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BRST Quantization and Coadjoint Orbit Theories

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Abstract

A new 'harmonic' BRST method is presented for quantizing those dynamical systems having second-class constraints which split into holomorphic and antiholomorphic algebras. These theories include those whose phase spaces are coadjoint orbits of a compact semisimple Lie group. The method also applies to theories with holomorphic first-class constraints which have nonvanishing brackets with their antiholomorphic conjugates. An operatorial quantization, resembling supersymmetric quantum mechanics, is presented. In addition, a general path integral is given and is shown to reduce to that given by Batalin, Fradkin, and Vilkovisky.

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1. Introduction

Certain constrained systems, notably the $D = 10$ harmonic superstring and superparticle [1,2], the Brink-Schwarz superparticle in four dimensions [2,3], and certain coadjoint orbit theories such as particle spin dynamics on a Lie group [4], admit the separation of the second-class constraints into two sets, one holomorphic and one antiholomorphic, each of which not only closes under Poisson brackets with itself, but has the property that either set and the first-class constraints together are also closed under Poisson brackets. In these systems the holomorphic half of the second-class constraints may be implemented as if they were first-class. This is fine for operator quantization, but a BRST quantization is problematic, as the BRST charge would not be hermitian. Operatorial implementation of such BRST charges has been considered previously [5] but in a formalism with a larger number of ghosts. Other authors, most notably Batalin, Fradkin and Fradkina [6] and Egorian and Manvelian [7], have considered generally the much harder feat of embedding theories into larger phase spaces and transmuting the second-class constraints into first, avoiding the problem altogether, but without taking into account any natural holomorphic structure or its preservation or use in implementing the constraints. A new implementation of the BRST-BFV technique for these second-class constraints is given here for systems admitting such holomorphic separations, epitomized by theories whose phase spaces are the coadjoint orbits of semisimple* Lie groups.

In the following sections the method is illustrated starting from the simplest possible case. A more general system, spin dynamics on both $SU(2)$ and $SU(3)$, is used to illustrate the general method. Finally, the implementation of the harmonic BRST-BFV quantization in path integral form is discussed.

2. Harmonic States and Second-Class BRST charges

The harmonic BRST-BFV method starts from the same point as ref. [5] but differs in its ghost structure. In order to make clear the method for dealing with second-class constraints we start first with a simple example. The method supposes that the second-class constraints may be separated into two sets of constraints each of which is closed under the Poisson bracket operation, and whose Poisson brackets with the Hamiltonian of the theory is included in its span. Consider the second-class constraints $q \approx 0$ and $p \approx 0$. We may form them into annihilation and creation operators $a = p - iq$, $a^* = p + iq$ and try imposing them on states. Of course, one may not impose both operators, since their commutator is never zero. In the Gupta-Bleuler method, one would impose the condition that the matrix elements of $\hat{a}$ and $\hat{a}^\dagger$

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* The construction of dynamical systems, especially two-dimensional gravity, directly from the coadjoint orbits of infinite dimensional or non-compact Lie groups is also possible [8].

vanish between physical states. This would leave the vacuum state, $|0\rangle$, as the only physical state.

The harmonic BRST method introduces "ghost" operators $\hat{\xi}$ and $\hat{\bar{\xi}}$ which are complex conjugates, anticommuting and canonical conjugates as well. The method requires one to construct the nilpotent operators $\hat{\Theta} = \hat{\xi}\hat{a}$ and $\hat{\bar{\Theta}} = \hat{\bar{\xi}}\hat{a}^\dagger$, and impose both. A typical (in general, unphysical) state is of the form

$$|\psi\rangle = |\psi_0\rangle + \xi\,|\psi_1\rangle\,. \tag{2.1}$$

The harmonicity conditions $\hat{\Theta}\,|\psi\rangle = \hat{\bar{\Theta}}\,|\psi\rangle = 0$ yield two conditions

$$\begin{aligned} \hat{a}\,|\psi_0\rangle &= 0, \\ \hat{a}^\dagger\,|\psi_1\rangle &= 0. \end{aligned} \tag{2.2}$$

Now the two constraints are not applied to the same state, so there exists a solution to the harmonicity conditions.

Upon taking the Poisson bracket of the two BRST charges, one finds a curious thing. That is, the "Laplacian,"

$$-i\{\Theta, \bar{\Theta}\} = 2\bar{\xi}\xi + aa^* = 2\bar{\xi}\xi + p^2 + q^2, \tag{2.3}$$

is the $OSp(1,1|2)$ invariant quadratic form. We recall the Parisi-Sourlas relation [9],

$$\int d\bar{\xi}\, d\xi\, dp\, dq\, f(2\bar{\xi}\xi + p^2 + q^2) = 2\pi f(0), \tag{2.4}$$

for differentiable functions f which vanish at infinity. This gives meaning to the imprecise statement that the ξ and $\bar{\xi}$ have negative dimension and cancel the dynamical variables p and q. Moreover, for exponential functions, we find directly the interesting relation

$$\lim_{\beta\to\infty} e^{-\beta(2\bar{\xi}\xi + p^2 + q^2)} = 2\pi\, \xi\bar{\xi}\, \delta(p)\delta(q). \tag{2.5}$$

The usual BRST operator is analogous to the exterior derivative operator d on differential forms (with the ghosts playing the role of basis one-forms), except that the inner product on the Hilbert space is not positive definite. This corresponds to the product on forms (of mixed degree) $(\alpha, \beta) = \int \alpha \wedge \beta$ which makes the exterior derivative operator d, like the BRST charge, self-adjoint. If one were to consider the constraint $a \approx 0$ alone, the BRST operator would not be self-adjoint so one is forced to consider the charges $\hat{\Theta}$ and $\hat{\bar{\Theta}}$ together. Now these operators are adjoints of each other, which must be reflected in the inner product. The correct inner product [10] on the ghost Hilbert space of quantum states (which are simply functions of the ghost coordinates ξ) is

$$(f, g) = \int d\xi \, d\bar{\xi} \, e^{-\xi\bar{\xi}} f^*(\bar{\xi}) g(\xi). \tag{2.6}$$

For this simple system one can prove an analog of the Hodge theorem. That is, any state in the Hilbert space of functions of q and ξ can be written uniquely as a harmonic state plus a $\hat{\Theta}$-exact state plus a $\hat{\bar{\Theta}}$-exact state. (The states $|\chi_\pm\rangle$ below are not unique, of course, but the terms containing them are.)

$$\begin{aligned} |\phi\rangle &= |\phi_0\rangle + \hat{\Theta}\,|\chi_+\rangle + \hat{\bar{\Theta}}\,|\chi_-\rangle\,, \\ \hat{\Theta}\,|\phi_0\rangle &= \hat{\bar{\Theta}}\,|\phi_0\rangle = 0. \end{aligned} \tag{2.7}$$

From now on we will drop the hats over operators and trust that this will cause no confusion. Whenever $\bar{\xi}$ appears as a quantum operator, it is to be interpreted as the derivative $\frac{\partial}{\partial \xi}$. When $\bar{\xi}$ appears as a classical variable it is a Grassmann variable and satisfies the Poisson bracket relations $\{\xi, \bar{\xi}\} = \{\bar{\xi}, \xi\} = 1$.

3. SU(2) Particle Spin Dynamics

As a toy system we consider the action for a particle spin written in terms of group variables [4], $g \in SU(2)$,

$$I_{\text{spin}} = i\lambda \int dt \ \mathrm{Tr}(\sigma_3 g^{-1}\dot{g}). \tag{3.1}$$

Being first-order in time derivatives, this model is reparametrization invariant and has its dynamics determined solely by its constraints.

These constraints are

$$\begin{aligned} J_1 &\approx 0, \\ J_2 &\approx 0, \\ J_3 - \lambda &\approx 0, \end{aligned} \tag{3.2}$$

where the J_i are the generators of right translations of g. The first two are obviously second-class while the last is first-class, and generates the only gauge invariance of the model: $g \to g\, e^{i\beta(t)\sigma_3}$.

In the Gupta-Bleuler method for quantizing this system [4], there are nontrivial physical states only when λ is an integer. In this case, there are $2\lambda + 1$ independent states, which can be represented with rotation matrices

$$\psi(g) = \sum_{m=-\lambda}^{\lambda} \psi_m D^{\lambda}_{m,\lambda}(g). \tag{3.3}$$

These states are obtained from the left action of $SU(2)$ on a state which is a highest weight state for the right action. Our goal is to find them from a BRST approach.

Here we wish to find nilpotent operators Θ and $\bar{\Theta}$ as before, but now there is also a true first-class constraint involved. Thus we must find, in addition, a hermitian BRST charge Ω which has Θ and $\bar{\Theta}$ as BRST invariant quantities:

$$\begin{aligned} &\{\Theta, \Omega\} = 0, \\ &\{\bar{\Theta}, \Omega\} = 0, \\ &\{\Omega, \Omega\} = \{\Theta, \Theta\} = \{\bar{\Theta}, \bar{\Theta}\} = 0. \end{aligned} \tag{3.4}$$

These operators are easily found to be

$$\begin{aligned} \Theta &= \xi J_+, \\ \bar{\Theta} &= \bar{\xi} J_-, \\ \Omega &= c(J_3 - \lambda) - c\xi\bar{\xi}. \end{aligned} \tag{3.5}$$

The physical states must satisfy

$$\begin{aligned} \Theta \left|\psi\right\rangle &= 0, \\ \bar{\Theta} \left|\psi\right\rangle &= 0, \\ \Omega \left|\psi\right\rangle &= 0, \end{aligned} \tag{3.6}$$

and are defined up to the addition of null states $\Omega \left|anything\right\rangle$. If the polarization for the ξ, $\bar{\xi}$ dependence is chosen such that the states are functions of ξ only, then the physical states are precisely those of (3.3).

In this case we should notice that there is an ordering ambiguity in the last term of the BRST operator Ω given in (3.5). For the case $\lambda > 0$ the operator is correct as written, while if $\lambda < 0$ the order of ξ and $\bar{\xi}$ must be reversed (and the sign changed because of the Grassmann character of these ghosts.) In effect the normal ordering ambiguity means that there is no restriction that the coupling λ in the model be an integer, since a normal ordering constant, taking any value between 0 and -1, can be added to $\xi\bar{\xi}$.

$$\Omega = c(J_3 - \lambda) - c(\xi\bar{\xi} + \alpha), \quad -1 \leq \alpha \leq 0. \tag{3.7}$$

4. More General Systems

The $SU(2)$ spin system is too trivial to display all of the features of the general case. More complicated behavior is displayed by the dynamics on a higher rank (compact) group manifold. The action is similar to the spin case.

$$I_{\mathcal{G}} = i \int dt \; \mathrm{Tr}(K g^{-1} \dot{g}). \tag{4.1}$$

Here K is some fixed Lie algebra element of the group $\mathcal{G}$. The constraints of this model [4] are the various components of $I^R + K \approx 0$ where I^R is the generator of $\mathcal{G}$ acting on the right. The first-class constraints may given by the components

$$\begin{aligned} \mathrm{Tr}(KI^R) + \mathrm{Tr}(K^2) &\approx 0, \\ \mathrm{Tr}(\bar{K}(\rho) I^R) &\approx 0, \end{aligned} \tag{4.2}$$

where $\bar{K}(\rho)$ and K span the Lie algebra $\mathcal{C}_K$ of the stability group of K. The $\bar{K}(\rho)$ may be chosen to satisfy $\mathrm{Tr}(\bar{K}(\rho)K) = 0$ and form an algebra by themselves. The second-class constraints are the remaining components

$$\mathrm{Tr}(E_{\pm\alpha} I^R) \approx 0, \tag{4.3}$$

with $E_{\pm\alpha} \in \mathcal{C}_K^{\perp}$.

As explained in ref. [4], there must be conditions on K to ensure that the space of physical states in the Gupta-Bleuler approach is not empty. If K satisfies these conditions, then there will be a unique state in the representation of I^R satisfying the constraints.

Before we proceed to the examination of the $SU(3)$ case, let us prove that there always exist operators satisfying the relations (3.4). We assume there is an algebra of holomorphic constraints $\mathcal{A}$ and an algebra $\mathcal{F}$ of first-class constraints such that $\mathcal{A} \oplus \mathcal{F}$ is also closed under Poisson brackets. For the spin model and its generalizations, the algebra $\mathcal{A} \oplus \mathcal{F}$ is the Borel subalgebra of the Lie group, while $\mathcal{F}$ contains the Cartan subalgebra and perhaps some of the raising operators. The crucial observation is that each set $t\mathcal{A} = \{ta_i | a_i \in \mathcal{A}\}$ is also an algebra with Poisson bracket relations

$$\{ta_i, ta_j\} = t f_{ij}^k \, ta_k, \tag{4.4}$$

and so, therefore, is $t\mathcal{A} \oplus \mathcal{F}$. Strictly speaking, the object t is an abstract variable which may take values in the ground field, but is not required to do so. It should be thought of as some scalar generator of the polynomials over the ground field. We also note that the BRST charge for the algebra $t\mathcal{A}$ is homogeneous of degree one.

$$Q(t\mathcal{A}) = tQ(\mathcal{A}). \tag{4.5}$$

It follows that if we construct the BRST charge for the algebra $t\mathcal{A} \oplus \mathcal{F}$, we obtain

$$Q(t\mathcal{A} \oplus \mathcal{F}) =: t\Theta + \Omega, \tag{4.6}$$

and the nilpotency of $Q(t\mathcal{A} \oplus \mathcal{F})$ for all t yields the relations

$$\{\Theta, \Theta\} = \{\Omega, \Omega\} = \{\Theta, \Omega\} = 0. \tag{4.7}$$

The rest of the relations in (3.4) follow from complex conjugation. As an example we construct the operators Θ and Ω for the $SU(3)$ model with $K = \alpha T_8$ and examine the physical states. We choose the Cartan subalgebra $\{T_3, T_8\}$, the simple roots as $\alpha_1 = (1, 0)$ and $\alpha_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, and the generators of $SU(3)$ as $T_a = \frac{1}{2}\lambda_a$.

$$\begin{aligned}
\Omega = {} & c_8(T_8 - \alpha - \tfrac{\sqrt{3}}{2}(\xi_2\bar{\xi}_2 + \xi_{12}\bar{\xi}_{12})) \\
& + c_1(T_1 - \tfrac{1}{2}(\xi_{12}\bar{\xi}_2 + \xi_2\bar{\xi}_{12})) \\
& + c_2(T_2 - \tfrac{i}{2}(\xi_{12}\bar{\xi}_2 - \xi_2\bar{\xi}_{12})) \\
& + c_3(T_3 - \tfrac{1}{2}(\xi_{12}\bar{\xi}_{12} - \xi_2\bar{\xi}_2)) \\
& - \tfrac{i}{2}\epsilon_{ijk}b_i c_j c_k, \\
\Theta = {} & \xi_2 E_{\alpha_2} + \xi_{12}E_{\alpha_2+\alpha_1}.
\end{aligned} \tag{4.8}$$

If we expand out the general state as

$$|\psi\rangle = |\psi_0\rangle + \xi_2\,|\psi_2\rangle + \xi_{12}\,|\psi_{12}\rangle + \xi_2\xi_{12}\,|\psi_3\rangle\,, \tag{4.9}$$

and impose the harmonicity conditions and BRST conditions (3.6), we obtain the following relations for the zero ghost number sector.

$$\begin{aligned}
(T_8 - \alpha)\,|\psi_0\rangle &= T_3\,|\psi_0\rangle = 0, \\
E_{\alpha_2}\,|\psi_0\rangle &= E_{\alpha_2+\alpha_1}\,|\psi_0\rangle = 0, \\
T_-\,|\psi_{12}\rangle &= |\psi_2\rangle\,, \\
E_{\alpha_2}\,|\psi_{12}\rangle &= E_{\alpha_2+\alpha_1}\,|\psi_2\rangle = 0, \\
E_{-\alpha_2}\,|\psi_2\rangle + E_{-\alpha_2-\alpha_1}\,|\psi_{12}\rangle &= 0, \\
E_{-\alpha_2}\,|\psi_3\rangle &= E_{-\alpha_2-\alpha_1}\,|\psi_3\rangle = 0, \\
(T_8 - \alpha - \tfrac{\sqrt{3}}{2})\,|\psi_2\rangle &= (T_8 - \alpha - \tfrac{\sqrt{3}}{2})\,|\psi_{12}\rangle = 0, \\
(T_3 - \tfrac{1}{2})\,|\psi_{12}\rangle &= 0, \\
(T_8 - \alpha)\,|\psi_3\rangle &= T_3\,|\psi_3\rangle = 0.
\end{aligned} \tag{4.10}$$

The only solutions to these conditions are that $|\psi_0\rangle$ is a highest weight state with weight $(0, \alpha)$, $|\psi_3\rangle$ is a lowest weight state of the same weight, and all other components vanish.

In general there might be more than one physical state, but this situation occurs already in ordinary BRST quantization, where one must choose the ghost number of the states also. To this end we introduce the "second-class ghost number" operator N_2,

$$N_2 = \sum \xi_\alpha \bar{\xi}_\alpha, \tag{4.11}$$

which satisfies

$$\begin{aligned} [N_2, \Theta] &= \Theta, \\ [N_2, \bar{\Theta}] &= -\bar{\Theta}, \\ [N_2, \Omega] &= 0. \end{aligned} \tag{4.12}$$

On harmonic states we may diagonalize the operator N_2 as well as the ghost number operator

$$\begin{aligned} N_{\mathrm{gh}} &= \sum c_\beta b_\beta, \\ [N_{\mathrm{gh}}, \Theta] &= [N_{\mathrm{gh}}, \bar{\Theta}] = 0, \\ [N_{\mathrm{gh}}, \Omega] &= \Omega. \end{aligned} \tag{4.13}$$

In an operator quantization the second-class ghost number is superselected, because the operators are conserved by the time evolution under the Hamiltonian and the inner product (2.6) does not mix different ghost numbers. Thus we may choose these ghost numbers appropriately for each physical state, if it is necessary.

5. Path Integral Formulation

An operatorial quantization is quite useful in itself, but the real power of the BRST quantization lies in the flexibility it gives in the path integral formulation. A criterion for constructing a path integral quantization is that the there should be some formulation of these theories which reduces to the BFV Hamiltonian path integral in the appropriate limit. A crucial clue is given by the delta function relation (2.5). Since each of Θ and $\bar{\Theta}$ generates its own "BRST" transformation, we would like a path integral which is invariant under all of the BRST transformations.

In the BFV approach to quantization, one is allowed to deform the BFV invariant Hamiltonian by any BRST exact function, since they both describe the same physics.

$$H_{BRST} \cong H_{BRST} + \{\Omega, \Psi\} = H_{BFV}. \tag{5.1}$$

It would be nice simply to mimic this for the holomorphic constraints and deform the

Hamiltonian by any Θ exact function.

$$H_{BFV} \to H_{BFV} + \{\Theta, \Phi\}. \tag{5.2}$$

This general Θ deformation, however, is not allowed because it would spoil the invariance under $\bar{\Theta}$ transformations. The Hamiltonian, then, is deformable only by very special Θ exact functions; those that are also $\bar{\Theta}$ exact.

$$H_{BFV} \cong H_{BFV} + \{\Theta, \beta\bar{\Theta}\}. \tag{5.3}$$

Here β is any constant, which should be imaginary if the Hamiltonian is to be real. Our first ansatz, then, is that a correct path integral is

$$\mathcal{Z}_{\Psi,\beta} = \int \mathcal{D}\mu \, \exp \frac{i}{\hbar} \int dt (i\bar{\xi}\dot{\xi} + p\dot{q} + b\dot{c} + \bar{c}\dot{\bar{b}} + \pi\dot{\lambda} - H_{BRST} - \beta\{\Theta, \bar{\Theta}\} - \{\Omega, \Psi\}). \tag{5.4}$$

In this path integral, the "measure," $\mathcal{D}\mu$, is the canonical measure over all of the original phase space variables, ghost phase space variables, Lagrange multiplier phase space variables and second-class ghost phase space variables. We will use the rest of this section to argue its equivalence to the BFV path integral with second-class constraints [11] and, using it, to construct a more general path integral in the Batalin-Fradkin-Vilkovisky form.

There are restrictions on the gauge fermion Ψ in order that there be a BFV theorem guaranteeing the path integral's independence of the parameter β and the gauge fermion Ψ. To conserve the ghost numbers N_{gh} and N_2, Ψ must have ghost number -1 and second-class ghost number zero. To guarantee manifest unitarity, both Ψ and β must be imaginary. We impose the conditions

$$\{\Theta, \Psi\} = \{\bar{\Theta}, \Psi\} = 0, \tag{5.5}$$

to ensure that the generating functional $\mathcal{Z}_{\Psi,\beta}$ is both Θ and $\bar{\Theta}$ invariant.

Because the new term in the Hamiltonian, $\beta\{\Theta, \bar{\Theta}\}$, is BRST invariant, we may use the usual argument [12,13] to shift Ψ infinitesimally. We change variables in the path integral by the shift

$$\delta z_i = \{z_i, \Omega\}\epsilon, \quad \epsilon = \frac{i}{\hbar} \int \delta\Psi \, dt. \tag{5.6}$$

The generating functional $\mathcal{Z}_{\Psi,\beta}$ is therefore independent of the gauge fermion Ψ. The Hamiltonian, under the conditions (5.5), is invariant under Θ:

$$\{(H_{BRST} + \{\Omega, \Psi\} + \beta\{\Theta, \bar{\Theta}\}), \Theta\} = 0, \tag{5.7}$$

so that we may treat $\beta\bar{\Theta}$ as a separate gauge fermion and shift β by the same argument. (Strictly speaking, we need only one of the conditions $\{\Theta, \Psi\} = 0$ or $\{\bar{\Theta}, \Psi\} = 0$

to be able to shift β by this argument.)

$$\delta z_i = \{z_i, \Theta\}\epsilon, \quad \epsilon = \frac{i}{\hbar}\int \bar{\Theta}\,\delta\beta\,dt. \tag{5.8}$$

It is also not difficult to prove for the general case of Grassmann even constraints that the delta function relation (2.5) becomes

$$\lim_{\beta\to-\infty} e^{-i\beta\{\Theta,\bar{\Theta}\}} = \pi^N \delta^N(\xi_i)\delta^N(\bar{\xi}_j)\det(i\{a_i, a_j^*\})\delta^N(\Re a_i)\delta^N(\Im a_j). \tag{5.9}$$

Here the symbols a_i and a_j^*, $i,j = 1,\ldots,N$ stand for the holomorphic and anti-holomorphic constraints whose real and imaginary parts, $\Re a_i$, $\Im a_i$, are the original second-class constraints. This expression has the virtue of having the correct measure factor for the second-class constraints while also providing delta functions to fix out the second-class ghosts. If, in addition to requiring the gauge fermion to satisfy the Θ invariance condition (5.5), we also specify

$$\Psi = \Psi|_{\xi=\bar{\xi}=0}, \tag{5.10}$$

then the Poisson bracket $\{\Omega, \Psi\}$ is identical to the Dirac bracket $\{\Omega, \Psi\}_{DB}$. Since the generating functional $\mathcal{Z}_{\Psi,\beta}$ is independent of β, we take the limit as in (5.9) and restrict the gauge fermion as in (5.10), and we recover the BFV generating functional.

It is possible to implement the path integral (5.4) in the most general BFV form if the BRST charge is modified with the help of a single canonically conjugate pair of real anticommuting ghosts. With these ghosts, denoted by ρ and σ, having ghost numbers -1 and 1 respectively, a new nilpotent BRST charge, Q, may be constructed.

$$Q = \Omega + \sigma\{\Theta, \bar{\Theta}\}. \tag{5.11}$$

This BRST charge allows a most general form for the generating functional

$$\mathcal{Z}_{\widetilde{\Psi}} = \int \mathcal{D}\mu \, \exp \frac{i}{\hbar}\int dt(i\bar{\xi}\dot{\xi} + p\dot{q} + b\dot{c} + \bar{c}\dot{\bar{b}} + \pi\dot{\lambda} + \rho\dot{\sigma} - H_{BRST} - \{Q, \widetilde{\Psi}\}). \tag{5.12}$$

This path integral reduces to (5.4) if the new gauge fermion is related to the old one by

$$\widetilde{\Psi} = \Psi + \rho\beta. \tag{5.13}$$

6. Discussion

The main result of the present work is the construction of a formalism which can handle holomorphic and antiholomorphic second-class constraints along with first-class constraints, or holomorphic first-class constraints which have nonvanishing brackets with their antiholomorphic conjugates. Only the case of Grassmann even irreducible constraints has been considered. The generalization to the reducible case appears to pose no difficulty, while the case of some or all Grassmann odd second-class constraints ought to parallel the Grassmann even case exactly, with Grassmann even ghosts for the odd constraints, although this has not been explicitly checked.

The BRST invariant operators Θ, $\bar{\Theta}$ and $\{\Theta, \bar{\Theta}\}$ have interesting analogs in other systems. Besides the analog of the exterior derivative operator d, and its adjoint δ, there is the analog of supersymmetric quantum mechanics where the Hamiltonian is $\{\Theta, \bar{\Theta}\}$. These operators also formally resemble holomorphic and antiholomorphic exterior connections having "curvature" form $\{\Theta, \bar{\Theta}\}$. Such analogs might yield useful insights for systems which can be described by the present formalism.

This brings up the question of the range of applicability of the harmonic BRST-BFV method. There are many systems which have a holomorphic structure. All of these systems owe their holomorphic structures to some inherent group structure. The more general question of how to impose holomorphic constraints is also interesting. There are related questions, involving reality conditions [14], in Ashtekar's new variables approach to gravity. The author knows of no characterization of the systems which admit harmonic BRST-BFV quantizations and conjectures that the method is applicable to systems more general than those following from a Lie group action.

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