, \xi_{2N})$ is a negative norm ghost state for $D = 2N$ variables with definite

Grassmann parity. When $\psi_{2N}^-(\xi_{2N-1}, \xi_{2N})$ is the negative norm ghost state for a system consisting of just the variable ξ_{2N+1} , then

$$| - - \rangle_{2N+1} = \Psi_{2N}^- \psi_{2N+1}^- \quad (4.26)$$

is also a positive norm physical state of mixed Grassmann parity for the system consisting of $D = 2N + 1$ anticommuting variables. Again, one has to check that the norms work as stated and that the Gupta-Bleuler conditions hold, which is somewhat tedious. Similarly, the states

$$\begin{aligned} | + - \rangle_{2N+1} &= \Psi_{2N}^+ \psi_{2N+1}^- , \\ | - + \rangle_{2N+1} &= \Psi_{2N}^- \psi_{2N+1}^+ , \end{aligned} \quad (4.27)$$

are negative norm ghost states. Thus we have $2^{2N} = 2^{D-1}$ physical states and an equal number of ghost states. The total number of physical and ghost states is $2^{D-1} + 2^{D-1} = 2^D$, the total number of terms in a function of D anticommuting variables.

V. LORENTZIAN METRICS

If the metric for the ξ variables is not Euclidean but Lorentzian with signature $(-, +, +, \dots)$, we can map the system in variables $\xi_0, \xi_1, \dots, \xi_{D-1}$ to the Euclidean case with variables $\xi_1, \xi_2, \dots, \xi_D$, for instance by defining a new real Grassmann variable $\xi_D = i\xi_0$. We saw at the end of section IV A that a time-like Grassmann variable must be imaginary, or no physical states of just that one variable can exist. With this redefinition to map to the Euclidean case, the analysis can then proceed for the D real Grassmann variables $\xi_1, \xi_2, \dots, \xi_D$ with Euclidean signature. In $D = 3 + 1$ dimensions, for example, in terms of the original Lorentzian variables the positive norm physical states are

$$\begin{aligned} |0\rangle &= \left(1 + \frac{i}{2}\xi_1\xi_2\right) \left(1 + \frac{1}{2}\xi_0\xi_3\right) , \\ |1\rangle &= \frac{1}{\sqrt{2}} (\xi_1 + i\xi_2) \left(1 + \frac{1}{2}\xi_0\xi_3\right) , \\ |2\rangle &= \frac{1}{\sqrt{2}} \left(1 + \frac{i}{2}\xi_1\xi_2\right) (\xi_3 - \xi_0) , \\ |3\rangle &= \frac{1}{2} (\xi_1 + i\xi_2) (\xi_3 - \xi_0) , \\ |0'\rangle &= \left(1 - \frac{i}{2}\xi_1\xi_2\right) \left(1 - \frac{1}{2}\xi_0\xi_3\right) , \\ |1'\rangle &= \frac{1}{\sqrt{2}} (\xi_1 - i\xi_2) \left(1 - \frac{1}{2}\xi_0\xi_3\right) , \\ |2'\rangle &= \frac{1}{\sqrt{2}} \left(1 - \frac{i}{2}\xi_1\xi_2\right) (\xi_3 + \xi_0) , \\ |3'\rangle &= \frac{1}{2} (\xi_1 - i\xi_2) (\xi_3 + \xi_0) , \end{aligned} \quad (5.1)$$

while the ghost states are

$$\begin{aligned} |\bar{0}\rangle &= \left(1 + \frac{i}{2}\xi_1\xi_2\right) \left(1 - \frac{1}{2}\xi_0\xi_3\right) , \\ |\bar{1}\rangle &= \frac{1}{\sqrt{2}} (\xi_1 + i\xi_2) \left(1 - \frac{1}{2}\xi_0\xi_3\right) , \\ |\bar{2}\rangle &= \frac{1}{\sqrt{2}} \left(1 + \frac{i}{2}\xi_1\xi_2\right) (\xi_3 + \xi_0) , \\ |\bar{3}\rangle &= \frac{1}{2} (\xi_1 + i\xi_2) (\xi_3 + \xi_0) , \\ |\bar{0}'\rangle &= \left(1 - \frac{i}{2}\xi_1\xi_2\right) \left(1 + \frac{1}{2}\xi_0\xi_3\right) , \\ |\bar{1}'\rangle &= \frac{1}{\sqrt{2}} (\xi_1 - i\xi_2) \left(1 + \frac{1}{2}\xi_0\xi_3\right) , \\ |\bar{2}'\rangle &= \frac{1}{\sqrt{2}} \left(1 - \frac{i}{2}\xi_1\xi_2\right) (\xi_3 - \xi_0) , \\ |\bar{3}'\rangle &= \frac{1}{2} (\xi_1 - i\xi_2) (\xi_3 - \xi_0) . \end{aligned} \quad (5.2)$$

VI. SUPERSELECTION SECTORS

A. Even dimensions

As explained in Sec. (III), in the case that D is even, the physical wave functions are the products

$$\psi_{\text{phys}}(\xi_1, \xi_2, \dots, \xi_D) = \prod_{\substack{n=1 \\ \prod s_m=1}}^{D/2} \psi_{i_n}^{s_n}(\xi_{2n-1}, \xi_{2n}) \quad (6.1)$$

of 2D wave functions with an even number of ghost factors. Here the superscript s_n is the sign of the norm of the state, with $s_n = +1$ for physical states, and $s_n = -1$ for ghost states. Because ghost states are orthogonal to physical states, we can see that the Hilbert space of physical states is a sum of superselection sectors

$$\mathfrak{H}_{\text{phys}} = \bigoplus_{\substack{k=0 \\ 1 \leq j_1 < j_2 < \dots < j_{2k} \leq \lfloor D/2 \rfloor}}^{\lfloor D/2 \rfloor} \mathfrak{H}_{j_1 j_2 \dots j_{2k}} , \quad (6.2)$$

where $\mathfrak{H}_{i_1 i_2 \dots i_{2k}}$ is spanned by products of the form (6.1) with the negative norm factors in “slots” j_1, j_2, \dots, j_{2k} . In other words, $s_{j_1} = s_{j_2} = \dots = s_{j_{2k}} = -1$ and the rest of the $s_m = +1$. Each of the superselection sectors $\mathfrak{H}_{i_1 i_2 \dots i_{2k}}$ has dimension $2^{\lfloor D/2 \rfloor}$,

$$\dim \mathfrak{H}_{i_1 i_2 \dots i_{2k}} = 2^{\lfloor D/2 \rfloor} , \quad (6.3)$$

and there are

$$\sum_{n=0}^{2^{\lfloor D/4 \rfloor}} \binom{\lfloor D/2 \rfloor}{2n} = 2^{\lfloor D/2 \rfloor - 1} \quad (6.4)$$

different superselection sectors.

The ghost states are of similar form,

$$\psi_{\text{ghost}}(\xi_1, \xi_2, \dots, \xi_D) = \prod_{\substack{n=1 \\ \prod s_m = -1}}^{D/2} \psi_{i_n}^{s_n}(\xi_{2n-1}, \xi_{2n}) , \quad (6.5)$$

but with an odd number of negative norm factors. There are also $2^{\lfloor D/2 \rfloor - 1}$ different ghost superselection sectors.

B. Odd dimensions

In the case of odd D , we introduce the obvious notation for the single-variable wave functions $\psi^s(\zeta)$ of Eqs. (4.8) and (4.9),

$$\psi^\pm(\zeta) = \frac{1}{\sqrt{2}} \left(1 \pm \frac{1}{\sqrt{2}} \zeta \right) , \quad (6.6)$$

that have inner products

$$\int \psi^{s_1*}(\zeta) \psi^{s_2}(\zeta) d\zeta = s_1 \delta_{s_1 s_2} . \quad (6.7)$$

The physical states can be seen as states of form

$$\psi_{\text{phys}}(\xi_1, \dots, \xi_D) = \left(\prod_{\substack{n=1 \\ \prod s_m = s_D}}^{\lfloor D/2 \rfloor} \psi_{i_n}^{s_n}(\xi_{2n-1}, \xi_{2n}) \right) \psi^{s_D}(\xi_D) , \quad (6.8)$$

while the ghost states satisfy the product condition $\prod s_m = -s_D$. There are now $2^{\lfloor D/2 \rfloor}$ physical superselection sectors, twice as many as in the even D case. Each superselection sector has dimension $2^{\lfloor D/2 \rfloor}$ as in the even D case. The counting is the same for the ghost states.

VII. GENERAL HAMILTONIANS AND STABILITY OF THE GUPTA-BLEULER CONDITIONS

One can check that the $\hat{\xi}'_i$ operators acting on a physical state yield a physical state, and acting on a ghost state yield a ghost state; that is, the $\hat{\xi}'_i$ operators do not change the sign of the norm of a state. Further, they map any superselection sector, whether ghost or physical, onto itself. In contrast, the constraint operators $\hat{\phi}_i$ do change the sign of the norm of a state, so they map physical states into ghost states, and vice versa. The $\hat{\xi}'_i$ and $\hat{\phi}_j$ operators anticommute with each other,

$$\hat{\xi}'_i \hat{\phi}_j + \hat{\phi}_j \hat{\xi}'_i = 0 , \quad (7.1)$$

which can be expressed through the commutative diagram 7.2 below.

$$\begin{array}{ccc} \mathfrak{H}_{\text{phys}} & \xrightarrow{\hat{\xi}'_i} & \mathfrak{H}_{\text{phys}} \\ \downarrow \hat{\phi}_j & & \downarrow \hat{\phi}_j \\ \mathfrak{H}_{\text{ghost}} & \xrightarrow{-\hat{\xi}'_i} & \mathfrak{H}_{\text{ghost}} \\ \downarrow \hat{\phi}_j & & \downarrow \hat{\phi}_j \\ \mathfrak{H}_{\text{phys}} & \xrightarrow{\hat{\xi}'_i} & \mathfrak{H}_{\text{phys}} \end{array} \quad (7.2)$$

The maps are all onto and, if done twice, give back the state scaled by $1/2$, as expressed by the anticommutation relations

$$\hat{\xi}'_i \hat{\xi}'_j + \hat{\xi}'_j \hat{\xi}'_i = \delta_{ij} \quad (7.3)$$

and

$$\hat{\phi}_i \hat{\phi}_j + \hat{\phi}_j \hat{\phi}_i = \delta_{ij} , \quad (7.4)$$

which extend the commutative diagram 7.2 to the larger toroidal diagram 7.5 below.

$$\begin{array}{ccccc} & & \frac{1}{2} \cdot \text{id} & & \\ & \swarrow & & \searrow & \\ \mathfrak{H}_{\text{phys}} & \xrightarrow{\hat{\xi}'_i} & \mathfrak{H}_{\text{phys}} & \xrightarrow{\hat{\xi}'_i} & \mathfrak{H}_{\text{phys}} \\ \downarrow \hat{\phi}_j & & \downarrow \hat{\phi}_j & & \downarrow \hat{\phi}_j \\ \frac{1}{2} \cdot \text{id} & \mathfrak{H}_{\text{ghost}} & \xrightarrow{-\hat{\xi}'_i} & \mathfrak{H}_{\text{ghost}} & \xrightarrow{-\hat{\xi}'_i} & \mathfrak{H}_{\text{ghost}} & \frac{1}{2} \cdot \text{id} \\ \downarrow \hat{\phi}_j & & \downarrow \hat{\phi}_j & & \downarrow \hat{\phi}_j \\ \mathfrak{H}_{\text{phys}} & \xrightarrow{\hat{\xi}'_i} & \mathfrak{H}_{\text{phys}} & \xrightarrow{\hat{\xi}'_i} & \mathfrak{H}_{\text{phys}} \\ & \swarrow & & \searrow & \\ & & \frac{1}{2} \cdot \text{id} & & \end{array} \quad (7.5)$$

In either commutative diagram 7.2 or 7.5, the space $\mathfrak{H}_{\text{phys}}$ can be any physical superselection sector $\mathfrak{H}_{i_1 i_2 \dots i_{2k}}$ or any sum of physical superselection sectors, so it is consistent to restrict the physical space to any of the isomorphic physical superselection sectors, $\mathfrak{H}_{i_1 i_2 \dots i_{2k}}$, yielding the physical Hilbert space as a $2^{\lfloor D/2 \rfloor}$ -dimensional irreducible module of the Clifford algebra generated by the $\hat{\xi}'_i$, Eq. (7.3).

Because the general Hamiltonian $\hat{H}_{\text{phys}} = H(\hat{\xi}')$, as in the $D = 2$ case Eq. (2.11), is built from the primed $\hat{\xi}'_i$ operators, the energy eigenstates will span the physical Hilbert space, no matter which of the superselection sectors is chosen for the physical Hilbert space. The Gupta-Bleuler conditions,

$$\langle \phi_{\text{phys}} | \hat{\phi}_i | \psi_{\text{phys}} \rangle = 0 , \quad (7.6)$$

will thus hold for all times if they hold at any one time, because the time evolution of an initially physical state in one superselection sector remains in the same physical superselection sector. Thus, the matrix elements (7.6) are zero for all time.

The commutative diagram 7.5 also makes clear that there is a symmetry between the physical and ghost sectors. The determination of which sectors are physical and which are ghost is an artifact of the choice of the sign of the inner product. If the other sign of the inner product (from Eq. (3.1) or Eq. (4.24)) is chosen, the physical and ghost sectors are swapped.

VIII. EQUIVALENCE TO DIRAC-KÄHLER FERMIONS

Wave functions taking values in the space of antisymmetric tensors, equivalent to being valued in the space of differential forms, and a wave equation for a spin 1/2 particle in terms of them has a very long history, going back to Ivanenko and Landau²⁴ in 1928. In the early 1960s, Kähler^{25–27} found a mapping of the Dirac equation onto inhomogeneous differential forms. These ideas have been further developed by Graf²⁸ and many others. Dirac-Kähler fermions have also been proposed²⁹ as a solution to the fermion doubling problem on the lattice. The existence of the identical sectors was noticed by Benn and Tucker³⁰; Banks, Dothan, and Horn³¹; and Becher and Joos.²⁹ In $D = 3 + 1$, the four sectors—in our case, two physical and two ghost—have been posited³¹ as a solution to the family problem of the standard model. Jourjine³² argued recently that when the masses of the four generations are degenerate or sufficiently close, only three of the generations are observable and the fourth is hidden.

The space of complex-valued functions of D Grassmann variables, \mathcal{F}_D , is isomorphic to the space of inhomogeneous differential forms on \mathbb{R}^D with complex coefficients,

$$\mathcal{F}_D \cong \Lambda_{\mathbb{C}}(\mathbb{R}^D) = \bigoplus_{p=0}^D \Lambda_{\mathbb{C}}^p(\mathbb{R}^D), \quad (8.1)$$

by the bijection

$$\begin{aligned} a_{\mu_1 \mu_2 \dots \mu_k} \xi^{\mu_1} \xi^{\mu_2} \dots \xi^{\mu_k} &\longleftrightarrow \\ a_{\mu_1 \mu_2 \dots \mu_k} (\sqrt{2})^k dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_k}. \end{aligned} \quad (8.2)$$

Left multiplication of a function on Grassmann space by a Grassmann variable ξ^ν , which is the effect of the operator $\hat{\xi}^\nu$, corresponds to wedging the corresponding form by dx^ν ,

$$\hat{\xi}^\nu \longleftrightarrow \sqrt{2} dx^\nu \wedge, \quad (8.3)$$

while the action of the associated momentum, $\hat{\pi}_\nu = i \frac{\partial}{\partial \xi^\nu}$, corresponds to a contraction with the corresponding vector,

$$\hat{\pi}_\nu \longleftrightarrow i \frac{1}{\sqrt{2}} e_\nu \lrcorner, \quad (8.4)$$

where $e_\nu \lrcorner dx^\mu = \delta_\nu^\mu$. Thus the operator for the primed coordinates, $\hat{\xi}'^\mu = \frac{\partial}{\partial \xi_\mu} + \frac{1}{2} \eta^{\mu\nu} \xi_\nu$, corresponds exactly to the Clifford product^{25–27} applied to the corresponding form,

$$\hat{\xi}'^\mu \longleftrightarrow \frac{1}{\sqrt{2}} dx^\mu \vee. \quad (8.5)$$

Specifically, under this correspondence we have

$$\varphi(\hat{\xi}'^\mu \psi) = \frac{1}{\sqrt{2}} dx^\mu \vee \varphi(\psi), \quad (8.6)$$

and

$$\varphi(\sqrt{2} \hat{\xi}'^\mu \partial_\mu \psi) = dx^\mu \vee \partial_\mu \varphi(\psi). \quad (8.7)$$

When the commuting variables x^μ are added to the action⁷, the wave functions become functions of both the x^μ and the ξ_ν . The inner product between states becomes

$$\langle \phi | \psi \rangle = \int_{\mathbb{R}^D} d^D x \, i^{\lfloor \frac{D}{2} \rfloor} \int \phi^*(x, \xi) \psi(x, \xi) d\xi_1 \dots d\xi_D. \quad (8.8)$$

If we denote the action of the bijection $\varphi: \mathcal{F}_D \rightarrow \Lambda_{\mathbb{C}}(\mathbb{R}^D)$ on an element $\psi \in \mathcal{F}_D$ as $\varphi(\psi)$, the inner products in the two spaces, which lead to indefinite norms, are related by

$$\langle \phi | \psi \rangle = i^{\lfloor \frac{D}{2} \rfloor} 2^{-D/2} \int_{\mathbb{R}^D} \varphi(\phi^*) \wedge \varphi(\psi). \quad (8.9)$$

This inner product differs from the usual positive definite inner product for Dirac-Kähler wave functions given by

$$\langle \varphi(\phi) | \varphi(\psi) \rangle = \int_{\mathbb{R}^D} \overline{\varphi(\phi)} \wedge * \varphi(\psi), \quad (8.10)$$

where $\overline{\varphi(\phi)}$ is the complex conjugate of the form $\varphi(\phi)$ and $*$ is the Hodge star operator that maps p -forms to $(D - p)$ -forms. Note that $\varphi(\phi^*) = (-1)^{p(p-1)/2} \overline{\varphi(\phi)}$ when $\varphi(\phi^*)$ is a form of degree p . Mankoč Borštnik and Nielsen¹⁰ have also found the relationship between Grassmann quantum mechanical wave functions and Dirac-Kähler fermions, but do so from the operatorial point of view and take the standard positive-definite inner product, (8.10).

IX. DISCUSSION AND CONCLUSIONS

Quantized pseudoclassical systems in the Schrödinger realization using the generalized Gupta-Bleuler method exhibit rich interdependences among the reality of the variables, the Grassmann parity of the wave functions, and the split between physical and ghost states, though this structure has heretofore been hidden because the quantization of pseudoclassical theories in the Schrödinger realization has been relatively less studied in comparison to the path integral quantization. The existence of the Schrödinger realization has been assumed by Bordi, Casalbuoni, and Barducci,^{22,23} who also first found the physical states given in Eq. (4.8), and by Mankoč Borštnik.⁸ Delbourgo³³ considered nonrelativistic spin systems represented by the Schrödinger picture quantum mechanics of two anticommuting variables, and relativistic systems represented by four anticommuting variables. He also considered more general involutions on these variables.

The present quantization of pseudoclassical theories in the Schrödinger realization using Dirac's machinery for constrained systems appears to be the first to examine these systems not purely in terms of operators and their representations, but also to construct explicit wave functions and an explicit indefinite inner product, and to examine the adjointness properties of the operators under that inner product following from the involution properties of the Grassmann variables. The present quantization also explicitly realizes the Dirac-Kähler formulation of fermions in the language of Grassmann calculus rather than differential forms; the two descriptions are isomorphic.

In $D = 3 + 1$, the physical states (5.1) have been looked at from a more abstract operatorial point of view by Mankoč Borštnik^{9,34} and Mankoč Borštnik and Nielsen,^{10,35,36} which is closer to our Gupta-Bleuler quantization than to the reduced phase space quantization that the abstract approach of Berezin and Marinov⁷ most closely resembles. Crucially, Mankoč Borštnik and Nielsen¹⁰ examine operators, \tilde{a}^a and $\tilde{\tilde{a}}^a$, that play a central role in their analysis and correspond (up to constant factors) to our constraints $\hat{\phi}_j \approx 0$ and primed variables $\hat{\xi}'_j$, respectively.

We have seen that in this Gupta-Bleuler quantization, adding one more real Grassmann coordinate to a system with an even number of Grassmann variables has two effects. The first is that the number of physical states will double because the ghost state for the new variable can pair with ghost states of the previous system to make physical states in the combined system. In terms of the quantum mechanics, these new states are in a different superselection sector; one can then choose

whether to include one or both of these superselection sectors. The second effect is to make the physical states be of mixed Grassmann parity, because the "measure" in the integral defining the inner product will now have odd Grassmann parity. By contrast, the reduced phase space quantization has a positive definite inner product and so always produces an irreducible representation of the Clifford algebra; adding one more Grassmann coordinate to the system does not lead to a doubling of the number of physical states and there are no superselection sectors in a reduced phase space quantization. This should not be surprising as it is well known³⁷⁻³⁹ that reduced phase space quantizations are not always equivalent to other quantizations of the same constrained system.

We have also seen that the behavior of the Grassmann coordinates under the involution, in other words, whether the variables are taken to be real or imaginary, has an effect on the quantum system. In the trivial case, the quantum mechanics of an imaginary Grassmann variable cannot have a Schrödinger realization unless the kinetic term is negative because the constraint cannot otherwise be imposed.

Since the behavior of the pseudoclassical variables under the involution determines the adjointness properties of the corresponding quantum operators, the timelike ξ_0 must have reality properties opposite to the spacelike ξ_i , and their corresponding quantum operators, the gamma matrices γ^0 and γ^i , therefore must have opposite self-adjointness properties.

We have not paid much attention in the present work to systems with more than just anticommuting degrees of freedom, nor have we considered the physical meaning of the Lorentzian case, but Berezin and Marinov⁷ argue that while the ξ_0 needs to be present for manifest Lorentz invariance, one needs to remove it dynamically by imposing a pseudoclassical "Dirac equation" constraint, which dynamically removes the ξ_0 from the system in a covariant way. Doing so leads to the Dirac equation and world-line supersymmetry. In the present work we have not examined the rich systems with both anticommuting and commuting variables, only mentioning them in passing in Sec. VIII in order to make a connection to Dirac-Kähler fermions. Nonetheless, our methods can be fruitfully brought to bear on the full range of actions that include both commuting and anticommuting classical variables.

Appendix A: $D = 3 + 1$ Gamma matrix representation

In the $3 + 1$ dimensional case, the primed variables lead to operators

$$\begin{aligned}\xi'^\mu &= \xi^\mu - \{\xi^\mu, \varphi_\alpha\} \Delta^{\alpha\beta} \varphi_\beta = \xi^\mu - i(\pi^\mu - \frac{i}{2} \xi^\mu), \\ \hat{\xi}'^\mu &= \frac{\partial^L}{\partial \xi_\mu} + \frac{1}{2} \eta^{\mu\nu} \xi_\nu,\end{aligned}\quad (\text{A1})$$

that satisfy the anticommutation relations

$$\hat{\xi}'^\mu \hat{\xi}'^\nu + \hat{\xi}'^\nu \hat{\xi}'^\mu = \eta^{\mu\nu} = \text{diag}(-, +, +, +). \quad (\text{A2})$$

The $\hat{\xi}'^\mu$ can be represented by scaled Dirac gamma matrices; $\sqrt{2} \hat{\xi}'^\mu \rightarrow \gamma^\mu$. In the unprimed (*i.e.* $|++\rangle$ sector) physical basis (5.1), we define

$$\psi_0|0\rangle + \psi_1|1\rangle + \psi_2|2\rangle + \psi_3|3\rangle = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \quad (\text{A3})$$

and find that the representation of the $\hat{\xi}'^\mu$ in this basis is

$$\begin{aligned}\sqrt{2} \hat{\xi}'^0 &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = -i\sigma_2 \otimes \sigma_3, \\ \sqrt{2} \hat{\xi}'^1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \mathbb{1} \otimes \sigma_1, \\ \sqrt{2} \hat{\xi}'^2 &= \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} = -\mathbb{1} \otimes \sigma_2, \\ \sqrt{2} \hat{\xi}'^3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_3.\end{aligned}\quad (\text{A4})$$

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