BRST QUANTIZATION AND SELF-DUAL GRAVITY

by

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"Few students, teachers, or citizens have any awareness of the history of [the term 'gravity']—that gravity started as the designation of a teleological effect, a 'drive' or 'desire' on the part of the 'heavy' elements earth and water (and their mixtures) to seek the center of the earth; that the opposite 'desire' on the part of air and fire to rise was called 'levity;' that seventeenth-century natural philosophy banished both the teleological view and the word 'levity;' that Newton, explicitly eschewing knowledge of mechanism or process of interaction, made the grand surmise that, however it might work, the same effect that makes the apple fall binds the moon to the earth and planets to the sun; that, despite the beauty and elegance of the General Theory of Relativity, we have, to this day, no idea of how gravity 'works.' "

Arnold B. Arons, *A Guide to Introductory Physics Teaching* (Wiley, New York, 1990).

Abstract

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We investigate the BRST quantization of the self-dual formulation of gravity. The constraints in the self-dual formulation are complex and the standard BRST methods do not apply. We therefore extend BRST methods to systems with complex constraints. After reviewing standard BRST methods, we investigate two types of extension to systems with complex constraints. We first consider theories with complex constraints in which holomorphic and anti-holomorphic constraints are together second class and show that a harmonic BRST method applies to systems with both bosonic (commuting) and fermionic (anticommuting) holomorphic constraints. We next consider theories with complex constraints in which holomorphic and anti-holomorphic constraints are together first class and show that the generator of the BRST transformation is necessarily complex for such theories. This is not acceptable because, upon quantization, a complex BRST generator becomes a nonhermitian BRST operator which fails to achieve the primary goal of the BRST method, namely, the separation of physical from unphysical states. The self-dual formulation of gravity is of this second type of system with complex constraints. We review self-dual gravity in the Ashtekar variables and then show that the standard BRST charge is complex and therefore not useful for BRST quantization. We give two methods by which real BRST charges can be constructed. The first method involves an extension to a reducible system of constraints and the second involves the construction of real constraints by a simple recombination of the original complex constraints. The BRST charges constructed by these methods are real but nonpolynomial in the Ashtekar variables, although in the second method the constraints and the BRST charge are very nearly polynomial.

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Contents

List of Symbols

$\{\ ,\ \}$	Poisson bracket
$:=$	definition
$\equiv$	identity
$\approx$	weak equality (modulo constraints)
$\Gamma_{aA}{}^B$	spin connection 1-form of $\sigma^a{}_A{}^B$
ϵ_{AB}	alternating tensor in spinor space
$\Pi_{aA}{}^B$	imaginary part of $A_{aA}{}^B$ - difference between $A_{aA}{}^B$ and $\Gamma_{aA}{}^B$
$\sigma^a{}_A{}^B,\ \tilde{\sigma}^a{}_A{}^B$	$SU(2)$ soldering form on Σ (densitized)
Σ	3-dimensional manifold
$\tau^i{}_A{}^B$	Pauli matrices
$A_{aA}{}^B$	$SU(2)$ connection 1-form on Σ
∂	reference derivative operator on Σ
D	spatial derivative operator on Σ associated with q_{ab} or $\sigma^a{}_A{}^B$
$\mathcal{D}$	gauge-covariant spatial derivative operator
$F_{abA}{}^B$	curvature of the spin connection $\mathcal{D}$
K_{ab}	extrinsic curvature of Σ
$\widetilde{M}_{aA}{}^B$	momentum conjugate to $\sigma^a{}_A{}^B$ in extended phase space
p	trace of p^{ab}; momentum variable
p^{ab}	momentum conjugate to q_{ab}

q	determinant of q_{ab}; configuration space variable
q_{ab}	spatial metric on Σ
$R_{abcd},\ R$	Riemann tensor and scalar curvature of q_{ab}

Chapter 1

Introduction

1.1 Motivation for a quantum theory of gravity

Gravity is the most conspicuous force of nature. In our search for understanding of the world around us, gravity was the first influence to be recognised. When Isaac Newton gave us his two great contributions to our understanding of nature, the laws of motion and the universal law of gravitation, these gave an essentially complete description of all physical phenomena known to science at that time. The two centuries following Newton were characterized by the successful application of Newton's laws to a wider range of phenomena. The study of electricity and magnetism led to a successful theory describing these forces and to a unification of these forces into a single electromagnetic field. At the end of the nineteenth century, many physicists believed that all physical phenomena could, in principle, be described by the theories of gravity and electromagnetism, and that physics was effectively completed.

A shock to this complacency came when Einstein proposed his special theory of relativity in 1905. This theory grew out of the failure to detect the luminiferous aether, which was assumed to be the medium in which electromagnetic waves propagated. The first attempts by Fitzgerald and Lorentz to explain this failure were compatible with Newtonian mechanics and involved an actual physical contraction of the measuring instruments in the direction of motion relative to the absolute reference frame of the aether (as well as an actual time dilation). Einstein, on the other hand, proposed abandoning the aether altogether and interpreting this contraction as an apparent contraction of an object in one reference frame as observed by an observer in a reference frame moving relative to the first, leaving the question of the physical interpretation of this contraction unanswered. This theory was adopted by the physics community, in spite of the apparent absurdity of the existence of physically detectable electromagnetic waves propagating in the absence of a physical medium, and in spite of the apparent contradictions and counterintuitive notions it introduced into physics. Since then, it has been accepted that any fundamental theory of nature must be consistent with the special theory of relativity.

Maxwell's theory of the electromagnetic field, with its finite propagation velocity, was already consistent with special relativity. Newtonian gravitation, on the other hand, with its infinite propagation velocity, was not. Clearly, the next step was to modify Newtonian gravity in some way to make it consistent with special relativity. But Einstein followed a much more radical path. Motivated by what he has referred to as "the happiest thought of my life," namely the observation that an observer falling freely in a gravitational field does not feel the pull of gravity, Einstein developed a theory of gravity based on the equivalence between the gravitational field and an accelerating reference frame in the absence of external fields. This theory of general relativity attributes the properties of the gravitational field to the geometry of spacetime itself. General relativity has received wide acclaim for its aesthetic beauty and is generally accepted as the "correct" theory of gravity. As revolutionary as Einstein's theories of relativity are, it must be emphasized that they are classical theories in the sense that they accurately describe phenomena on a macroscopic scale but do not by themselves extend to the atomic and subatomic domains.

As our search for understanding penetrated to smaller and smaller domains, the limits of the classical Newtonian physics were discovered. Even before the end of the nineteenth century, spectroscopists were collecting data which could not be fit into the Newtonian model. At first it was hoped that these discrepancies would be resolved and the Newtonian edifice maintained, but it proved impossible to do this. An intellectual crisis, begun in 1900 with Planck's paper on blackbody radiation, was resolved a quarter of a century later with Heisenberg's formulation of matrix mechanics [1], Schrödinger's formulation of wave mechanics [2], and Dirac's abstract quantum algebra [3]. It was soon recognized that these are equivalent formulations of the same underlying theory, and collectively they are now referred to as quantum mechanics. This leads to the second criterion which any fundamental theory of nature must satisfy, namely it must be quantum mechanical.

For the last three quarters of the twentieth century, the program of fundamental theoretical physics has been to develop a unified theory of physical phenomena which is consistent with relativity theory and quantum theory. There has been substantial progress, but this program is still far from completion.

The first major success was the development of quantum electrodynamics, a relativistic quantum theory of the electromagnetic field and its sources. The subsequent discoveries of the weak and strong nuclear forces have largely been incorporated into this program with the development of the Weinberg-Salam electroweak theory and quantum chromodynamics. For reasons which are not fully understood, these theories have degrees of freedom on which physically measurable quantities do not depend. These are referred to as gauge degrees of freedom, and their values can be specified arbitrarily. Collectively, these theories are referred to as gauge theories.

Ironically, gravity, the first force to be "understood," has proven to be the most resistant to incorporation into this program. Despite repeated efforts, a quantum theory of gravity has not successfully been formulated. While the weakness of the force of gravity means that its effects at the atomic scale are orders of magnitude smaller than the effects of the other forces, and hence are unmeasurable in practice, the development of a quantum theory of gravity is essential from a philosophical point of view. A truly unified theory of nature must include the relativistic quantum theory of *all* the fundamental forces. This thesis represents a small step in the program to find a quantum theory of gravity.

1.2 BRST quantization of gauge theories

Gauge theories are characterized by the existence of unphysical degrees of freedom. In the Hamiltonian formulation of gauge theories, these unphysical degrees of freedom manifest themselves as constraints on the phase space.

One approach to solving such theories is to solve the constraint equations, $\phi(q,p) = 0$, directly and to work only on the constraint surface, *i.e.*, the physical subspace of the phase space. In the 1950s, Dirac [4] developed a generalized Hamiltonian method for dealing with constrained systems. In quantizing a theory by Dirac's method, the constraint functions $\phi(q,p)$ become quantum operators. Physical states are those which are annihilated by these quantum constraint operators. In the mid-1970s, Becchi, Rouet, Stora, and Tyutin (BRST) [5] discovered a global symmetry which fully captures the information contained in the local gauge transformations and the constraint functions. Physical states in the BRST theory are those which are annihilated by the BRST operator.

It is characteristic of these three methods that at each step, by enlarging the phase space, the dynamical equations become easier to solve. Solving the constraints of the Dirac formalism, for example, and going to the reduced phase space can introduce nonlinearities into the dynamical equations. The BRST method offers a powerful alternative to the reduced phase space method and the Dirac method for solving gauge theories. For simple systems that can be solved at the reduced phase space or Dirac levels the power of the BRST method is unnecessary, but in more complicated theories such as canonical general relativity, it can be the preferred method.

We wish to consider the rather formidable task of quantizing general relativity. This task has proven intractable in the original Einstein formulation. With the introduction by Ashtekar [11,12] of new variables for general relativity, the constraints and Hamiltonian equations have been made polynomial. This gives hope of making quantum general relativity more tractable. A new feature of the Ashtekar formulation, however, is that the constraints are complex. The theory of BRST quantization for Hamiltonian systems with real constraints has been worked out in great detail, [6,7] but new methods are needed for dealing with systems with complex constraints. After reviewing the standard BRST formalism in Chap. 2, we develop new methods for dealing with complex constraints in Chaps. 3 and 4. In Chap. 3, we consider theories with complex constraints in which holomorphic and anti-holomorphic constraints are together second class and show that a harmonic BRST method applies to systems with both bosonic (commuting) and fermionic (anticommuting) holomorphic constraints. In Chap. 4, we consider theories with complex constraints in which holomorphic and anti-holomorphic constraints are together first class and show that the generator of the BRST transformation is necessarily complex for such theories. This is not acceptable because, upon quantization, a complex BRST generator becomes a nonhermitian BRST operator which fails to achieve the primary goal of the BRST method, namely, the separation of physical from unphysical states. The methods developed in Chap. 4, in particular, are directly applicable to the BRST treatment of canonical general relativity in Ashtekar's variables.

1.3 Self-dual gravity

The traditional Hamiltonian formulation of general relativity [8,9] is in terms of a 3-metric q_{ab} and its conjugate momentum $\widetilde{p}^{ab}$ defined on a three-dimensional spatial slice Σ of four-dimensional space-time. This is a constrained Hamiltonian system and the biggest stumbling block to quantizing it is that the expression of the constraints is very complicated. The standard derivation of the constraints leads to a *nonpolynomial* dependence on q_{ab}. In 1992 Tate [10] showed that a rescaling of these constraints yields a polynomial form, but that they are of very high order (ten) in the phase space variables. In both cases the process of quantizing the constraints and solving the quantum constraint equations is too difficult to be accomplished successfully, because of nonpolynomiality in the first case and because of the high order in the second case.

In 1986 Abhay Ashtekar [11,12] introduced a new Hamiltonian formulation of general relativity based on certain self-dual spinorial variables. This new formulation has the feature that the constraints are *polynomial* and at most quadratic in the phase space variables. This simplification of the constraints offers hope for the construction of a tractable quantum theory of gravity. A new feature of the Ashtekar formulation is, however, that the constraints are complex. Furthermore, the constraints together with their complex conjugates are all first-class, so that they are of the type discussed in Chap. 4. This presents a problem for the application of BRST methods to the quantization of general relativity in the Ashtekar variables.

In 1987, Ashtekar, Mazur, and Torre (AMT) [13] investigated the BRST structure of self-dual gravity and constructed three BRST charges from three different combinations of constraints. We first review the Ashtekar formalism and then show that the BRST charges constructed by AMT are complex and therefore not useful for BRST quantization. We then demonstrate two methods by which real BRST charges can be constructed. The first method involves an extension to a reducible system of constraints and the second involves a simple recombination of the original constraints into a set which is fully real. The BRST charges constructed by these methods are real but nonpolynomial in the Ashtekar variables, although in the second method the constraints and the BRST charge are very nearly polynomial.

Chapter 2

BRST Quantization of Gauge Theories

The theory of BRST quantization of gauge theories is developed in great detail in the review article by Henneaux [6] and in the excellent book by Henneaux and Teitelboim [7]. In the standard BRST formalism, the assumption is made that the theory is real, in the sense that it involves real functions on a real phase space and, therefore, that the constraints in the theory are real. This chapter is a review of the BRST formalism for real theories. In Chaps. 3 and 4 we explore extensions of the BRST formalism to complex theories. In Chap. 3 we consider theories with complex constraints in which holomorphic and anti-holomorphic constraints are together second class. In Chap. 4 we consider complex extensions of real gauge theories in which real constraints on a real phase space are transformed (usually by a canonical transformation) to complex constraints on the real phase space.

2.1 Gauge symmetries and constraints

Gauge degrees of freedom, in a Hamiltonian formulation, appear as unphysical degrees of freedom in the phase space. The evolution in phase space of the physical system thus takes place in a subspace determined by a set of *constraint equations*. We begin with a review of the Dirac theory of constrained Hamiltonian systems.

2.1.1 Constrained Hamiltonian systems

Although our primary interest will be with Hamiltonian formulations of classical theories and their operator quantizations, it is useful to consider here how Hamiltonian theories arise from Lagrangian theories. For a system with N degrees of freedom which has a Lagrange function $L(q, \dot{q})$, the dynamics of the system is determined by the Euler-Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^n}\right) - \frac{\partial L}{\partial q^n} = 0, \qquad n = 1, \ldots, N. \tag{2.1}$$

By expanding the time derivative, Eq. (2.1) can be rewritten as

$$\ddot{q}^{n'} \frac{\partial^2 L}{\partial \dot{q}^{n'} \partial \dot{q}^n} = \frac{\partial L}{\partial q^n} - \dot{q}^{n'} \frac{\partial^2 L}{\partial q^{n'} \partial \dot{q}^n}. \tag{2.2}$$

The accelerations can be solved uniquely in terms of the positions and velocities if and only if the Jacobian $\frac{\partial^2 L}{\partial \dot{q}^{n'} \partial \dot{q}^n}$ is invertible; that is, if the determinant of the Jacobian

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^{n'} \partial \dot{q}^n}\right) \tag{2.3}$$

does not vanish. If the determinant (2.3) is zero, then there are constraints on the system. If some of these are satisfied identically, the solutions of the equations of motion will contain arbitrary functions of time. This is the situation that is encountered in theories with gauge degrees of freedom.

In constructing the Hamiltonian from the Lagrangian, one starts by defining the canonical momenta,

$$p_n(q,\dot{q}) := \frac{\partial L}{\partial \dot{q}^n}, \tag{2.4}$$

and performing a Legendre transformation to the canonical variables

$$H_0(q,p) = p_n \dot{q}^n(p,q) - L(q,\dot{q}(p,q)). \tag{2.5}$$

A necessary step in this procedure is to solve Eqs. (2.4) for the velocities $\dot{q}(q,p)$ as functions of the canonical variables and then substitute them into Eq. (2.5). The condition that this can be done is again that the determinant (2.3) not vanish. In other words, if the determinant (2.3) is zero, not all of the momenta (2.4) are independent, *i.e.*, $\dim(p) < \dim(\dot{q})$. In this case, there will be some constraint relations

$$\phi_j(q,p) \approx 0, \qquad j = 1,\ldots,J, \tag{2.6}$$

among the the canonical variables that follow from Eq. (2.4). In the Dirac terminology, these are called *primary constraints*. (The curly equals sign, read as *weakly equals*, is a reminder that Eqs. (2.6) are not identities on the phase space and that the ϕ_j should not be set equal to zero when they appear inside Poisson brackets.)

The primary constraints (2.6) define a subspace of the full phase space to which the physical states and their dynamics are confined. The canonical Hamiltonian (2.5) is ambiguous on the constraint surface defined by Eqs. (2.6) since one may add any multiple of the constraints ϕ_j to it without changing its value on the constraint surface. Thus, the canonical Hamiltonian can be arbitrarily extended off the constraint surface,

$$H_1 = H_0 + u^j(q,p)\phi_j, \tag{2.7}$$

where H_0 is the canonical Hamiltonian (2.5) and the u^j are, as yet, arbitrary functions on the phase space. A necessary consistency condition is that the primary constraints be preserved in time,

$$\dot{\phi}_n = \{\phi_n, H_1\} \approx 0, \tag{2.8}$$

where $\{\ ,\ \}$ is the Poisson bracket defined by

$$\{F, G\} = \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial q}. \tag{2.9}$$

These consistency conditions are of three types: (*i*) they may be satisfied identically, (*ii*) they may impose conditions on the u^j, or (*iii*) they may impose additional constraints,

$$\phi_k(q,p) \approx 0, \qquad k = J{+}1, \ldots, J{+}K, \tag{2.10}$$

called *secondary constraints*, on the phase space. Secondary constraints are treated the same as primary constraints. The Hamiltonian is again unique only up to the addition of these secondary constraints, and the secondary constraints must again be preserved in time by these Hamiltonians. This can lead to further constraints. We iterate this process until it terminates with a set of constraints,

$$\phi_m \approx 0, \qquad m = 1, \ldots, M, \tag{2.11}$$

a Hamiltonian

$$H = H_0 + u^m(q,p)\phi_m, \tag{2.12}$$

and possibly some restrictions on the Lagrange multipliers u^m.

Assuming now that we have a complete set of constraints ϕ_m, we can consider the restrictions on the Lagrange multipliers u^m,

$$\dot{\phi}_j = \{\phi_j, H_0\} + u^m\{\phi_j, \phi_m\} \approx 0, \tag{2.13}$$

where $1 \leq j \leq J$ and $1 \leq m \leq M$ and J now includes all the constraints, not just the primary ones. Equations (2.13) form a system of J inhomogeneous linear equations in the $M \leq J$ unknowns u^m. The general solution of (2.13) is of the form

$$u^m = U^m + v^a V_a{}^m, \tag{2.14}$$

where U^m is a particular solution of the inhomogeneous equation (2.13) and $v^a V_a{}^m,\ a = 1, \ldots, A$, is a linear combination of linearly independent solutions $V_a{}^m$ of the associated homogeneous equation.

Finally, substituting (2.14) into (2.12), and defining

$$\begin{aligned} H' &= H_0 + U^m \phi_m \\ \phi_a &= V_a{}^m \phi_m, \end{aligned} \tag{2.15}$$

we get the *total Hamiltonian*

$$H_T = H' + v^a \phi_a. \tag{2.16}$$

In terms of the total Hamiltonian, the equations of motion for any function F are simply

$$\dot{F} \approx \{F, H_T\} = \{F, H'\} + \{F, v^a \phi_a\}, \tag{2.17}$$

and the dependence of the dynamics on arbitrary functions of time, v^a, has been made explicit.

2.1.2 First-class constraints and gauge symmetries

Among the constraints – and, more generally, among functions defined on phase space – there is an important classification into *first-class* and *second-class* functions. A function $F(q,p)$ is called first class if its Poisson bracket with every constraint vanishes weakly,

$$\{F, \phi_m\} \approx 0, \qquad m = 1, \ldots, M. \tag{2.18}$$

A function of the canonical variables is called second class if it is not first class, that is, if there is at least one constraint such that its Poisson bracket with F does not vanish weakly. In this section we consider the properties of first-class constraints. Second-class constraints will be considered in Sec. 2.1.3.

An important feature of the first-class property is that it is preserved by the Poisson bracket operation, that is, the Poisson bracket of two first-class functions is again first class. From this it follows that H' and ϕ_a, defined by Eqs. (2.15), are first class.

The existence of arbitrary functions v^a in the total Hamiltonian leads to arbitrariness in the evolution of the dynamical variables. Consider, in particular, the difference δF in the values of a dynamical variable F at time $t_2 = t_1 + \delta t$ resulting from two different choices v^a, $\tilde{v}^a$ of the arbitrary functions at time t_1. From Eq. (2.16), it follows that

$$\delta F = \delta v^a \{F, \phi_a\}, \tag{2.19}$$

with $\delta v^a = (v^a - \tilde{v}^a)\delta t$. Since the *physical* state at time t_2 is independent of the arbitrary functions v^a, the transformation (2.19) of the dynamical variables likewise leaves the physical state unchanged. Adopting the terminology used in gauge field theories, we then say that *first-class constraints generate gauge transformations.*

2.1.3 Second-class constraints

In contrast to first-class constraints, which play a central role in the BRST treatment of gauge theories, second-class constraints result from irrelevant phase space variables which play no role in the dynamics of the system. This is illustrated by a simple example. Consider the set of constraints

$$q^1 \approx 0, \quad p_1 \approx 0, \tag{2.20}$$

They are second class because the Poisson bracket $\{q^1, p_1\} = 1$ does not vanish weakly. The physical interpretation of these constraints is clear in this simple example - the degrees of freedom q^1 and p_1 do not really take part in the dynamics of the system. We may simply ignore them and redefine Poisson brackets that do not include the $n = 1$ degree of freedom,

$$\{F, G\}^* = \sum_{n=2}^{N} \left(\frac{\partial F}{\partial q^n} \frac{\partial G}{\partial p_n} - \frac{\partial G}{\partial q^n} \frac{\partial F}{\partial p_n} \right). \tag{2.21}$$

We then obtain the correct dynamics by using these redefined Poisson brackets.

In the general case, eliminating the irrelevant phase space variables from the theory may be technically difficult. Dirac, however, developed a technique for consistently redefining the Poisson brackets that yields a consistent dynamics. For simplicity, we assume that the constraints ϕ_m are completely split into first-class γ_a and second-class χ_α constraints. (This can always be done, in principle, by a suitable redefinition of the constraints.) The Poisson bracket matrix then is *weakly*

$$\begin{array}{c c} & \begin{array}{cc} \gamma_a & \chi_\alpha \end{array} \\ \begin{array}{c} \gamma_b \\ \chi_\beta \end{array} & \begin{pmatrix} 0 & 0 \\ 0 & C_{\beta\alpha} \end{pmatrix} \end{array} \tag{2.22}$$

where $C_{\beta\alpha}$ is an antisymmetric matrix which is invertible everywhere on the constraint surface determined by the second-class constraints $\chi_\alpha \approx 0$. Since $C_{\alpha\beta}$ is invertible, it has an inverse $C^{\alpha\beta}$,

$$C^{\alpha\beta}C_{\beta\gamma} = \delta^\alpha{}_\gamma. \tag{2.23}$$

The Dirac bracket is now defined as

$$\{F, G\}_{DB} = \{F, G\} - \{F, \chi_\alpha\}C^{\alpha\beta}\{\chi_\beta, G\}. \tag{2.24}$$

The Dirac bracket has all of the same algebraic properties, including the Jacobi identity,

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0, \tag{2.25}$$

as the Poisson bracket.

The Dirac bracket projects the system dynamics onto the constraint surface determined by the second-class constraints, thereby effectively eliminating them from the problem. In particular, we notice that the Dirac bracket of any function F of the phase space variables with any second-class constraint vanishes *strongly*,

$$\{\chi_\alpha, F\}_{DB} = 0 \qquad \text{for any } F. \tag{2.26}$$

It follows that the second-class constraints can be set equal to zero either before or after evaluating a Dirac bracket. The situation is thus the following. The Poisson bracket is used to separate the constraints into first and second class. Then all the dynamical equations are rewritten in terms of the Dirac bracket, and the second-class constraints become *strong* identities.

One interesting way in which second class constraints arise naturally in gauge theories is through gauge fixing. As we have seen, gauge transformations are generated by first-class constraints. Fixing the gauge means imposing additional conditions, that is, additional *constraints*, on the system. The gauge-fixing conditions effectively eliminate the gauge degrees of freedom from the dynamics. As one might expect, the gauge-generating first-class constraints and the gauge-fixing constraints are, together, second class. A gauge-fixing constraint converts a first-class constraint into a second-class constraint. In order to fully fix the gauge, the number of gauge-fixing conditions must equal the number of first-class constraints.

2.1.4 Gauge-invariant functions

A classical "observable" A_0 is a function on the constraint surface which is invariant under gauge transformations. Its Dirac brackets with the first-class constraints γ_a therefore vanish weakly,

$$\{A_0, \gamma_a\}_{DB} \approx 0, \tag{2.27}$$

or, equivalently,

$$\{A_0, \gamma_a\}_{DB} = A_a{}^b \gamma \tag{2.28}$$

Since the system is constrained to lie on the constraint surface $\gamma_a \approx 0$, two gauge invariant functions ("observables") A_0 and A_0' which have the same value on the constraint surface should be identified,

$$A_0' \sim A_0 \quad \text{iff} \quad A_0' = A_0 + k^a \gamma_a. \tag{2.29}$$

Thus, the concept of observable involves two steps: (i) the restriction to the constraint surface; (ii) the gauge invariance condition (2.27), *i.e.*, the condition that the observable be first class with respect to the Dirac bracket.

2.2 Classical BRST structure

The BRST transformation was first discovered [5] in the context of quantum field theory, and it was only later realized that the BRST construction can be fully understood within classical mechanics. The BRST symmetry captures the structure of the first-class constraint surfaces in phase space and could well have been discovered in the nineteenth century if the study of mechanics over Grassmann algebras, involving anticommuting variables, had been of interest. Although the BRST construction can be applied to both Lagrangian and Hamiltonian methods, it is more transparent within the Hamiltonian formalism. We examine the classical BRST structure in this section and will briefly discuss BRST quantization in section 2.3.

2.2.1 Ghost extended phase space

As emphasized in the previous section, gauge theories have unphysical degrees of freedom which must be "removed" from the theory in order to extract physically meaningful quantities. When quantizing a Hamiltonian formulation of a gauge theory, there are two obvious ways in which the undesirable degrees of freedom can be eliminated. The first is the obvious *reduced phase space* approach in which the phase space constraints are solved and used to eliminate the gauge degrees of freedom from the classical theory. This approach is often computationally very difficult. The second way is the *Dirac constrained dynamics* method in which the constraint equations are not solved at the classical level, but are instead quantized along with the physical part of the theory. The resulting Hilbert space, initially "too large," is reduced to the physical Hilbert space by the requirement that the quantum *constraint operators* annihilate the physical states. Naively, one might think that these two methods exhaust the possible methods of eliminating the extra degrees of freedom. It is quite remarkable, however, that a third method, the *BRST* method, is available which involves first *expanding* the phase space. For each gauge degree of freedom, a *ghost* degree of freedom with the opposite statistics is added in such a way that the two degrees of freedom cancel each other.

Although the BRST approach is most natural in the quantum context where we have the familiar bosonic and fermionic statistics, it can also make sense in a classical context by introducing Grassmann numbers ξ^i, *i.e.*, classical anticommuting numbers that obey the rule $\xi^1\xi^2 = -\xi^2\xi^1$. In fact, the quantum terminology is often adopted in the classical setting, and classical commuting variables are referred to as "bosonic," while classical anticommuting variables are referred to as "fermionic." We define the Grassmann parity ϵ by

$$\epsilon(z) = 0, \qquad \epsilon(\xi) = 1, \tag{2.30}$$

for any commuting variable z and anticommuting variable ξ. The Grassmann parity of the product of two numbers is defined by

$$\epsilon(\alpha\beta) = \epsilon(\alpha) + \epsilon(\beta), \qquad \text{mod } 2. \tag{2.31}$$

To see how Grassmann variables can effectively cancel commuting variables, we consider the integral

$$\int d\xi^1\, d\xi^2\, dp\, dq\, f(2\xi^1\xi^2 + p^2 + q^2). \tag{2.32}$$

To evaluate this integral, we perform a Taylor series expansion of f in powers of $2\xi^1\xi^2$,

$$f(2\xi^1\xi^2 + p^2 + q^2) = f(p^2 + q^2) + 2\xi^1\xi^2 f'(p^2 + q^2). \tag{2.33}$$

The Taylor series terminates because the anticommutativity of ξ requires $\xi\cdot\xi = 0$. Next we recall the definitions for integrating Grassmann variables,

$$\int d\xi\, 1 = 0, \qquad \int d\xi\, \xi = 1. \tag{2.34}$$

Performing the integrals in (2.32), we find

$$\begin{aligned}
&\int d\xi^1\, d\xi^2\, dp\, dq\, f(2\xi^1\xi^2 + p^2 + q^2) \\
&\qquad = \int dp\, dq \int d\xi^1\, d\xi^2\, [f(p^2 + q^2) + 2\xi^1\xi^2 f'(p^2 + q^2)] \\
&\qquad = -2 \int dp\, dq\, f'(p^2 + q^2) \\
&\qquad = 2\pi f(0),
\end{aligned} \tag{2.35}$$

where the last step is performed by transforming to polar coordinates. Equation (2.35) is the Parisi-Sourlas [14] relation. So, speaking loosely, ξ^1 and ξ^2 have "negative" dimension and cancel the dynamical variables q and p.

Thus BRST analysis is characterized by the presence of "ghost" degrees of freedom, the number of which equals the number of (gauge) degrees of freedom that need to be cancelled. For each constraint $G_a \approx 0$, we introduce a ghost η^a and its conjugate momentum $\mathcal{P}_a$, both of which are anticommuting. They satisfy the complex conjugation properties

$$(\eta^a)^* = \eta^a, \qquad \mathcal{P}_a^* = -\mathcal{P}_a, \tag{2.36}$$

and the Poisson bracket relation

$$\{\mathcal{P}_a, \eta^b\} = -\delta_a^b = \{\eta^b, \mathcal{P}_a\}. \tag{2.37}$$

The ghosts η^a are taken to be real by convention. The definition for complex conjugation of the Poisson bracket,

$$\{A, B\}^* = -\{B^*, A^*\} \tag{2.38}$$

then forces the momenta $\mathcal{P}_a$ to be imaginary. Note that the Poisson bracket is symmetric for Grassmann variables. The Poisson brackets of η^b and $\mathcal{P}_a$ with the original phase space variables $z^A = q^A, p_A$ vanish,

$$\{\eta^a, z^A\} = 0 = \{\mathcal{P}_a, z^A\}, \tag{2.39}$$

and the Poisson brackets $\{z^A, z^B\}$ of the original phase space variables are left unchanged. For convenience, all of the rules for operating with fermionic (Grassmann) variables are collected in Appendix A.

It is also convenient to define an additional structure on the extended phase space, that of *ghost number*, by

$$\begin{aligned} \mathrm{gh}(z^A) &= 0, \\ \mathrm{gh}(\eta^a) = -\mathrm{gh}(\mathcal{P}_a) &= 1. \end{aligned} \tag{2.40}$$

A sum of terms with different ghost numbers is said to have a ghost number which is not well defined or indefinite. The ghost number of a product of variables (with definite ghost number) is equal to the sum of their ghost numbers. We observe that the product $\eta^a \mathcal{P}_a$ is real and has ghost number zero. Similarly, we define the *antighost number* by

$$\begin{aligned} \mathrm{antigh}(z^A) &= 0 \\ \mathrm{antigh}(\mathcal{P}_a) = -\mathrm{antigh}(\eta^a) &= 1. \end{aligned} \tag{2.41}$$

2.2.2 Construction of the BRST charge

Consider a Hamiltonian system with independent bosonic first-class constraint functions G_a. That is, the constraint equations are

$$G_a \approx 0. \tag{2.42}$$

The first-class condition implies that the constraint functions form a Poisson algebra,

$$\{G_a, G_b\} = C_{ab}{}^c G_c. \tag{2.43}$$

Although it is conventional to call the $C_{ab}{}^c$ the structure functions for this algebra, for calculating the BRST charge it is convenient to define

$${}^{(1)}U_{ab}{}^c := -\tfrac{1}{2} C_{ab}{}^c, \tag{2.44}$$

and, with a slight abuse of terminology, call the ${}^{(1)}U_{ab}{}^c$ the *first-order structure functions* associated with the constraints G_a. It is also convenient to call the G_a the *zeroth-order structure functions*,

$${}^{(0)}U_a := G_a. \tag{2.45}$$

This terminology is consistent in that (2.44) and (2.45) turn out to be special cases of a hierarchy of nth-order structure functions ${}^{(n)}U$ that appear during the construction of the BRST charge.

Since the constraints G_a must satisfy the Jacobi identity for the Poisson bracket, which can be written as

$$\{\{G_a, G_b\}, G_c\}_{\mathrm{A}} = 0 \tag{2.46}$$

(A denotes antisymmetrization), the first-order structure constants ${}^{(1)}U$ can not all be independent, but instead must satisfy the conditions

$$\{ {}^{(1)}U_{[bc}{}^{a},\ {}^{(0)}U_{d]}\} + 2\,{}^{(1)}U_{[bc}{}^{k}\,{}^{(1)}U_{d]k}{}^{a} = 2\,{}^{(2)}U_{bcd}{}^{ak}\,{}^{(0)}U_{k}, \qquad (2.47)$$

where the square brackets indicate antisymmetrization on the indices bcd. This relation defines the second-order structure functions ${}^{(2)}U_{b_1b_2b_3}{}^{a_1a_2}$ which are completely antisymmetric in both $(b_1b_2b_3)$ and (a_1a_2). Note that, if the ${}^{(1)}U$ happen to be constants on the phase space, the constraints form a Lie algebra and the left side of (2.47) vanishes identically. In this case the ${}^{(2)}U$ can be set to zero.

The second-order structure functions must again satisfy similar conditions. By taking the Poisson bracket of Eq. (2.47) with the constraints ${}^{(0)}U$ and antisymmetrizing in the lower indices, we get

$$\begin{aligned} &\{ {}^{(2)}U_{[b_1b_2b_3}{}^{[a_1a_2]},\ {}^{(0)}U_{b_4]}\} - \tfrac{1}{2}\{ {}^{(1)}U_{[b_1b_2}{}^{[a_1},\ {}^{(1)}U_{b_3b_4]}{}^{a_2]}\} \\ &- 3\,{}^{(1)}U^{k}_{[b_1b_2}\,{}^{(2)}U_{b_3b_4k]}{}^{[a_1a_2]} + 4\,{}^{(2)}U_{[b_1b_2b_3}{}^{[a_1|k|}\,{}^{(1)}U_{b_4k}{}^{a_2]} \\ &\qquad\qquad = 3\,{}^{(3)}U_{b_1b_2b_3b_4}{}^{a_1a_2a_3}\,{}^{(0)}U_{a_3}. \end{aligned} \qquad (2.48)$$

This relation defines the third-order structure functions ${}^{(3)}U_{b_1b_2b_3b_4}{}^{a_1a_2a_3}$ which are completely antisymmetric in both $(b_1b_2b_3b_4)$ and $(a_1a_2a_3)$. The vertical lines around the index k indicate that it is excluded from the antisymmetrization.

The hierarchy of structure functions may terminate at a low order, or may continue to a high, or even infinite, order. At each order, the structure functions of rank $n+1$ are determined by the structure functions of rank 1 through n by equations similar to Eqs. (2.47) and (2.48),

$$\frac{1}{2}\sum_{p=0}^{n}\{\,{}^{(p)}U_{[b_1\ldots b_{p+1}}{}^{[a_1\ldots a_p},\ {}^{(n-p)}U_{b_{p+2}\ldots b_{n+2}]}{}^{a_{p+1}\ldots a_n]}\}$$
$$-\sum_{p=0}^{n-1}(p+1)(n-p+1)\,{}^{(p+1)}U_{[b_1\ldots b_{p+2}}{}^{[a_1\ldots a_p k}\,{}^{(n-p)}U_{b_{p+3}\ldots b_{n+2}k]}{}^{a_{p+1}\ldots a_n]}$$
$$=(n+1)\,{}^{(n+1)}U_{b_1\ldots b_{n+2}}{}^{a_1\ldots a_{n+1}}\,{}^{(0)}U_{a_{n+1}}. \tag{2.49}$$

The ${}^{(n+1)}U$ are antisymmetric in both the lower indices $(b_1 \ldots b_{n+2})$ and the upper indices $(a_1 \ldots a_{n+1})$.

This, then, gives us a procedure for calculating the structure functions of all orders, although the calculation of the structure functions quickly becomes quite involved as n increases.

Assuming that the structure functions have been calculated, the BRST charge Ω can be constructed,

$$\Omega = \sum_{p\geq 0} {}^{(p)}\Omega, \qquad {}^{(0)}\Omega = \eta^a G_a, \tag{2.50}$$

where the function ${}^{(p)}\Omega$ is a polynomial of order p in the ghost momenta,

$${}^{(p)}\Omega = \eta^{b_1}\ldots\eta^{b_{p+1}}\,{}^{(p)}U_{b_{p+1}\ldots b_1}{}^{a_1\ldots a_p}\mathcal{P}_{a_p}\ldots\mathcal{P}_{a_1}. \tag{2.51}$$

We note that the structure functions ${}^{(p)}U_{b_{p+1}\ldots b_1}{}^{a_1\ldots a_p}$ are functions only of the original phase space variables, and that the leading ($p = 0$) term in the expansion, shown explicitly in Eq. (2.50), follows from Eq. (2.51) and the definition, ${}^{(0)}U_a = G_a$, of the zero-order structure functions. The importance of the BRST charge Ω is that it can be used to find the physical states of a dynamical system, as we will see in Sec. 2.3.

We point out two useful specific cases of the general BRST charge (2.50). In the case of an abelian gauge group, the constraints all commute, $\{G_a, G_b\} = 0$, the first-order structure functions all vanish, and the BRST charge is simply

$$\Omega_{\text{abelian}} = \eta^a G_a. \tag{2.52}$$

In the case of a gauge group which is a true Lie group, the constraints form a Lie algebra, $\{G_a, G_b\} = C_{ab}{}^c G_c$ with the $C_{ab}{}^c$ constant, the second- and higher-order structure functions vanish, and the BRST charge is given by

$$\Omega_{\text{Lie}} = \eta^a G_a - \tfrac{1}{2}\eta^b \eta^c C_{cb}{}^a \mathcal{P}_a. \tag{2.53}$$

2.2.3 Reducible systems

It may occur that the number of first-class constraints is greater than the number of gauge degrees of freedom. In that case, not all of the constraints are independent. If we have m' first-class constraints,

$$G_a \approx 0, \qquad a = 1, \ldots, m', \tag{2.54}$$

and there are m gauge degrees of freedom, then there must be $m' - m$ relations among the constraints,

$$Z_i{}^a G_a = 0, \qquad i = 1, \ldots, m' - m. \tag{2.55}$$

Such a system is called *reducible* and the relations among the constraints are called *reducibility conditions*. We assume that the reducibility conditions are themselves independent. Although one can, in principle, solve the reducibility conditions and extract an irreducible set of n constraints from the original set, this may not be desirable. The irreducible set of constraints may involve equations that are more difficult to solve, for instance. We review here the BRST construction for reducible systems [7].

The reducibility conditions can be thought of as "constraints on the constraints." They decrease the number of constraints to an independent subset just as the constraints themselves reduce the number of phase space degrees of freedom to the physical subspace. And just as we introduced a ghost pair $\eta^a, \mathcal{P}_a$ for each constraint G_a, we also introduce a *ghost of ghost* pair ϕ^i, π_i for each reducibility condition Z_i. The ghost of ghost ϕ^i has ghost number 2 and its conjugate momentum π_i has ghost number -2. The commutativity properties of ϕ^i, π_i are opposite that of the ghosts so that, for commuting constraints, $\eta^a, \mathcal{P}_a$ are fermionic (anticommuting) and ϕ^i, π_i are bosonic (commuting). The ghosts of ghosts are assumed to have canonical Poisson brackets,

$$\{\Pi_i, \phi^j\} = -\delta_i{}^j, \tag{2.56}$$

and are a further extension of the phase space.

The construction of the BRST generator must include the reducibility conditions as well as the original constraints, so the BRST charge given by Eqs. (2.50) and (2.51) must be modified. The BRST generator Ω must satisfy the following requirements.

$$(i) \qquad \mathrm{gh}(\Omega) = 1, \quad \epsilon(\Omega) = 1, \quad \Omega^* = \Omega, \tag{2.57}$$

$$(ii) \qquad \Omega = \eta^a G_a + \phi^i Z_i{}^a \mathcal{P}_a + \text{“more,”} \tag{2.58}$$

$$(iii) \qquad \{\Omega, \Omega\} = 0, \tag{2.59}$$

where "more" now refers to terms containing at least two η's and one $\mathcal{P}$, or two $\mathcal{P}$'s and one η.

To construct the higher-order terms of the BRST charge we follow a procedure similar to the irreducible case. We expand the BRST charge according to the antighost number,

$$\Omega = \sum_{p\geq 0} {}^{(p)}\Omega, \qquad \text{antigh}\, {}^{(p)}\Omega = p, \tag{2.60}$$

and evaluate the Poisson bracket of Ω with itself,

$$\{\Omega, \Omega\} = \sum_{p\geq 0} {}^{(p)}B, \qquad \text{antigh}\, {}^{(p)}B = p, \tag{2.61}$$

with

$${}^{(p)}B = \sum_{k=0}^{p} \left\{ {}^{(p-k)}\Omega, {}^{(k)}\Omega \right\}_{\text{orig}} + \sum_{k=0}^{p+1} \sum_{s=0}^{k} \left\{ {}^{(p-k+s+1)}\Omega, {}^{(k)}\Omega \right\}_{\eta a_s, \mathcal{P} a_s}, \tag{2.62}$$

where the bracket $\{\ ,\ \}_{\text{orig}}$ denotes the Poisson bracket in the original phase space variables z^A, which does not modify the antighost number, and where $\{\ ,\ \}_{\eta a_s, \mathcal{P} a_s}$ is the bracket with respect to $\eta^a, \mathcal{P}_a$ ($s = 0$) or ϕ^a, π_a ($s = 1$) only. The sum on s terminates at $s = 1$ because we have assumed that the reducibility conditions are independent. The latter bracket eliminates one $\mathcal{P}_a$ or π_a and thus reduces the antighost number. The nilpotency condition $\{\Omega, \Omega\} = 0$ thus leads to a set of equations

$${}^{(p)}B = 0, \qquad p = 0, 1, 2, \ldots, \tag{2.63}$$

which can be solved for the higher order terms of Ω.

2.3 BRST quantization

The classical BRST structure of a gauge theory involves an extension of the phase space to include the ghosts on the same footing as the original phase space variables $z^A = q^A, p_A$. BRST quantization requires the realization of not only the z's, but also the ghosts η and their conjugate momenta $\mathcal{P}$ as linear operators in a Hilbert space. In the Hilbert space where the ghosts are realized as operators, the BRST charge becomes a linear operator. Since the Poisson bracket of two anticommuting functions becomes an anticommutator upon quantization, the nilpotency condition reads,

$$[\Omega, \Omega]_+ = 2\Omega^2 = 0. \tag{2.64}$$

Similarly, the gauge invariance condition becomes a BRST invariance condition,

$$\Omega\psi = 0, \tag{2.65}$$

for any *physical* state ψ. BRST invariance does not completely describe the physical states. The nilpotency of Ω implies that any state of the form $\Omega\chi$ obeys

$$\Omega(\Omega\chi) = \Omega^2\chi = 0. \tag{2.66}$$

We can therefore define a BRST cohomology of states and the true physical states are equivalence classes of this BRST cohomology. Define

$$\begin{aligned} \Omega\psi = 0 &\Leftrightarrow \psi \text{ is BRST}-\text{closed}, \\ \psi = \Omega\chi &\Leftrightarrow \psi \text{ is BRST}-\text{exact}. \end{aligned} \tag{2.67}$$

The quantum state cohomology

$$H^*_{\text{st}}(\Omega) = \frac{\operatorname{Ker}\Omega}{\operatorname{Im}\Omega} \tag{2.68}$$

is defined as the set of equivalence classes of BRST-closed states modulo BRST exact states,

$$\psi \sim \psi + \Omega\chi. \tag{2.69}$$

We now make the important observation that the preservation of the inner product by the members of equivalence classes requires that the BRST charge be Hermitian,

$$\begin{aligned}\langle\phi|\psi\rangle &= \langle\phi|\left(|\psi\rangle + \Omega\,|\chi\rangle\right)\\ &= \langle\phi|\psi\rangle + \langle\phi|\Omega|\chi\rangle\\ &= \langle\phi|\psi\rangle + \left\langle\Omega^{\dagger}\phi|\chi\right\rangle,\end{aligned} \tag{2.70}$$

where ϕ and ψ are physical states and χ can be *any* state, not necessarily physical. From the last term we see that $\Omega^{\dagger}$ must equal Ω for the term to vanish by Eq. (2.65). Since the quantum BRST operator is derived from the classical BRST charge, we draw the important conclusion that, to construct a quantum theory with well-defined physical states, *the classical BRST charge* Ω *must be real.*

Chapter 3

Harmonic BRST [1]

We consider systems with first-class holomorphic constraints which are second-class with respect to their complex conjugates or, equivalently, second-class constraints which can be polarized into holomorphic and antiholomorphic subsets, each of which is separately first-class. We show that the harmonic Becchi-Rouet-Stora-Tyutin (BRST) method of quantizing systems with bosonic holomorphic constraints extends to systems having both bosonic and fermionic holomorphic constraints. The ghosts for bosonic holomorphic constraints in the harmonic BRST method have a Poisson bracket structure different from that of the ghosts in the usual BRST method, which applies to systems with real first-class constraints. Apart from this exotic ghost structure for bosonic constraints, the new feature of the harmonic BRST method is the introduction of two new holomorphic BRST charges, Θ and $\bar{\Theta}$ and the addition of an extra term $-\beta\{\Theta, \bar{\Theta}\}$ to the BRST-invariant Hamiltonian. We apply the Fradkin-Vilkovisky theorem to general systems with mixed bosonic and fermionic holomorphic constraints and show that, taking an appropriate limit, the extra term in the harmonic BRST modified path integral reproduces the correct Senjanovic measure [28].

[1] The contents of this chapter have previously been published by Allen and Crossley [15].

3.1 Introduction

A new implementation of Becchi-Rouet-Stora-Tyutin (BRST) quantization was introduced in 1991 by Allen [16] in order to quantize theories with bosonic holomorphic constraints. The main assumption of the method is that there is an algebra of first-class constraints, some of which are not real-valued, but rather are holomorphic, functions and that some subset of the holomorphic constraints, together with their complex conjugates, are second-class. That is, the matrix of Poisson brackets of the holomorphic with their anti-holomorphic partners is not weakly vanishing. If the matrix of Poisson brackets were weakly vanishing, one could simply take the real and imaginary parts of the constraints as being separate first-class constraints and ignore altogether the difficulties of any holomorphic structure. Keeping the holomorphic structure is impossible in a standard Becchi-Rouet-Stora-Tyutin—Batalin-Fradkin-Vilkovisky (BFV) [31] quantization. The quantum constraints, and hence the BRST-BFV charge operator, will not be Hermitian, making it impossible to decouple the unphysical states from the physical ones.

The number of systems to which the new method presented in this chapter applies is in principle quite large. Kalau [17] has shown that any set of an even number of second-class constraints may be split into holomorphic and anti-holomorphic algebras, but has noted that the split may not be computationally useful, either because the quantum algebra may have anomalies or because the holomorphic constraints are computationally intricate. Perhaps the most important example in which a holomorphic structure is useful is that of Ashtekar's new canonical variables reformulation of general relativity

[18,19], where all the constraints are first-class holomorphic and polynomial when written in terms of the self-dual spin connection. Other examples are the $D = 10$ harmonic superstring and superparticle [20,21], the Brink-Schwarz superparticle in four dimensions [21,22] and certain coadjoint orbit theories such as particle spin dynamics on a Lie group [23]. We remark that this splitting can also be used with the operatorial quantization in the case that classical first-class constraints become anomalous in the quantum theory, but our path-integral may need to be modified along the lines of Ref. [24]. Operatorial constructions somewhat different from ours have been given by Hasiewicz *et al.* [25] and recently by Marnelius [26]. The only example of a system known to us to which our method does not apply in principle is Berezin and Marinov's [27] action for a massive point fermion which has an odd number of second-class fermionic constraints.

In this chapter we extend the harmonic BRST-BFV quantization scheme to the case of both bosonic and fermionic constraints. In Sec. 2 we review the use of the harmonic BRST-BFV method for systems with bosonic constraints. In Sec. 3 we demonstrate the extension of this method to systems with fermionic constraints by applying it to the most trivial case, that of a single fermionic constraint. In Sec. 4 we treat the general case of an arbitrary number of bosonic and fermionic constraints. We demonstrate that in a certain limit, the modification of the path integral reproduces the correct Senjanovic [28] measure for second-class constraints. Throughout the analysis we have assumed that the constraints are irreducible.

3.2 The harmonic BRST-BFV method for bosonic constraints

The main assumption that we start from is the existence of an algebra of holomorphic constraints, $\mathcal{A}$, closed under Poisson brackets, whose matrix of Poisson brackets with the complex conjugate algebra, $\bar{\mathcal{A}}$, has non-vanishing determinant, even weakly,

$$\det_{(ij)}\{a_i, \bar{a}_j\} \not\approx 0, \quad a_i \in \mathcal{A}, \quad \bar{a}_j \in \bar{\mathcal{A}}. \tag{3.1}$$

If the determinant in (3.1) were zero, we could find a subalgebra, $\mathcal{B} \subset \mathcal{A}$, such that the determinant was non-vanishing for $a_i \in \mathcal{B}$, $\bar{a}_j \in \bar{\mathcal{B}}$. We could then take the real and imaginary parts of the remaining constraints $a_i \in \mathcal{A}$ $\mathcal{B}$ and treat them separately.

We also assume that there may be some first-class constraints, $\mathcal{F}$, as well, which we may take to be real and for which we assume $\mathcal{F} \oplus \mathcal{A}$ is also an algebra. To construct the necessary operators, we look at the one-parameter set of algebras, $\mathcal{F} \oplus t\mathcal{A}$, consisting of the first-class constraints and holomorphic constraints scaled by an arbitrary real parameter t. There exists a fermionic function, the formal BRST charge, which has vanishing Poisson bracket with itself [6],

$$\begin{aligned} Q(\mathcal{F} \oplus t\mathcal{A}) &= \eta_I f_I + t\eta_i a_i + \dots \\ &= \Omega + t\Theta, \\ \{Q, Q\} &= 0. \end{aligned} \tag{3.2}$$

In Eq. (3.2), the η's are ghost variables, which are anticommuting when the constraints are bosonic. It is worth pointing out that the charge Ω is not necessarily identical to the BRST charge used when the second-class constraints are implemented using Dirac brackets [Eq. (2.24)]. For an example of this, see Ref. [16].

Because the parameter t is arbitrary, the last equality of (3.2) implies the following relations,

$$\{\Omega, \Omega\} = \{\Omega, \Theta\} = \{\Theta, \Theta\} = 0. \tag{3.3}$$

Unfortunately, the charge Q defined in (3.2) is not real, so it is not suitable to use its quantum version, $\hat{Q}$, to define physical states. It is necessary for the existence of a BRST cohomology that the operator $\hat{Q}$ be either Hermitian or anti-Hermitian. This is because the equivalence relation $|\Psi\rangle \cong |\Psi\rangle + \hat{Q}\,|\Phi\rangle$, with $|\Phi\rangle$ arbitrary, must be compatible with the inner product on the Hilbert space. In other words, states that are $\hat{Q}$-exact must be orthogonal to the physical $\hat{Q}$-closed (and $\hat{Q}$-exact) states and, therefore, have zero norm. To have the decoupling

$$\left\langle \Psi \middle| \hat{Q}\Phi \right\rangle = \left\langle \hat{Q}^\dagger \Psi \middle| \Phi \right\rangle = 0, \tag{3.4}$$

for all physical states $|\Psi\rangle$ and all arbitrary $|\Phi\rangle$, it is necessary that $\hat{Q} = \pm\hat{Q}^\dagger$.

It is useful to use the operators $\hat{\Omega}$ and $\hat{\Theta}$ separately. Physical states are defined to be those which are in the cohomology of $\hat{\Omega}$ and are annihilated by both $\hat{\Theta}$ and its adjoint, $\hat{\bar{\Theta}}$,

$$\begin{aligned} \hat{\Theta}\,|\text{phys}\rangle = \hat{\bar{\Theta}}\,|\text{phys}\rangle = \hat{\Omega}\,|\text{phys}\rangle = 0, \\ |\text{phys}\rangle \cong |\text{phys}\rangle + \hat{\Omega}\,|\text{anything}\rangle\,. \end{aligned} \tag{3.5}$$

These states are harmonic in the sense that they are annihilated by the Laplacian $\{\hat{\Theta}, \hat{\bar{\Theta}}\}$.

The ghosts used in Θ and $\bar{\Theta}$ are different from the usual BRST ghosts. There are two inequivalent complex structures one can impose on fermionic phase space variables. The one closest in analogy to the bosonic oscillator has the canonically conjugate variables, ξ and $\xi^* = \bar{\xi}$, which are also complex conjugates of one another. The real and imaginary parts of ξ, in addition to being real, are also canonically self-conjugate,

$$\begin{aligned} &\{\xi, \bar{\xi}\} = -i, \quad \xi = \frac{1}{\sqrt{2}}(\rho + i\pi), \\ &\{\rho, \rho\} = \{\pi, \pi\} = -i, \quad \{\rho, \pi\} = 0. \end{aligned} \tag{3.6}$$

These are the ghosts that we will require. They differ from the so-called (b, c) ghosts [29,30] of the usual BRST-BFV formalism. The (b, c) ghosts are real and are not canonically self-conjugate,

$$\{b, c\} = -i, \quad \{b, b\} = \{c, c\} = 0, \quad b^* = b, \quad c^* = c. \tag{3.7}$$

The symmetry of the fermionic Poisson bracket allows the existence of these two inequivalent complex structures on a pair of canonically conjugate fermionic variables. Bosons have only one such structure. The ghosts used to construct Θ for the holomorphic constraints $a_i \in \mathcal{A}$ are of the first type (3.6), while the ghosts used in Ω are the (b, c) ghosts.

In Ref. [16] a Hamiltonian path integral construction was given, analogous to the BFV [31] construction explained in great detail in Ref. [6]. When the assumption of the harmonic BRST-BFV method applies, the Hamiltonian path integral Z is

$$\mathcal{Z}_{\Psi,\beta} = \int \mathcal{D}\mu \, \exp \frac{i}{\hbar} \int dt (i\bar{\xi}\dot{\xi} + p\dot{q} + b\dot{c} + \dot{\bar{c}}\dot{b} + \pi\dot{\lambda} - H_{\mathrm{BRST}} - \beta\{\Theta, \bar{\Theta}\} - \{\Omega, \Psi\}). \tag{3.8}$$

If the fermionic gauge-fixing parameter Ψ and the constant β are both imaginary, then the path integral is manifestly unitary. The term added by the standard BRST-BFV method, $\{\Omega, \Psi\}$, is the Poisson bracket of the standard BRST charge Ω and a fermionic gauge-fixing function Ψ. The Fradkin-Vilkovisky theorem states that the path integral (3.8) is invariant under infinitesimal variations of Ψ. The new feature of the harmonic BRST-BFV method is the introduction of the harmonic term, $\beta\{\Theta, \bar{\Theta}\}$, to the BRST-invariant Hamiltonian H_{BRST}. The holomorphic BRST charge, Θ, is associated with the holomorphic subalgebra of constraints and $\bar{\Theta}$, its complex conjugate, is the appropriate gauge-fixing function once we scale by the parameter β. Applying the Fradkin-Vilkovisky theorem to the harmonic term, the path integral (3.8) is also invariant under infinitesimal variations of the parameter β. The "extra" piece, $\exp(-i\beta\{\Theta, \bar{\Theta}\})$, in (3.8) in the limit of $\beta \to \infty$ becomes the correct Senjanovic measure for second-class constraints and eliminates the ghost degrees of freedom as well,

$$\lim_{\beta\to\infty} e^{-i\beta\{\Theta,\bar{\Theta}\}} = (-1)^N \pi^N \delta^N(\xi_i)\delta^N(\bar{\xi}_j)\det(i\{a_i, \bar{a}_j\})\delta^N(\operatorname{Re} a_i)\delta^N(\operatorname{Im} a_j). \tag{3.9}$$

3.3 Fermionic constraints

We first consider the simple case of a single holomorphic fermionic constraint, $\phi \approx 0$. Since both the real and imaginary parts of ϕ must weakly vanish, it follows that the complex conjugate constraint also vanishes, $\bar{\phi} \approx 0$. The harmonic BRST-BFV method then introduces a pair of bosonic ghosts, which are both complex conjugate and canonically conjugate. A single fermionic constraint can have only a very simple algebra, although one that is more general than that of a single bosonic constraint. That algebra is

$$\{\phi, \phi\} = \gamma\phi, \tag{3.10}$$

where γ is a fermionic function on phase space. For simplicity, we assume that the brackets between ϕ and $\bar{\phi}$ are those of the fermionic oscillator, $\{\phi, \bar{\phi}\} = -i$. We consider explicitly the case in which γ is a (Grassmann odd) constant. In this case the method yields a state identical to that of a single bosonic constraint, but with the roles of the original and the ghost variables interchanged. This is a consequence of the $OSp(1{,}1\,|\,2)$ invariance of the system [16].

The harmonic BRST charges for (3.10) are

$$\hat{\Theta} = \hat{c}\hat{\phi} + \frac{i}{2}\hat{\gamma}\hat{\bar{c}}\hat{c}\hat{c}, \quad \hat{\bar{\Theta}} = \hat{\bar{c}}\hat{\bar{\phi}} - \frac{i}{2}\hat{\bar{\gamma}}\hat{c}\hat{\bar{c}}\hat{\bar{c}}. \tag{3.11}$$

The most general state in the ghost-enlarged Hilbert space is a sum of products of ghost states $|n\rangle_c$ of occupation number n with states $|\psi_n\rangle$ of the original Hilbert space, $|\Psi\rangle = \sum_{n=0}^{\infty} |n\rangle_c |\psi_n\rangle$. The harmonicity conditions (3.5) yield the physical state

$$|\text{phys}\rangle = |0\rangle_c |\psi_0\rangle\,, \quad \hat{\bar{\phi}}\,|\psi_0\rangle = 0. \tag{3.12}$$

The Poisson bracket of Θ and $\bar{\Theta}$ is

$$i\{\Theta, \bar{\Theta}\} = \phi\bar{\phi} + c\bar{c} - ic\bar{c}(\phi\bar{\gamma} - \gamma\bar{\phi}) + \tfrac{3}{4}\gamma\bar{\gamma}c^2\bar{c}^2. \tag{3.13}$$

When $\gamma = 0$, this is simply the $OSp(1{,}1\,|\,2)$-invariant form. We can prove the relation similar to (3.9),

$$\lim_{\beta\to\infty} \exp(-\beta i\{\Theta, \bar{\Theta}\}) = -\pi\delta(\phi)\delta(\bar{\phi})\delta(c)\delta(\bar{c}). \tag{3.14}$$

To prove (3.14), we integrate the left side against a test function $\varphi(c,\bar{c})$, scale the ghosts and take the limit outside the integral,

$$
\begin{aligned}
&\lim_{\beta\to\infty}\int dc\,d\bar{c}\;\varphi(c,\bar{c})\;\exp(-\beta i\{\Theta,\bar{\Theta}\})\\
&=\lim_{\beta\to\infty}\int\frac{dc'\,d\bar{c}'}{\beta}\;\varphi\Big(\frac{c'}{\sqrt{\beta}},\frac{\bar{c}'}{\sqrt{\beta}}\Big)\;e^{-c'\bar{c}'}\Big(1-\beta\phi\bar{\phi}\Big)\\
&\qquad\qquad\times\Big(1+i\,c'\bar{c}'(\phi\bar{\gamma}-\gamma\bar{\phi})-\frac{3}{4\beta}(c'\bar{c}')^2\gamma\bar{\gamma}-(c'\bar{c}')^2\phi\bar{\gamma}\gamma\bar{\phi}\Big)\\
&=-\,\pi\,\phi\bar{\phi}\,\varphi(0,0).
\end{aligned}
\tag{3.15}
$$

We prove the general case of (3.9) in the next section.

3.4 The general case

We now consider a general constrained system with N fermionic and M bosonic constraints satisfying the assumption of the new method just presented. The general form of the BRST charge Θ is

$$
\begin{aligned}
\Theta =& c_I\phi_I+\xi_i a_i+\sum_{\substack{n=0\\ n+m\geq 1}}^{\infty}\sum_{m=0}^{M}\;\bar{c}_{\bar{I}_1}\cdots\bar{c}_{\bar{I}_n}c_{I_1}\cdots c_{I_n}\bar{\xi}_{\bar{\imath}_1}\cdots\bar{\xi}_{\bar{\imath}_m}\xi_{i_1}\cdots\xi_{i_m}\\
&\times\Big(c_{I_{n+1}}\Xi^{\bar{I}_1\cdots\bar{I}_nI_1\cdots I_{n+1}\bar{\imath}_1\cdots\bar{\imath}_mi_1\cdots i_m}+\xi_{i_{m+1}}X^{\bar{I}_1\cdots\bar{I}_nI_1\cdots I_n\bar{\imath}_1\cdots\bar{\imath}_mi_1\cdots i_{m+1}}\Big)\,,
\end{aligned}
\tag{3.16}
$$

where the indices run $I=1,\ldots,N$ and $i=1,\ldots,M$. We write this schematically as

$$
\Theta=c\phi+\xi a+\sum_{n,m}{}'\,\bar{c}^{\,n}c^{n+1}\bar{\xi}^{\,m}\xi^{\,m}\Xi_{n,m}+\sum_{n,m}{}'\,\bar{c}^{\,n}c^{n}\bar{\xi}^{\,m}\xi^{\,m+1}X_{n,m},
\tag{3.17}
$$

where c^n, for instance, denotes $c_{I_1}c_{I_2}\cdots c_{I_n}$ and the primed sum $\sum'_{n,m}$ is a sum on all multi-indices of positive length, $\{n+m\geq 1,\,n\geq 0,\,M\geq m\geq 0\}$. From

(3.16) it follows that the general BRST Laplacian is

$$
\begin{aligned}
i\{\Theta,\overline{\Theta}\} = a_i\overline{a}_{\bar{\imath}} + \phi_I\overline{\phi}_{\bar{I}} + i\,\xi_i\{a_i,\overline{a}_{\bar{\jmath}}\}\overline{\xi}_{\bar{\jmath}} + i\,c_I\{\phi_I,\overline{\phi}_{\bar{J}}\}\overline{c}_{\bar{J}} \\
+ i\,\xi_i\{a_i,\overline{\phi}_{\bar{J}}\}\overline{c}_{\bar{J}} + i\,c_I\{\phi_I,\overline{a}_{\bar{\jmath}}\}\overline{\xi}_{\bar{\jmath}} \\
+ \text{terms quadratic in ghosts, times constraints} \\
+ \text{terms with more than two ghosts,}
\end{aligned}
\tag{3.18}
$$

which we rewrite using DeWitt's supermatrix notation [32],

$$
i\{\Theta,\overline{\Theta}\} = a\overline{a} + \phi\overline{\phi} + i\xi A\overline{\xi} + icB\overline{c} + i\xi C\overline{c} + icD\overline{\xi} + \tilde{\Delta}. \tag{3.19}
$$

The last term, $\tilde{\Delta}$, contains all of the higher-order pieces.

To make the calculation of $e^{-\beta i\{\Theta,\overline{\Theta}\}}$ in the general case similar to the case of a single fermionic constraint considered above, it is useful to rescale the bosonic and fermionic constraints and ghosts,

$$
\begin{aligned}
a = a'/\sqrt{\beta}, \quad \bar{a} = \bar{a}'/\sqrt{\beta}, \quad c = c'/\sqrt{\beta}, \quad \overline{c} = \overline{c}'/\sqrt{\beta} \\
\phi = \phi'/\sqrt{\beta}, \quad \overline{\phi} = \overline{\phi}'/\sqrt{\beta}, \quad \xi = \xi'/\sqrt{\beta}, \quad \overline{\xi} = \overline{\xi}'/\sqrt{\beta}.
\end{aligned}
\tag{3.20}
$$

We find

$$
e^{-\beta i\{\Theta,\overline{\Theta}\}} = \exp\left(-\beta\phi\overline{\phi} - \beta a\overline{a} - i\,\xi' A\overline{\xi}' - i\,c' B\overline{c}' - i\,\xi' C\overline{c}' - i\,c' D\overline{\xi}' - \beta\tilde{\Delta}\right), \tag{3.21}
$$

which we rewrite in the suggestive form

$$
\begin{aligned}
e^{-\beta i\{\Theta,\overline{\Theta}\}} = \exp\Big(-\beta\phi\overline{\phi} - \beta a\overline{a} - i\,\xi'(A - CB^{-1}D)\overline{\xi}' \\
-i\,(c' + \xi' CB^{-1})B(\overline{c}' + B^{-1}D\overline{\xi}') - \beta\tilde{\Delta}\Big).
\end{aligned}
\tag{3.22}
$$

The last piece $\beta\tilde{\Delta}$ is the sum of two terms, $\beta\tilde{\Delta}^{(1)}$ and $\beta\tilde{\Delta}^{(2)}$, where

$$\begin{aligned}
\beta\tilde{\Delta}^{(1)} = {\sum_{n,m}}' \Big(& a'\,\overline{c}'^{\,n+1}\,c'^{\,n}\,\overline{\xi}'^{\,m-1}\,\xi'^{m}\,\overline{\Xi}_{n,m}\;\beta^{-n-m+\frac{1}{2}} \\
& + i\;\overline{c}'^{\,n+1}\,c'^{\,n}\,\overline{\xi}'^{\,m}\,\xi'^{m+1}\,\{a,\overline{\Xi}_{n,m}\}\;\beta^{-n-m} \\
& + a'\,\overline{c}'^{\,n}\,c'^{\,n}\,\overline{\xi}'^{\,m}\,\xi'^{m}\,\overline{X}_{n,m}\;\beta^{-n-m+\frac{1}{2}} \\
& + i\;\overline{c}'^{\,n}\,c'^{\,n}\,\overline{\xi}'^{\,m+1}\,\xi'^{m+1}\,\{a,\overline{X}_{n,m}\}\;\beta^{-n-m} \\
& + \phi'\,\overline{c}'^{\,n}\,c'^{\,n}\,\overline{\xi}'^{\,m}\,\xi'^{m}\,\overline{\Xi}_{n,m}\;\beta^{-n-m+\frac{1}{2}} \\
& + i\;\overline{c}'^{\,n+1}\,c'^{\,n+1}\,\overline{\xi}'^{\,m}\,\xi'^{m}\,\{\phi,\overline{\Xi}_{n,m}\}\;\beta^{-n-m} \\
& + \phi'\,\overline{c}'^{\,n-1}\,c'^{\,n}\,\overline{\xi}'^{\,m+1}\,\xi'^{m}\,\overline{X}_{n,m}\;\beta^{-n-m+\frac{1}{2}} \\
& + i\;\overline{c}'^{\,n}\,c'^{\,n+1}\,\overline{\xi}'^{\,m+1}\,\xi'^{m}\,\{\phi,\overline{X}_{n,m}\}\;\beta^{-n-m}\Big) \\
& + c.c.,
\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}
\beta\tilde{\Delta}^{(2)} = {\sum_{n,m}}'{\sum_{k,\ell}}' \Big(& \overline{c}'^{\,n+k}\,c'^{\,n+k}\,\overline{\xi}'^{\,\ell+m}\,\xi'^{\ell+m}\,\Xi_{n,m}\overline{\Xi}_{k,\ell}\;\beta^{1-k-n-\ell-m} \\
& + \overline{c}'^{\,n+k+1}\,c'^{\,n+k+1}\,\overline{\xi}'^{\,\ell+m-1}\,\xi'^{\ell+m-1}\,\Xi_{n,m}\overline{\Xi}_{k,\ell}\;\beta^{1-k-n-\ell-m} \\
& + i\;\overline{c}'^{\,n+k+1}\,c'^{\,n+k+1}\,\overline{\xi}'^{\,\ell+m}\,\xi'^{\ell+m}\,\{\Xi_{n,m},\overline{\Xi}_{k,\ell}\}\;\beta^{-k-n-\ell-m} \\
& + \overline{c}'^{\,n+k-1}\,c'^{\,n+k}\,\overline{\xi}'^{\,\ell+m+1}\,\xi'^{\ell+m}\,\Xi_{n,m}\overline{X}_{k,\ell}\;\beta^{1-k-n-\ell-m} + c.c. \\
& + \overline{c}'^{\,n+k}\,c'^{\,n+k+1}\,\overline{\xi}'^{\,\ell+m}\,\xi'^{\ell+m-1}\,\Xi_{n,m}\overline{X}_{k,\ell}\;\beta^{1-k-n-\ell-m} + c.c. \\
& + i\;\overline{c}'^{\,n+k}\,c'^{\,n+k+1}\,\overline{\xi}'^{\,\ell+m+1}\,\xi'^{\ell+m}\,\{\Xi_{n,m},\overline{X}_{k,\ell}\}\;\beta^{-k-n-\ell-m} + c.c. \\
& + \overline{c}'^{\,n+k-1}\,c'^{\,n+k-1}\,\overline{\xi}'^{\,\ell+m+1}\,\xi'^{\ell+m+1}\,X_{n,m}\overline{X}_{k,\ell}\;\beta^{1-n-k-\ell-m} \\
& + \overline{c}'^{\,n+k}\,c'^{\,n+k}\,\overline{\xi}'^{\,\ell+m}\,\xi'^{\ell+m}\,X_{n,m}\overline{X}_{k,\ell}\;\beta^{1-n-k-\ell-m} \\
& + i\;\overline{c}'^{\,n+k}\,c'^{\,n+k}\,\overline{\xi}'^{\,\ell+m+1}\,\xi'^{\ell+m+1}\,\{X_{n,m},\overline{X}_{k,\ell}\}\;\beta^{-n-k-\ell-m}\Big).
\end{aligned} \tag{3.24}$$

The terms in (3.23) and (3.24) are all at least $\mathcal{O}(\beta^{-\frac{1}{2}})$, which means that they can be ignored in the limit $\beta \to \infty$. To obtain the delta function relation analogous to the Senjanovic measure (3.9), we integrate against a test function φ of the bosonic variables $c, \overline{c}, a, \overline{a}$. The integral we wish to evaluate is

$$
\begin{aligned}
I_\beta &= \int d^N c\ d^N \overline{c}\ d^M a\ d^M \overline{a}\ e^{-\beta i\{\Theta,\overline{\Theta}\}}\ \varphi(c,\overline{c},a,\overline{a}) \\
&= \int \frac{d^N c'\ d^N \overline{c}'\ d^M a'\ d^M \overline{a}'}{\beta^{N+M}}\ \varphi\left(\frac{c'}{\sqrt{\beta}},\frac{\overline{c}'}{\sqrt{\beta}},\frac{a'}{\sqrt{\beta}},\frac{\overline{a}'}{\sqrt{\beta}}\right) \\
&\quad \times e^{-\left(\beta\phi\overline{\phi}+a'\overline{a}'+i\xi'(A-CB^{-1}D)\overline{\xi}'+i(c'+\xi' CB^{-1})B(\overline{c}'+B^{-1}D\overline{\xi}')+\mathcal{O}(\beta^{-\frac{1}{2}})\right)},
\end{aligned}
\tag{3.25}
$$

which, upon a shift of the ghost variables, $c' \to c' - \xi' CB^{-1}$, becomes

$$
\begin{aligned}
I_\beta = (-1)^{N+M} \int & \frac{d^N c'\ d^N \overline{c}'\ d^M a'\ d^M \overline{a}'}{\beta^{N+M}}\ \varphi\left(\frac{c'}{\sqrt{\beta}},\frac{\overline{c}'}{\sqrt{\beta}},\frac{a'}{\sqrt{\beta}},\frac{\overline{a}'}{\sqrt{\beta}}\right) \beta^N \delta^N(\phi)\delta^N(\overline{\phi}) \\
& \times \beta^M \delta^M(\xi)\delta^M(\overline{\xi}) \det[i(A - CB^{-1}D)] e^{-a'\overline{a}'}\ e^{-ic' B\overline{c}'} \\
& + \mathcal{O}(\beta^{-\frac{1}{2}}).
\end{aligned}
\tag{3.26}
$$

Because φ is a test function, we obtain in the limit

$$
\begin{aligned}
\lim_{\beta\to\infty} I_\beta =& (-\pi)^{N+M} \delta^N(\phi)\delta^N(\overline{\phi})\delta^M(\xi)\delta^M(\overline{\xi}) \\
& \times \det[i(A - CB^{-1}D)](\det(iB))^{-1} \varphi(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}),
\end{aligned}
\tag{3.27}
$$

which proves the general case of (3.9),

$$
\begin{aligned}
\lim_{\beta\to\infty} e^{-\beta i\{\Theta,\overline{\Theta}\}} =& (-\pi)^{N+M} \delta^N(\phi_I)\delta^N(\overline{\phi}_{\bar{J}})\delta^M(\xi_i)\delta^M(\overline{\xi}_{\bar{j}}) \\
& \times \operatorname{sdet}\begin{pmatrix} i\{a_i,\overline{a}_{\bar{j}}\} & i\{a_i,\overline{\phi}_{\bar{J}}\} \\ i\{\phi_I,\overline{a}_{\bar{j}}\} & i\{\phi_I,\overline{\phi}_{\bar{J}}\} \end{pmatrix} \\
& \times \delta^N(\operatorname{Re} c_I)\delta^N(\operatorname{Im} \overline{c}_{\bar{J}})\delta^M(\operatorname{Re} a_i)\delta^M(\operatorname{Im} \overline{a}_{\bar{j}}).
\end{aligned}
\tag{3.28}
$$

In summary, our main result is that the harmonic BRST-BFV method introduced in ref. [16] generalizes to the case of mixed bosonic and fermionic constraints. There is no problem in extending the operator formalism, but it is not trivial to show that the extension also works for the path integral Z. It is possible that the relation (3.28) is only valid for test functions and not for rapidly decreasing functions as well. This is because the bosonic part of the exponent $-\beta i\{\Theta, \overline{\Theta}\}$ might not be negative definite, although the quadratic piece by itself is negative definite. For test functions, this is irrelevant because in the limit the quadratic piece dominates all others on any finite interval. In the case of purely bosonic constraints, this is not an issue because the higher order pieces are fermionic and have no convergence problem. To be rigorous about a particular path integral, one must check in that specific case that the result also holds for functions of rapid decrease.

Our modification of the path integral and our limit argument are very similar to the method of equivariant localization recently introduced by Dykstra, Lykken and Raiten [33]. Invariance under the change of variables generated by the holomorphic BRST charge Θ in our formalism corresponds closely to the invariance under the equivariant exterior derivative d_χ in the formalism of Ref. [33].

The extension of the formalism to the reducible case is an open problem, but one whose solution is quite likely to follow the standard reducible BRST quantization for real constraints.

Chapter 4

BRST Quantization of Complex Extensions of Real Theories

The constrained Hamiltonian theories most commonly encountered involve real constraints on a real phase space. One of the novel features of self-dual gravity is that it involves an extension to complex constraint functions on a real phase space. The advantage of this is that certain calculations become much more manageable in this extended phase space. But ultimately one wants to recover the real theory from the complex theory, and to do this one imposes reality conditions. In the BRST formalism, this means that the complex constraint functions must have reality conditions associated with them. In this chapter, we present our progress toward the formal aspects of constructing a real BRST charge for systems with complex constraints and associated reality conditions.

4.1 Complex extensions of real theories – reality conditions

We consider systems of real constraints G°_a that are linearly recombined,

$$G_a = C_a^b G_b^\circ, \tag{4.1}$$

where the coefficients A_a^b are, in general complex quantities. Complex conjugation of Eq. (4.1),

$$G_a^* = C_a^{b*} G_b^\circ, \tag{4.2}$$

leads to reality conditions on the constraints,

$$G_a - G_a^* = (C_a^b - C_a^{b*}) G_b^\circ. \tag{4.3}$$

For a complete set of constraints, $*$ leads to a scrambling of pieces, so G^* can be written as $G^* = AG$. Since complex conjugation is an involution ($G^{**} = G$), it follows from

$$G^{**} = (AG)^* = A^* A G \equiv G, \tag{4.4}$$

that the coefficients A have the interesting property

$$A^* = A^{-1}. \tag{4.5}$$

As a simple example of reality conditions, consider the linear combination of constraints

$$\begin{aligned} G_1 &= G_1^\circ \\ G_2 &= G_2^\circ + iAG_1^\circ, \end{aligned} \tag{4.6}$$

where G_1° and G_2° are real constraint functions. Complex conjugation of these constraints leads to the reality conditions,

$$\begin{aligned} G_1^* &= G_1 \\ G_2^* &= G_2 - 2iAG_1. \end{aligned} \tag{4.7}$$

In some cases, the reality conditions on the constraints can be used to construct a real BRST charge by imposing corresponding reality conditions on the ghosts and their conjugate momenta. We consider separately the case of the coefficients A_a^b being constant on the phase space and the case of the coefficients being phase space functions.

4.2 Reality conditions with constant coefficients

The rank of a theory is defined as the highest order of nonzero structure functions ${}^{(n)}U$. We demonstrate the construction of real BRST charges for rank-zero and rank-one theories.

4.2.1 Rank zero

A rank-zero theory has no nonzero structure functions. This is the abelian case,

$$\{G_a, G_b\} = 0, \tag{4.8}$$

with the BRST charge given by Eq. (2.52),

$$\Omega_{\text{abelian}} = \eta^a G_a. \tag{4.9}$$

We impose the condition that the BRST charge be real, $\Omega^* = \Omega$, and use the reality conditions (4.3) on the constraints to derive reality conditions on the ghosts,

$$\begin{aligned} \eta^a G_a &= (\eta^a G_a)^* \\ &= \eta^{a*} A_a^b G_b \\ &= \eta^{b*} A_b^a G_a, \end{aligned} \tag{4.10}$$

where the last step involves a simple relabeling of the dummy indices. Since we want this to hold for arbitrary constraints G_a, we get the reality conditions on the ghosts,

$$\eta^{a*} = \eta^b A_b^{a*}. \tag{4.11}$$

Complex conjugation of the fundamental Poisson brackets then gives the reality conditions on the ghost momenta,

$$\mathcal{P}_a^* = -A_a^b \mathcal{P}_b. \tag{4.12}$$

It is easy to check that the BRST charge Ω and the Poisson bracket $\{\mathcal{P}_a, \eta^a\}$ are both preserved under complex conjugation by recalling equation (4.5).

To summarize, a real BRST charge can be constructed for an abelian theory with complex constraints in which the reality conditions have constant coefficients. This is accomplished by imposing reality conditions on the ghosts, which in turn imposes reality conditions on the ghost momenta through the Poisson bracket relations.

4.2.2 Rank one

A rank-one theory has first-order structure constants. This is the case of a Lie algebra. The BRST charge for a rank-one theory is given by Eq. (2.53),

$$\Omega_{\text{Lie}} = \eta^a G_a - \tfrac{1}{2}\eta^b \eta^c C_{cb}{}^a \mathcal{P}_a. \tag{4.13}$$

We assume the same reality conditions (4.1) on the constraints as in the abelian (rank zero) case. The requirement that the antighost number zero part of (4.13) be real leads to the *same* reality conditions on the ghosts and their momenta as in the abelian case. The new element is the first-order structure functions $C_{ab}{}^c$. Reality conditions on the first-order structure functions follow from complex conjugation of the Poisson brackets between the constraints. The resulting reality conditions are

$$C_{ab}{}^{c*} = A_a^d A_b^e C_{de}{}^f A_f^{C*}. \tag{4.14}$$

It is straightforward to check that the rank one term is also real,

$$(\eta^a \eta^b C_{ab}{}^c \mathcal{P}_c)^* = \eta^a \eta^b C_{ab}{}^c \mathcal{P}_c, \tag{4.15}$$

and therefore that the BRST charge (4.13) is real. In both the rank zero case and the rank one case, the constancy of the coefficients A_a^b has been essential in preserving the Poisson bracket relations.

4.3 Reality conditions with non-constant coefficients

We now consider reality conditions with coefficients that are phase space functions. We first demonstrate that, in this case, the standard BRST treatment of a complexified theory yields a complex BRST charge and is therefore unacceptable. We then give an alternative BRST method by which a real BRST charge can be constructed. This is accomplished by extending the phase space to include the constraints and their complex conjugates. This expanded system of constraints is inherently reducible and is dealt with using the method of Henneaux and Teitelboim [Ref. 7, Chap. 10] for BRST quantization of systems with reducible constraints.

4.3.1 Standard BRST treatment

For our starting point, we consider a simple example of two abelian constraints G_1° and G_2° which are assumed to be real and bosonic,

$$\{G_1^\circ, G_2^\circ\} = 0. \tag{4.16}$$

The BRST charge for this example is given by Eq. (2.52),

$$\Omega^\circ = \eta_\circ^1 G_1^\circ + \eta_\circ^2 G_2^\circ, \tag{4.17}$$

and is manifestly real if the ghosts $\eta_\circ^1$ and $\eta_\circ^2$ are taken to be real, which we are free to do.

What we want to consider is the complex extension of this real theory. By this we mean the analytic continuation of the set of real functions on phase space to the set of complex functions on phase space, with a transformation that takes real constraints into complex constraints. As a concrete example, consider replacing the real constraints G_1° and G_2° by

$$\begin{aligned} G_1^\circ &\quad\rightarrow\quad G_1 = G_1^\circ \\ G_2^\circ &\quad\rightarrow\quad G_2 = G_2^\circ + iA(q,p)G_1^\circ, \end{aligned} \tag{4.18}$$

where we have added a linear multiple of the first constraint to the second. (If we had added a completely arbitrary imaginary term we would have introduced a third constraint, since both the real and imaginary parts must separately vanish, and we would have a different theory.) The coefficient $A = A(q,p)$ is an arbitrary real, bosonic function of the phase space variables. We can think of this transformation as a "deformation" of the real constraints into complex constraints.

The Poisson bracket structure of the constraints becomes

$$\{G_1, G_1\} = \{G_2, G_2\} = 0, \tag{4.19}$$

$$\{G_1, G_2\} = \{G_1^\circ, G_2^\circ + iAG_1^\circ\} = i\{G_1, A\}G_1. \tag{4.20}$$

There is only one nonzero first-order structure function,

$$C_{12}{}^1 = i\{G_1, A\}, \tag{4.21}$$

and the second-order structure functions ${}^{(2)}U_{abc}{}^{de}$ necessarily vanish because they are antisymmetric in (abc) and we have only two indices available. The BRST charge (4.17) is thus deformed into

$$\Omega^\circ \quad \rightarrow \quad \Omega = \eta^1 G_1 + \eta^2 G_2 - i\eta^2\eta^1\{G_1, A\}\mathcal{P}_1. \tag{4.22}$$

We now want to investigate the reality properties of Ω. In particular, we want to see if Ω can remain real. If we hope to accomplish this we must allow the ghosts and ghost momenta to become complex, but there is no *a priori* reason why this should not be allowed. We complex conjugate Ω,

$$\Omega^* = G_1^*\eta^{1*} + G_2^*\eta^{2*} + i\mathcal{P}_1^*(-\{A^*, G_1^*\})\eta^{1*}\eta^{2*}, \tag{4.23}$$

and use the reality condition on G_2,

$$G_2^* = G_2 - 2iAG_1, \tag{4.24}$$

derived from the definition (4.18) and the reality of G_1°, G_2°, and A, to rearrange (4.23) into

$$\Omega^* = (\eta^{1*} - 2iA\eta^{2*})G_1 + \eta^{2*}G_2 - i\eta^{2*}\eta^{1*}\{G_1, A\}\mathcal{P}_1^*. \tag{4.25}$$

Requiring $\Omega^* = \Omega$, we find the reality properties of the ghosts η^1 and η^2 from the first two terms,

$$\begin{aligned} \eta^{2*} &= \eta^2 \\ \eta^{1*} - 2iA\eta^{2*} &= \eta^1 \qquad \text{or} \qquad \eta^{1*} = \eta^1 + 2iA\eta^2. \end{aligned} \tag{4.26}$$

A straightforward calculation yields the transformation of the original ghosts which is consistent with these reality properties,

$$\begin{aligned} \eta^1_\circ \quad &\rightarrow \quad \eta^1 = \eta^1_\circ - iA\eta^2_\circ, \\ \eta^2_\circ \quad &\rightarrow \quad \eta^2 = \eta^2_\circ, \end{aligned} \tag{4.27}$$

and requiring that the fundamental Poisson brackets between the ghosts be preserved ($\{\mathcal{P}_a, \eta^b\} = -\delta_a^b$) gives the corresponding transformation of the ghost momenta,

$$\begin{aligned} \mathcal{P}_1^\circ \quad &\rightarrow \quad \mathcal{P}_1 = \mathcal{P}_1^\circ, \\ \mathcal{P}_2^\circ \quad &\rightarrow \quad \mathcal{P}^2 = \mathcal{P}_2^\circ + iA\mathcal{P}_1^\circ. \end{aligned} \tag{4.28}$$

In particular, we observe that $\mathcal{P}_1$ remains unchanged in the deformed theory and is therefore pure imaginary. The consequence of this is that the last term in Eq. (4.22) is intrinsically complex because η^1 has nonzero real and imaginary parts while all the factors multiplying it are pure real or pure imaginary. We now draw our *first important conclusion*:

> The addition of an imaginary piece to a set of real constraints introduces an imaginary piece to the BRST charge. Therefore, the standard BRST treatment of a complexified theory will not work in quantum theory and another approach is required.

4.3.2 Inclusion of complex conjugate constraints

To eliminate the imaginary piece of the BRST charge, we can contemplate two approaches. The first is to transform the complex constraints into purely real constraints. This, however, simply returns us to the initial constraints G_1° and G_2° and we have assumed that there is some reason that this is undesirable. In the case of self-dual gravity, this reason is that the real constraints are not polynomial in the phase space variables, while the complex constraints are polynomial. So we reject this first approach.

The second approach to making the BRST charge real is to add the complex conjugates of the imaginary constraints to the set of constraints in the hope that the imaginary terms added to the BRST charge will then appear in complex conjugate pairs, making the BRST charge manifestly real. This procedure, however, introduces an additional complication. The complex conjugate constraints that we add are not independent of the original constraints and we therefore end up with a reducible set of constraints. This is not a serious problem, since we know how to handle reducible constraints (see Sec. 2.2.4).

To see how this approach works, we continue with the example of the previous section and add the constraint $G_{\overline{2}}$, complex conjugate of G_2, to the constraints G_1 and G_2,

$$\begin{aligned} G_1 &= G_1^{\circ}, \\ G_2 &= G_2^{\circ} + iA(q,p)G_1^{\circ}, \\ G_{\overline{2}} &= G_2^{\circ} - iA(q,p)G_1^{\circ}. \end{aligned} \tag{4.29}$$

$A = A(q,p)$ is again assumed to be a real function on the phase space. However, to avoid the unnecessary complication of second-order structure functions, we assume that the Poisson brackets of A with the original constraints are constant,

$$\{G_1^{\circ}, A\} := \Gamma_1 = \text{constant}, \qquad \{G_2^{\circ}, A\} := \Gamma_2 = \text{constant}. \tag{4.30}$$

The Poisson brackets of A with the modified constraints are then

$$\begin{aligned} \{G_1, A\} &= \Gamma_1, \\ \{G_2, A\} &= \Gamma_2 + iA\Gamma_1, \\ \{G_{\overline{2}}, A\} &= \Gamma_2 - iA\Gamma_1, \end{aligned} \tag{4.31}$$

and the nonconstant Poisson brackets among the constraints are

$$\begin{aligned}\{G_1, G_2\} &= i\Gamma_1 G_1,\\ \{G_1, G_{\overline{2}}\} &= -i\Gamma_1 G_1,\\ \{G_2, G_{\overline{2}}\} &= -2i\Gamma_2 G_1.\end{aligned} \tag{4.32}$$

The nonzero first-order structure functions can be read off directly from equations (4.32),

$$\begin{aligned}C_{12}{}^1 &= i\Gamma_1,\\ C_{1\overline{2}}{}^1 &= -i\Gamma_1,\\ C_{2\overline{2}}{}^1 &= -2i\Gamma_2.\end{aligned} \tag{4.33}$$

Since the first-order structure functions are all constant, the second-order structure functions can be taken to vanish.

In addition to the constraint algebra, we now have the reducibility condition

$$Z := Z^a G_a = -2iAG_1 + G_2 - G_{\overline{2}} = 0, \tag{4.34}$$

with reducibility coefficients

$$Z^1 = -2iA, \qquad Z^2 = 1, \qquad Z^{\overline{2}} = -1. \tag{4.35}$$

The last step before constructing the BRST charge Ω is to extend the phase space with a ghost and its canonically conjugate momentum for each constraint and for the reducibility condition,

$$\begin{aligned}&\eta^1,\ \mathcal{P}_1 &&\text{(associated with } G_1\text{)},\\ &\eta^2,\ \mathcal{P}_2 &&\text{(associated with } G_2\text{)},\\ &\eta^{\overline{2}},\ \mathcal{P}_{\overline{2}} &&\text{(associated with } G_{\overline{2}}\text{)},\\ &\phi,\ \pi &&\text{(associated with } Z\text{)}.\end{aligned} \tag{4.36}$$

The ghosts η^i and their momenta $\mathcal{P}_i$ are anticommuting (fermionic) variables as before. The ghost of ghost ϕ and its conjugate momentum π have statistics opposite that of the ghosts and are therefore commuting (bosonic) variables. The Poisson bracket structure among the ghosts can be taken to be canonical,

$$\begin{aligned} \{\mathcal{P}_i, \eta^j\} = \{\eta^j, \mathcal{P}_i\} = -\delta_i^j, \\ \{\pi, \phi\} = -\{\phi, \pi\} = -1, \end{aligned} \tag{4.37}$$

with all other brackets among the ghosts vanishing. In addition, we assume the brackets of the original phase space variables are unchanged and that the brackets between the ghosts and the original phase space variables vanish.

We now have all of the building blocks for the BRST charge, which we now construct according to the rules detailed in Sec. 2.2.3,

$$\begin{aligned} \Omega =& \eta^1 G_1 + \eta^2 G_2 + \eta^{\overline{2}} G_{\overline{2}} - i\Gamma_1 \eta^2 \eta^1 \mathcal{P}_1 + i\Gamma_1 \eta^{\overline{2}} \eta^1 \mathcal{P}_1 + 2i\Gamma_2 \eta^{\overline{2}} \eta^2 \mathcal{P}_1 \\ & - 2iA\phi\mathcal{P}_1 + \phi\mathcal{P}_2 - \phi\mathcal{P}_{\overline{2}}. \end{aligned} \tag{4.38}$$

There could, in principle, be additional terms to the BRST charge arising from the nonconstant reducibility coefficient Z^1, but a straightforward (though somewhat tedious) calculation shows that the BRST charge (4.38) is nilpotent, $\{\Omega, \Omega\} = 0$, and that it is therefore the complete BRST charge.

We now consider the reality of the BRST charge (4.38). For the sum of the zero-order terms $\eta^i G_i$ to be real, it is sufficient that the ghost η^1 be taken to be real and that the ghosts η^2 and $\eta^{\overline{2}}$ be complex conjugates,

$$(\eta^2)^* = \eta^{\overline{2}}. \tag{4.39}$$

Indeed, there is no need to assume, as in the previous section, that η^2 and $\eta^{\overline{2}}$ are complex; and they can be taken to be real if one chooses, in which case $(\eta^2) = \eta^{\overline{2}}$, although this is not necessary. Complex conjugation of the fundamental Poisson bracket between $\mathcal{P}_1$ and η^1 and between $\mathcal{P}_2$ and η^2 then requires that $\mathcal{P}_1$ be pure imaginary (as in the standard BRST treatment) and that $i\mathcal{P}_2$ and $i\mathcal{P}_{\overline{2}}$ be complex conjugates, since

$$-1 = \{\mathcal{P}_2, \eta^2\}^* = -\{\eta^{2*}, \mathcal{P}_2^*\} = -\{\mathcal{P}_2^*, \eta^{\overline{2}}\}, \tag{4.40}$$

implies

$$(\mathcal{P}_2)^* = -\mathcal{P}_{\overline{2}}. \tag{4.41}$$

As with the ghosts η^i, there is no need to assume that the ghost momenta $\mathcal{P}_i$ are complex and they can be taken to be pure imaginary as in the standard BRST formalism, although again this is not necessary. Finally, there is no restriction on the reality of the ghost of ghost ϕ and it can be taken to be real. With these complex conjugation rules for the ghosts and their momenta, we can rewrite the BRST charge (4.38) in the form

$$\Omega = \Big(\tfrac{1}{2}\eta^1 G_1 + \eta^2 G_2 - i\Gamma_1 \eta^2 \eta^1 \mathcal{P}_1 + i\Gamma_2 \eta^{\overline{2}} \eta^2 \mathcal{P}_1 - iA\phi\mathcal{P}_1 + \phi\mathcal{P}_2\Big) + \text{c.c.}, \tag{4.42}$$

where c.c. stands for the complex conjugate of everything inside the parentheses. Thus, the BRST charge (4.38) contains terms which are either real or occur as sums of complex conjugate pairs and we have explicitly demonstrated that the BRST charge (4.38) is real.

It is clear that this procedure generalizes to an arbitrary complexification of a set of real constraints into a set of complex constraints of the form

$$G_i^\circ \quad \rightarrow \quad G_i = G_i^\circ + iA^{ij} G_j^\circ, \tag{4.43}$$

and we state our *second important conclusion*:

A real BRST charge for a system with an arbitrary set of complex first-class constraints which are also first-class with their complex conjugates can be constructed by adding to the complex constraints their complex conjugates and treating the extended system of constraints as a standard reducible set of constraints.

Chapter 5

Self-dual Gravity and Ashtekar Variables

This chapter is a review of the Ashtekar formulation of general relativity in terms of self-dual variables. We begin by reviewing the traditional Hamiltonian formulation of general relativity. The Ashtekar variables are then introduced via a canonical transformation of the phase space. The new phase space variables are a self-dual connection $A_{aA}{}^{B}$ and an SU(2) (densitized) "soldering" form $\tilde{\sigma}^{a}{}_{A}{}^{B}$.[1] In the last section of this chapter we review the algebra of constraints in the Ashtekar variables. This review lays the foundation for the new results on the classical BRST structure of the Ashtekar theory that are presented in Chap. 6.

[1] Throughout, we use the Penrose [34,35] abstract index notation in which the tensor indices are used to indicate the tensor structure of a tensor object and are not to be thought of as components in a particular basis.

5.1 Hamiltonian formulation of Einstein's theory

A Hamiltonian formulation of a physical theory involves dynamics on a phase space. Time is therefore treated in a fundamentally different way from either the spatial degrees of freedom or the dynamical variables defined on the spatial manifold. A generally relativistic theory, on the other hand, puts time on the same level as the spatial degrees of freedom. So the first step in constructing a Hamiltonian formulation of general relativity is to perform a "3 + 1" split of spacetime into a three-dimensional space, which has the structure of a Riemannian manifold, and time, which is treated as a real parameter. Dynamics of the three-dimensional gravitational field then involves the statement of initial conditions and the equations of time evolution of the gravitational field. This is the well-known [8,9] Cauchy initial value problem for general relativity, and is reviewed in Sec. 5.1.1. This forms the basis for the Hamiltonian phase space formulation which is reviewed in Sec. 5.1.2.

5.1.1 The initial value formulation

Spacetime is assumed to be a four-dimensional pseudo-Riemannian manifold M, with metric g_{ab}. The covariant derivative on M compatible with g_{ab} is ${}^4\nabla_a$,

$$ {}^4\nabla_a g_{bc} = 0. \tag{5.1} $$

In terms of ${}^4\nabla_a$, the curvature tensor is defined as

$$ 2\, {}^4\nabla_{[a}\, {}^4\nabla_{b]} k_c = {}^4R_{abc}{}^d k_d \tag{5.2} $$

for all 4-vector fields k_d.

We assume that the four-dimensional spacetime manifold (M, g_{ab}) with signature (-+++) has the structure $\Sigma \times \mathbb{R}$, so that we can consider the foliation of spacetime into a set of Cauchy surfaces Σ_t, parameterized by a global time function t. Let n_a be a unit normal to the spatial manifold Σ. The 3-metric q_{ab} on Σ is then related to the 4-metric by

$$q_{ab} = g_{ab} + n_a n_b. \tag{5.3}$$

The "flow of time" is defined by a vector field t_a on M satisfying the condition $t^a \, {}^4\nabla_a \, t = 1$. In general, this "flow of time" is not in the direction of the time-like normal n_a, and it is convenient to decompose t_a into components perpendicular and parallel to Σ,

$$t^a = N n^a + N^a. \tag{5.4}$$

Thus, the "flow of time" t^a in the four-dimensional manifold involves a spatial shift given by a *shift function* N^a as well as a time-like shift in the direction n^a whose magnitude is given by the *lapse function* N.

The 3-metric q_{ab} is taken to be the configuration variable in the initial value formulation. The notion of time derivative is given by the extrinsic curvature

$$\begin{aligned} K_{ab} &= q_a{}^m q_b{}^n \, {}^4\nabla_m n_n \\ &= \tfrac{1}{2} \mathcal{L}_{\overrightarrow{n}} q_{ab}, \end{aligned} \tag{5.5}$$

where $\mathcal{L}_{\overrightarrow{n}}$ is the Lie derivative in the direction of $\overrightarrow{n}$. This gives the change in the metric along the time-like direction n_a perpendicular to Σ. The data $(\Sigma,\ q_{ab},\ K_{ab})$ are sufficient to completely specify the initial conditions of the 3-manifold, which is reasonable considering that Einstein's equation is a second order differential equation on the metric.

To construct the dynamics of q_{ab} and K_{ab}, we introduce the covariant differentiation operator ∇_a on Σ compatible with q_{ab},

$$\nabla_a q_{bc} = 0, \tag{5.6}$$

and the Riemann tensor of ∇_a, defined by

$$2\nabla_{[a}\nabla_{b]}k_c = R_{abc}{}^d k_d, \tag{5.7}$$

for all space-like k_c ($k_c n^c = 0$). $R_{abc}{}^d$ is related to the Riemann tensor ${}^4R_{abc}{}^d$ of the 4-metric g_{ab} by

$$R_{abcd} = q_a^m q_b^n q_c^r q_d^s \, {}^4R_{mnrs} - 2K_{c[a}K_{b]d}. \tag{5.8}$$

Contracting (5.8) twice leads to a scalar constraint equation

$$2G_{ab}n^a n^b \equiv R + K^2 - K^{ab}K_{ab}, \tag{5.9}$$

where $K := K_a{}^a$, $R := R_a{}^a$ is the scalar 3-curvature, and where we have used the definition of the Einstein tensor,

$$G_{ab} = {}^4R_{ab} - \tfrac{1}{2}\,{}^4R g_{ab}. \tag{5.10}$$

Similarly, contracting (5.2) on a and c and using the 3-metric to project b into Σ leads to a vector constraint equation

$$G_{ab}n^a q^b{}_m \equiv \nabla^a(K_{am} - Kq_{am}). \tag{5.11}$$

Assuming that the 4-metric g_{ab} satisfies the vacuum Einstein's equation,

$$G_{ab} = 0, \tag{5.12}$$

these constraint equations become

$$C(q, \dot{q}) := R + K^2 - K^{ab}K_{ab} \approx 0, \tag{5.13}$$

$$C_m(q, \dot{q}) := \nabla^a(K_{am} - Kq_{am}) \approx 0. \tag{5.14}$$

Equations (5.13) and (5.14) are the four *constraint equations* of the initial value formulation of the vacuum Einstein's theory. They are constraints on the initial value data that can be specified.

Given a timelike vector field t_a (and its corresponding flow parameter t), the evolution of q_{ab} and K_{ab} is given by the Lie derivatives of these fields with respect to t_a,

$$\dot{q}_{ab} = 2NK_{ab} + \mathcal{L}_{\vec{N}} q_{ab}, \tag{5.15}$$

$$\dot{K}_{ab} = -NR_{ab} + 2NK_a{}^m K_{bm} - NKK_{ab} + \nabla_a \nabla_b N + \mathcal{L}_{\vec{N}} K_{ab}. \tag{5.16}$$

Equation (5.15) displays that the total time derivative of q_{ab} involves a component in the unique timelike direction n_a involving the extrinsic curvature K_{ab} and a horizontal component in the direction of the shift function N_a. It is important to note that Eqs. (5.15) and (5.16) involve only the three-dimensional fields. These equations, together with the constraint equations (5.13) and (5.14), are fully equivalent to the vacuum Einstein equation on the 4-metric.

5.1.2 The Hamiltonian Formulation

To construct the Hamiltonian formulation of Einstein's theory of gravity, we want to replace the velocity $\dot{q}_{ab}$ with the momentum $\widetilde{p}^{ab}$ canonically conjugate to q_{ab}. To do this we use the standard procedure and set $\widetilde{p}^{ab} = \frac{\delta L}{\delta \dot{q}_{ab}}$, where L

is an appropriately chosen Lagrangian. We start with the familiar Einstein-Hilbert action S,

$$S = \int d^4x \sqrt{-g}\, {}^4R, \tag{5.17}$$

where g is the determinant of the spacetime metric $g_{\mu\nu}$ and ${}^4R$ is the scalar curvature of the four-dimensional spacetime. This leads to the obvious choice of Lagrangian,

$$L = \int d^3x \sqrt{-g}\, {}^4R. \tag{5.18}$$

The next step is to express this Lagrangian in terms of (q_{ab}, N, N^a). The scalar curvature, ${}^4R$, can be rewritten as

$${}^4R = 2(G_{ab}n^a n^b - {}^4R_{ab}n^a n^b). \tag{5.19}$$

The first term of Eq. (5.19) is given in terms of the 3-curvature R and the exterior curvature K_{ab} by Eq. (5.13),

$$2G_{ab}n^a n^b = R + K^2 - K^{ab}K_{ab}. \tag{5.20}$$

The second term of Eq. (5.19) can be rewritten as

$$\begin{aligned} {}^4R_{ab}n^a n^b &= -\, n^a({}^4\nabla_a {}^4\nabla_m - {}^4\nabla_m {}^4\nabla_a)n^m \\ &= ({}^4\nabla_a n^a)({}^4\nabla_m n^m) - ({}^4\nabla_m n^a)({}^4\nabla_a n^m) \\ &\qquad + {}^4\nabla_m(n^a\, {}^4\nabla_a n^m) - {}^4\nabla_a(n^a\, {}^4\nabla_m n^m) \\ &= K^2 - K_m{}^a K_a{}^m + \text{total divergence terms}. \end{aligned} \tag{5.21}$$

The total divergence terms contain second time derivatives of q_{ab} which unnecessarily complicate the Hamiltonian formulation, so we assume the boundary conditions are such that the total divergence terms can be neglected. Then, up to total divergence terms, ${}^4R$ is given by

$${}^4R \simeq R + K^{ab}K_{ab} - K^2. \tag{5.22}$$

This is the Lagrangian density that we choose. The Lagrangian is then given by

$$L(q,\dot{q}) \simeq \int_\Sigma d^3x\,\sqrt{q}\,N(R+K^{ab}K_{ab}-K^2), \tag{5.23}$$

where we have used the fact that the 4-metric and the 3-metric are related by $\sqrt{g}=N\sqrt{q}$. The variation of Eq. (5.23) with respect to $\dot{q}_{ab}$ gives the conjugate momentum

$$\widetilde{p}^{ab} = \frac{\delta L}{\delta \dot{q}_{ab}} = \sqrt{q}(K^{ab}-Kq^{ab}), \tag{5.24}$$

where the tilde indicates that p is weighted by a density factor $\sqrt{q}$, *i.e.*, $\widetilde{p}^{ab} = \sqrt{q}p^{ab}$.

We now perform a Legendre transformation to construct the Hamiltonian function. Following the standard procedure, we define the Hamiltonian by

$$H(q,\widetilde{p}) = \int_\Sigma d^3x\,(\widetilde{p}^{ab}\dot{q}_{ab}) - L. \tag{5.25}$$

We need to express the right hand side of Eq. (5.25) in terms of the canonical variables, q_{ab} and $\widetilde{p}^{ab}$. We use Eq. (5.15) to express $\dot{q}_{ab}$ in terms of K_{ab}, and then Eq. (5.24) to express K_{ab} in terms of $\widetilde{p}^{ab}$. This gives

$$\widetilde{p}^{ab}\dot{q}_{ab} = 2N\sqrt{q}(\widetilde{p}^{ab}\widetilde{p}_{ab} - \tfrac{1}{2}\widetilde{p}^2) + 2\nabla_a(\widetilde{p}^{ab}N_b) - 2N_b\nabla_a\widetilde{p}^{ab}. \tag{5.26}$$

and

$$\begin{aligned} L &= \int_\Sigma d^3x\,\sqrt{q}N(R+K^{ab}K_{ab}-K^2) \\ &= \int_\Sigma d^3x\,N\left\{q^{1/2}R + q^{-1/2}(\widetilde{p}^{ab}\widetilde{p}_{ab} - \tfrac{1}{2}\widetilde{p}^2)\right\}. \end{aligned} \tag{5.27}$$

Substituting Eqs. (5.26) and (5.27) into Eq. (5.25) gives the Hamiltonian in terms of the canonical variables,

$$H(q,\widetilde{p}) \simeq \int_\Sigma d^3x\,N\left\{-q^{1/2}R + q^{-1/2}(\widetilde{p}^{ab}\widetilde{p}_{ab} - \tfrac{1}{2}\widetilde{p}^2)\right\} + \int_\Sigma d^3x\,N_b(-2\nabla_a\widetilde{p}^{ab}), \tag{5.28}$$

where $\simeq$ is a reminder that boundary terms have been neglected.

To complete the Hamiltonian formulation, we need to reexpress the constraint equations (5.13) and (5.14) in terms of the canonical variables. In the process, it will become clear that the integrand of Eq. (5.28) is just a combination of the constraints. Using Eq. (5.24) to substitute for K^{ab} in terms of $\widetilde{p}^{ab}$, the vector constraint (5.14) becomes

$$C_a(q,\widetilde{p}) := -2q_{am}\nabla_n\widetilde{p}^{mn} \approx 0. \tag{5.29}$$

Similarly, the scalar constraint 5.13 becomes

$$C(q,\widetilde{p}) := q^{1/2}R + q^{-1/2}(\widetilde{p}^{ab}\widetilde{p}_{ab} - \tfrac{1}{2}\widetilde{p}^2) \approx 0. \tag{5.30}$$

These constraints have physical interpretations. The vector constraint (5.29) is the generator of canonical transformations in $(q_{ab}, \widetilde{p}^{ab})$ resulting from spatial diffeomorphisms on Σ, while the scalar constraint (5.30) is the generator of "time-evolution" of the initial data $(q_{ab}, \widetilde{p}^{ab})$. The Hamiltonian (5.28) can be rewritten as

$$H(q,\widetilde{p}) \simeq \int_\Sigma d^3x\ [NC(q,\widetilde{p}) + N^a C_a(q,\widetilde{p})]\,. \tag{5.31}$$

Thus, modulo surface terms, the Hamiltonian is just a linear combination of constraints. This is a general characteristic of a relativistic theory with reparameterization invariance.

5.1.3 Constraint algebra in Einstein's theory

For later comparison with the Ashtekar formulation of general relativity, we now discuss the Poisson bracket algebra of constraints in the traditional variables. For the sake of brevity, only the results are included here. Details can be found in Ref. [9]. We note in particular that the constraints C_a and C are, respectively, covector densities and scalar densities on the 3-manifold Σ. Since Poisson brackets are defined only between *real-valued functions* on the phase space, it is first necessary to construct scalar-valued constraints from C_a and C by integrating them against the shift and lapse functions,

$$\begin{aligned} C_{\overrightarrow{N}}(q,\widetilde{p}) &:= \int_\Sigma d^3x\, N^a C_a(q,\widetilde{p}) \\ C_N(q,\widetilde{p}) &:= \int_\Sigma d^3x\, NC(q,\widetilde{p}). \end{aligned} \tag{5.32}$$

The Poisson bracket algebra of these real valued constraints is

$$\begin{aligned} \{C_{\overrightarrow{N}}, C_{\overrightarrow{M}}\} &= -C_{\overrightarrow{K}} \quad \text{with } K^a = (\mathcal{L}_{\overrightarrow{N}}\overrightarrow{M})^a \\ \{C_{\overrightarrow{N}}, C_M\} &= -C_K \quad \text{with } K = (\mathcal{L}_{\overrightarrow{N}}M) \\ \{C_N, C_M\} &= -C_{\overrightarrow{L}} \quad \text{with } L^a = q^{ab}(N\nabla_b M - M\nabla_b N). \end{aligned} \tag{5.33}$$

The Poisson brackets between all of the constraints are thus linear combinations of the constraints themselves, and the constraints are therefore first class.

5.2 Spinor extended phase space

The self-dual formulation of gravity requires the extension of the traditional phase space to incorporate spinor fields. In vacuum gravity, the only case we consider, spinors are not strictly necessary since the spinor formulation and the more familiar tetrad formulation are equivalent. The spinor formulation is necessary, however, if one wants to couple Dirac fields to gravity. Furthermore, in self-dual gravity, one of the canonical variables is a self-dual connection; and the spinorial notation is especially well-suited for calculations involving self-dual fields. Thus before considering the Ashtekar theory of self-dual gravity in Sec. 5.3, we review in this section the spinor extension of canonical general relativity.

5.2.1 $SU(2)$ spinors

As in traditional canonical gravity, we consider the phase space on a three-dimensional manifold Σ. Since Σ is a manifold, tensor fields $T^{a...b}{}_{c...d}$ are well defined. We consider, in addition, objects such as $T^{a...b}{}_{c...d}{}^{A...B}{}_{C...D}$ with internal $SU(2)$ indices $A\ldots B, C\ldots D$. We can regard $T^{a...b}{}_{c...d}{}^{A...B}{}_{C...D}$ as a *generalized tensor* in the sense of Ref. [36], with the Penrose abstract index notation extended to include the internal spinor indices. Objects such as $\lambda^{A...B}{}_{C...D}$ with only internal indices can be thought of as $SU(2)$ "Higgs scalars."

$SU(2)$ spinors can loosely be thought of as "square roots" of 3-vectors in the sense that the space of bispinors $\lambda_A{}^B$ is isomorphic to the space of 3-vectors λ^a. At each point of the 3-manifold Σ, the isomorphism between the 3-real dimensional tangent space and the 3-real dimensional vector space of 2×2 trace-free Hermitian matrices is given by

$$\lambda^a = \sigma^a{}_A{}^B \lambda_B{}^A. \tag{5.34}$$

The $\sigma^a{}_A{}^B$ are thus said to "solder" the spinor space to the tangent space, and are therefore called *soldering forms*. An explicit representation can be given in terms of the Pauli matrices $\tau^i{}_A{}^B$, satisfying the relation

$$\tau^i{}_A{}^B \tau^j{}_B{}^D \equiv (\tau^i \tau^j)_A{}^D = i\epsilon^{ijk} \tau_{kA}{}^D + \delta^{ij} \delta_A{}^D. \tag{5.35}$$

We can express $\sigma^a{}_A{}^B$ in terms of the Pauli matrices and a real triad E_i^a by

$$\sigma^a{}_A{}^B = -\frac{i}{\sqrt{2}} E_i^a \tau^i{}_A{}^B. \tag{5.36}$$

From (5.35) we then get the important result

$$-\operatorname{tr}(\sigma^a \sigma^b) \equiv -\sigma^a{}_A{}^B \sigma^b{}_B{}^A = E_i^a E^{bi} = q^{ab} \tag{5.37}$$

In other words, the soldering form $\sigma^a{}_A{}^B$ can loosely be thought of as the "square root" of the 3-metric q^{ab}. It is this relationship between the soldering form and the 3-metric which allows the reformulation of metric gravity into a spinor formulation.

The $SU(2)$ indices are lowered by the antisymmetric 2-form ϵ_{AB} and raised by its inverse ϵ^{AB} according to the following conventions,

$$\lambda^A = \epsilon^{AB} \lambda_B, \qquad \lambda_B = \lambda^A \epsilon_{AB}, \tag{5.38}$$

where care must be taken with the order of the indices because of the antisymmetry of ϵ_{AB}.

Tensor analysis is extended to generalized tensors by the introduction of a connection D_a that acts on both tensor and spinor indices. Its action on tensor objects is the same as that of ∇_a; in particular, it is compatible with the metric. In addition, it is compatible with σ^a. Thus we have

$$D_a q_{bc} = 0, \qquad D_a \sigma^b{}_A{}^B = 0. \tag{5.39}$$

It is convenient to introduce the spinor connection 1-form $\Gamma_{aA}{}^B$ by the relation

$$\begin{aligned} D_a \lambda_{bA} &= {}^3\nabla_a \lambda_{bA} + \Gamma_{aA}{}^B \lambda_{bB} \\ &= \partial_a \lambda_{bA} + \Gamma_{ab}{}^c \lambda_{cA} + \Gamma_{aA}{}^B \lambda_{bB} \end{aligned} \tag{5.40}$$

The curvature constructed from D_a contains both a tensor part $R_{abm}{}^n$ and a spinor part $F_{abM}{}^N$, given by

$$2D_{[a}D_{b]}\lambda_{mM} = R_{abm}{}^n\lambda_{nM} + F_{abM}{}^N\lambda_{mN}. \tag{5.41}$$

We are now ready to consider the spinor extended phase space.

5.2.2 Extended phase space

We now show how it is possible to consider $\sigma^a{}_A{}^B$ as the basic dynamical variable and q_{ab} as a derived object. The momentum conjugate to $\sigma^a{}_A{}^B$ is a density of weight one, $\widetilde{M}_{aA}{}^B := \sqrt{q}M_{aA}$. The extended phase space consists of the points $(\sigma^a{}_A{}^B,\ \widetilde{M}_{aA}{}^B)$. The Poisson brackets among the fundamental variables are

$$\begin{gathered}\{\sigma^a{}_A{}^B,\ \sigma^b{}_C{}^D\} = 0, \qquad \{\widetilde{M}_{aA}{}^B,\ \widetilde{M}_{bC}{}^D\} = 0,\\ \{\sigma^a{}_A{}^B(x),\ \widetilde{M}_{bC}{}^D(y)\} = \delta^b_a\delta^C_{(A}\delta^D_{B)}\delta^3(x,y).\end{gathered} \tag{5.42}$$

The Poisson brackets between any two observables f and g are, consequently,

$$\{f,\ g\} = \int \mathrm{tr}\left\{\frac{\delta f}{\delta\sigma^a}\frac{\delta g}{\delta\widetilde{M}_a} - \frac{\delta f}{\delta\widetilde{M}_a}\frac{\delta g}{\delta\sigma^a}\right\}. \tag{5.43}$$

The dynamical variable $\sigma^a{}_A{}^B$ has $3 \times 3 = 9$ real degrees of freedom. As a result, in the transition from metric gravity, which has six real degrees of freedom, to the spinorial representation of gravity, we have added three degrees of freedom to the configuration variables. Since the physical degrees of freedom have not changed (we are still considering vacuum Einstein gravity) we have three new constraints. These constraints are associated with the $SU(2)$ rotations, which are "pure gauge." This exactly parallels the situation in Yang-Mills theory. The new constraints are

$$C_{ab}(\sigma, \widetilde{M}) := -\,\mathrm{tr}(\widetilde{M}_{[a}\sigma_{b]}) \equiv \widetilde{M}_{[ab]} \approx 0, \tag{5.44}$$

or

$$C^{AB}(\sigma, \widetilde{M}) := \sigma^a{}_C{}^{(A}\widetilde{M}_{aD}{}^{B)}\epsilon^{CD} \approx 0. \tag{5.45}$$

Indeed, the constraints (5.45) are the generators of infinitesimal rotations in the $SU(2)$ space; *i.e.*, they generate small $SU(2)$ gauge transformations on the dynamical variables. It follows, in particular, that these three constraints form a first-class set.

Just as the dynamical variable $\sigma^a{}_A{}^B$ is related to the 3-metric q^{ab} by Eq. (5.37), the canonical momentum $\widetilde{M}_{aA}{}^b$ of the extended phase space is related to the canonical momentum $\widetilde{p}^{ab}$ and, indeed, we can set

$$\widetilde{p}^{ab} := \widetilde{M}^{(ab)}, \tag{5.46}$$

where the parentheses indicate symmetrization on the indices a and b, so that the constraint surface is characterized by the equation $\widetilde{M}^{ab} = \widetilde{p}^{ab}$.

The constraints of the standard Hamiltonian theory, Eqs. (5.29) and (5.30), can be "lifted" to the extended phase space since they are written in terms of q_{ab} and $\widetilde{p}^{ab}$ which themselves have been lifted by Eqs. (5.37) and (5.46). Thus, the "old" constraints become

$$C_a(\sigma, \widetilde{M}) := -2q_{am}D_n\widetilde{p}^{mn} \approx 0. \tag{5.47}$$

and

$$C(\sigma, \widetilde{M}) := -q^{1/2}R + q^{-1/2}(\widetilde{p}^{ab}\widetilde{p}_{ab} - \tfrac{1}{2}\widetilde{p}^2) \approx 0, \tag{5.48}$$

where q_{ab} and $\widetilde{p}^{ab}$ are now regarded as secondary variables, defined in terms of the dynamical variables $\sigma^a{}_A{}^B$ and $\widetilde{M}_{aA}{}^B$ by Eqs. (5.37) and (5.46) respectively.

The canonical transformations generated by the constraints (5.47) and (5.48) continue to have the same interpretations, namely the vector constraint (5.47) generates spatial diffeomorphisms and the scalar constraint (5.48) generates time-evolution. In all, we now have nine real degrees of freedom and $3+3+1 = 7$ constraints. We thus have two physical degrees of freedom per space-time point, in agreement with the standard Hamiltonian formulation of vacuum general relativity.

5.2.3 The Sen connection and self-duality

The covariant derivative operator D_a, introduced in Sec. 5.2.1, is the natural extension of ∇_a to spinor fields *on the 3-manifold* Σ. However, to formulate the dynamics of the 3-manifold in terms of spinors, it is not sufficient to know the geometry of the 3-manifold; it is also necessary to know something about how the 3-manifold is embedded in the 4-dimensional spacetime manifold M. In 1981, Sen [37] introduced a covariant derivative operator $\underline{\mathcal{D}}_a$ into the initial value formulation of Einstein's theory. The action of $\underline{\mathcal{D}}_a$ on an arbitrary spinor α_A is given by

$$\underline{\mathcal{D}}_a \alpha_A = D_a \alpha_A + \frac{i}{\sqrt{2}} K_{aA}{}^B \alpha_B, \tag{5.49}$$

in which the embedding of Σ in M is encoded in the extrinsic curvature $K_{aA}{}^B = K_{ab}\sigma^b{}_A{}^B$. We will refer to $\underline{\mathcal{D}}$ as the *Sen connection.* The Sen connection arises naturally from the extension ${}^4D_a$ of the covariant derivative operator ${}^4\nabla_a$ to $SL(2,\mathbb{C})$ spinor fields on the 4-manifold M. The Sen connection is the pullback of ${}^4D_a$ to Σ,

$$\underline{\mathcal{D}}_a \alpha_A := q_a{}^b \, {}^4D_b \alpha_A, \tag{5.50}$$

and is an $SU(2)$ derivative operator.

Since $\underline{\mathcal{D}}_a$ is the restriction of ${}^4D_a$ to Σ, the curvature $F_{abA}{}^B$ constructed from $\underline{\mathcal{D}}_a$ is related to the spinorial curvature ${}^4R_{abA}{}^B$ constructed from ${}^4D_a$ by

$$F_{abA}{}^B = q_a{}^m q_b{}^n\, {}^4R_{mnA}{}^B. \tag{5.51}$$

This equation has important consequences. It can be shown [9] that the spinorial curvature ${}^4R_{abA}{}^B$ is, in fact, equal to the self-dual part ${}^{+4}R_{abc}{}^d$ of the Riemann curvature, expressed in terms of $SL(2,\mathbb{C})$ spinors $\sigma_a{}^{AA'}$,

$${}^4R_{ab}{}^A{}_M \sigma_c{}^{MA'} \sigma^d{}_{AA'} = {}^{+4}R_{abc}{}^d := \tfrac{1}{2}({}^4R_{abc}{}^d - i\, {}^{*4}R_{abc}{}^d), \tag{5.52}$$

where ${}^{*4}R_{abc}{}^d$ is the dual of the Riemann tensor, defined by

$${}^{*4}R_{abc}{}^d := \tfrac{1}{2}\epsilon_c{}^{dm}{}_n\, {}^4R_{abm}{}^n. \tag{5.53}$$

Since *any* self-dual 2-form at a point of M is completely determined by its pull-back to a spacelike, 3-dimensional subspace of the tangent space at that point, the 3-curvature $F_{abA}{}^B$ contains the same information as the 4-curvature ${}^4R_{abA}{}^B$. The importance of the Sen connection and of the self-dual formulation of general relativity is thus that it projects down to the 3-manifold Σ the full information of 4-dimensional general relativity. We note that the choice of the self-dual part of the curvature was arbitrary. We could just as well have chosen the anti-self-dual part since the self-dual and anti-self-dual curvatures contain essentially the same information. The connection in the anti-self-dual case is constructed by replacing i in the Sen connection (5.49) by $-i$.

An important consequence of the Sen connection $\underline{\mathcal{D}}_a$ is that the curvature $F_{abA}{}^B$ constructed from $\underline{\mathcal{D}}_a$ leads to an especially simple form for the constraint equations. It has been shown [9] that

$$\begin{aligned} \mathrm{tr}(\sigma^b F_{ab}) &\equiv \sigma^b{}_A{}^B F_{abB}{}^A = \frac{i}{\sqrt{2}} q_a{}^b G_{bc} n^c \\ \mathrm{tr}(\sigma^a \sigma^b F_{ab}) &\equiv \sigma^a{}_A{}^B \sigma^b{}_B{}^C F_{abC}{}^A = G_{bc} n^b n^c, \end{aligned} \tag{5.54}$$

where G_{ab} is the Einstein tensor of the 4-metric g_{ab}. Thus, the constraints of the vacuum Einstein theory can be written in the simple form

$$\mathrm{tr}(\sigma^b F_{ab}) = 0, \qquad \mathrm{tr}(\sigma^a \sigma^b F_{ab}) = 0. \tag{5.55}$$

5.3 Self-dual gravity

The simplification of the constraint equations of Einstein's theory when they are expressed in terms of $\sigma^a{}_A{}^B$ and the Sen connection $\underline{\mathcal{D}}_a$ suggests that the phase space description of the theory may be simple in terms of these variables. However, since $\underline{\mathcal{D}}_a$ depends on *both* q_{ab} and p^{ab} (since the extrinsic curvature K^{ab} depends on p^{ab}), the transformation from the traditional canonical variables (q_{ab}, p^{ab}) to new variables $(\sigma^a{}_A{}^B, \underline{\mathcal{D}}_a)$ is non-trivial; it mixes configuration and momentum variables in a way which does not preserve the fundamental Poisson brackets. This situation was resolved by Ashtekar [11,12] in 1986 with the introduction of a new connection modeled on the Sen connection, but which leads to a simple canonical structure. The Ashtekar formulation of general relativity is commonly referred to as Ashtekar gravity or self-dual gravity, but has the same information content as the original Einstein theory.

5.3.1 Ashtekar variables

Using the Sen connection as motivation, Ashtekar introduced two new connections ${}^{\pm}\mathcal{D}_a$,

$$ {}^{\pm}\mathcal{D}_a \lambda_{bM} := D_a \lambda_{bM} \pm \frac{i}{\sqrt{2}} \Pi_{aM}{}^{N} \lambda_{bN}, \tag{5.56} $$

where $\Pi_{aM}{}^{N}$ is given by

$$ \Pi_{aM}{}^{N} := q^{-1/2} \left(\widetilde{M}_{aM}^{N} + \tfrac{1}{2}(\mathrm{tr}\, \widetilde{M}_b \sigma^b) \sigma_{aM}{}^{N} \right). \tag{5.57} $$

We recall that $\widetilde{M}_{aM}{}^{N}$ is the momentum canonically conjugate to $\sigma^a{}_M{}^N$, and note that $\Pi_{aM}{}^{N}$ is related to $M_{aM}{}^{N}$ in the same way that the extrinsic curvature K^{ab} is related to p^{ab}, see Eq. (5.24). In fact, when the constraint Eqs. (5.44) are satisfied, Π^{ab} reduces to K^{ab}, *i.e.*,

$$ \Pi^{(ab)} = K^{ab}. \tag{5.58} $$

It also follows from Eq. (5.58) that on the constraint surface the Ashtekar connection (5.56) reduces to the Sen connection (5.49). Thus, the difference between the Ashtekar connection and the Sen connection is the inclusion of the antisymmetric part $\Pi^{[ab]}$ of Π^{ab} in the Ashtekar connection.

Expanding Eq. (5.56), showing all the connection 1-forms explicitly,

$$ {}^{\pm}\mathcal{D}_a \lambda_{bM} := \partial_a \lambda_{bM} + \Gamma_{ab}{}^{c} \lambda_{cM} + \Gamma_{aM}{}^{N} \lambda_{bN} \pm \frac{i}{\sqrt{2}} \Pi_{aM}{}^{N} \lambda_{bN}, \tag{5.59} $$

we notice that, since $\Pi_{aM}{}^{N}$ has the same index structure as $\Gamma_{aM}{}^{N}$, it is convenient to combine the last two terms into new spinorial connection 1-forms ${}^{\pm}A_{aM}{}^{N}$,

$$ {}^{\pm}A_{aM}{}^{N} = \Gamma_{aM}{}^{N} \pm \frac{i}{\sqrt{2}} \Pi_{aM}{}^{N}, \tag{5.60} $$

so that the action of ${}^{\pm}\mathcal{D}_a$ on an object λ_M with only a spinor index is given by

$$ {}^{\pm}\mathcal{D}_a \lambda_M = \partial_a \lambda_M + {}^{\pm}A_{aM}{}^N \lambda_N. \tag{5.61} $$

We note also that ${}^{\pm}\mathcal{D}_a$ and D_a have the same action on tensors.

The new variables introduced by Ashtekar are the connection 1-forms ${}^{\pm}A_{aA}{}^B$ and the *densitized* soldering form $\tilde{\sigma}^a{}_A{}^B := q^{1/2}\sigma^a{}_A{}^B$. It was necessary to "densitize" $\sigma^a{}_A{}^B$ since one of the canonically conjugate pair of variables has to be a density and ${}^{\pm}A_{aA}{}^B$ has density weight zero. Because of the parallel nature of the self-dual and the anti-self-dual formulations, either ${}^{+}A_{aA}{}^B$ or ${}^{-}A_{aA}{}^B$ can be taken as the variable canonically conjugate to $\tilde{\sigma}^a{}_A{}^B$. In recent papers on Ashtekar gravity, a specific choice is usually made, although at the classical level the choice is arbitrary and both choices can be found in the literature. In our discussion of the BRST structure of self-dual gravity, it will be convenient to use both ${}^{+}A_{aA}{}^B$ and ${}^{-}A_{aA}{}^B$, so we will continue to keep both in this section.

The Poisson-brackets relations between the new variables are, see Ref. [12] for details,

$$ \begin{gathered} \{\tilde{\sigma}^a{}_{AB}(x), \tilde{\sigma}^m{}_{MN}(y)\} = 0, \qquad \{\,{}^{\pm}A_a{}^{AB}(x),\, {}^{\pm}A_m{}^{MN}(y)\} = 0, \\ \{\,{}^{\pm}A_a{}^{AB}(x), \tilde{\sigma}^m{}_{MN}(y)\} = \pm\frac{i}{\sqrt{2}}\delta^m{}_a \delta_{(M}{}^A \delta_{N)}{}^B \delta(x,y), \end{gathered} \tag{5.62} $$

so the Poisson bracket structure turns out to be especially simple and ${}^{\pm}A_{aA}{}^B$ and $\tilde{\sigma}^a{}_A{}^B$ are, indeed, viable canonical variables.

Let us summarize the steps from the spinor variables of Sec. 5.2.2 to the Ashtekar variables. First, a straightforward canonical transformation $(\sigma^a{}_A{}^B, \widetilde{M}_{aA}{}^B) \to (\tilde{\sigma}^a{}_A{}^B, \Pi_{aA}{}^B)$ gives a density weight to $\sigma^a{}_A{}^B$ and removes an appropriate trace factor from $\widetilde{M}_{aA}{}^{B}$. In a second step, $(\tilde{\sigma}^{a}{}_{A}{}^{B}, \Pi_{aA}{}^{B}) \to (\tilde{\sigma}^{a}{}_{A}{}^{B}, {}^{\pm}A_{aA}{}^{B})$, the configuration and momentum variables are mixed, but in such a way that the Poisson bracket structure (5.62) remains canonical. The resulting pair of Ashtekar variables, $(\tilde{\sigma}^{a}{}_{A}{}^{B}, {}^{\pm}A_{aA}{}^{B})$, is thus arrived at by the composition of two canonical transformations.

5.3.2 Constraints in the Ashtekar theory

The power of the Ashtekar variables lies in the simplification that they bring to the constraints and the dynamical equations, which we now proceed to investigate. We begin with the new constraints (5.44) or (5.45). Using the definition of the Ashtekar connection (5.56) and Eq. (5.57), it is easy to show that

$$ {}^{\pm}\mathcal{D}_{a}\tilde{\sigma}^{a}{}_{A}{}^{B} = \pm\sqrt{2} i M_{[ab]}\sigma^{a}{}_{A}{}^{M}\sigma^{b}{}_{M}{}^{B}. \tag{5.63} $$

Hence, the constraint (5.44) is completely equivalent to

$$ {}^{\pm}\mathcal{D}_{a}\tilde{\sigma}^{a}{}_{A}{}^{B} = 0. \tag{5.64} $$

Equation (5.64) will be referred to as the *Gauss constraint*, since it requires the divergence of the field variable $\tilde{\sigma}^{a}{}_{A}{}^{B}$ to vanish and parallels the Gauss law of electromagnetism or Yang-Mills theory. Because the covariant derivative operator ${}^{\pm}\mathcal{D}_{a}$ contains connection coefficients $\Gamma_{ab}{}^{c}$ which act on tensor indices, Eq. (5.64) is not manifestly expressed in terms of the Ashtekar variables. However, using the definition of the covariant derivative of the densitizing factor $q^{1/2}$, see Ref. [38],

$$ D_{a}q^{1/2} := \partial_{a}q^{1/2} - \Gamma_{ab}{}^{b}q^{1/2}, \tag{5.65} $$

it follows that the divergence of a vector density of weight one, $\tilde{\lambda}^a$ is independent of the covariant derivative operator,

$$D_a\tilde{\lambda}^a = \partial_a\tilde{\lambda}^a. \tag{5.66}$$

As a result, Eq. (5.64) can be rewritten as

$$^{\pm}\mathcal{D}_a\tilde{\sigma}^a{}_A{}^B = \partial_a\tilde{\sigma}^a{}_A{}^B + [A_a, \tilde{\sigma}^a]_A^B = 0, \tag{5.67}$$

which involves only the canonical variables $(\tilde{\sigma}^a{}_A{}^B, A_{aA}{}^B)$.

To re-express the constraints (5.47) and (5.48), we first need to calculate the spinorial curvature $^{\pm}F_{abM}{}^N$ of $^{\pm}\mathcal{D}_a$ in terms of the curvature of q_{ab} and the extrinsic curvature K^{ab}. Using the definition (5.56) of $^{\pm}\mathcal{D}_a$ in terms of D_a and $\Pi_{aM}{}^N$ we find that

$$^{\pm}F_{abM}{}^N\lambda_N := 2\,^{\pm}\mathcal{D}_{[a}\,^{\pm}\mathcal{D}_{b]}\lambda_M = R_{abM}{}^N\lambda_N - \Pi_{[aM}{}^P\Pi_{b]P}{}^N\lambda_N + \sqrt{2}iD_{[a}\Pi_{b]M}{}^N\lambda_N, \tag{5.68}$$

where $R_{abM}{}^N$ is the spinorial curvature of D_a, and the antisymmetrization is only over the tensor indices (a, b). Using $\sigma^a{}_A{}^B$ (and its inverse) to convert spinor indices into tensor indices, we get

$$\begin{aligned} \mathrm{tr}(\sigma^a\,^{\pm}F_{ab}) &= \frac{1}{2\sqrt{2}}(\Pi_{am}\Pi_{bn} - \Pi_{bm}\Pi_{an})\epsilon^{mna} \mp \frac{i}{\sqrt{2}}D^a(\Pi_{ba} - \Pi q_{ba}) \\ &\approx \mp\frac{i}{\sqrt{2}}D^a(K_{ab} - Kq_{ab}), \end{aligned} \tag{5.69}$$

where $\approx$ stands for weak equality, modulo the Gauss constraint (5.44). Recalling Eq. (5.24), we see that vector constraint (5.47) can be rewritten as

$$C_a(\tilde{\sigma}, A) = \mp 2\sqrt{2}i\,\mathrm{tr}(\tilde{\sigma}^b\,^{\pm}F_{ab}). \tag{5.70}$$

Similarly, we get

$$\begin{aligned} \mathrm{tr}(\sigma^a\sigma^b\,^{\pm}F_{ab}) &= \tfrac{1}{2}(R + \Pi^2 - \Pi_{ab}\Pi^{ab}) \mp i\epsilon^{abc}D_a\Pi_{bc} \\ &\approx \tfrac{1}{2}(R + K^2 - K_{ab}K^{ab}), \end{aligned} \tag{5.71}$$

where R is the scalar curvature of q_{ab}. Again using Eq. (5.24), we see that the scalar constraint (5.48) can be rewritten as

$$C(\tilde{\sigma}, A) = -2q^{-1/2}\,\mathrm{tr}(\tilde{\sigma}^a\tilde{\sigma}^b\,{}^{\pm}F_{ab}). \tag{5.72}$$

The right hand sides of Eqs. (5.70) and (5.72) are functionals of $\tilde{\sigma}^a{}_A{}^B$ and $A_{aA}{}^B$ only. Thus all the constraints have been re-expressed in terms of the Ashtekar variables.

To summarize, the set of constraints on the extended phase space becomes simply

$$\begin{aligned} {}^{\pm}\mathcal{D}_a\tilde{\sigma}^a{}_A{}^B &= \partial_a\tilde{\sigma}^a{}_A{}^B + [\,{}^{\pm}A_a, \tilde{\sigma}^a]_A^B = 0, \\ \mathrm{tr}(\tilde{\sigma}^b\,{}^{\pm}F_{ab}) &= 0, \\ \mathrm{tr}(\tilde{\sigma}^a\tilde{\sigma}^b\,{}^{\pm}F_{ab}) &= 0. \end{aligned} \tag{5.73}$$

5.3.3 Constraint algebra in Ashtekar's theory

We state briefly the algebra of constraints in the Ashtekar variables. Details can be found in Ref. [18]. To construct the Poisson bracket algebra of constraints in the Ashtekar variables, we again integrate the local constraint functions against arbitrary test functions over the 3-manifold,

$$\begin{aligned} C_N(\tilde{\sigma}, A) &= -\sqrt{2}i\int_\Sigma d^3x\ \mathrm{tr}(\underline{N}\mathcal{D}_a\tilde{\sigma}^a), \\ C_{\vec{N}}(\tilde{\sigma}, A) &= -\sqrt{2}i\int_\Sigma d^3x\, N^a\,\mathrm{tr}(\tilde{\sigma}^b F_{ab} - A_a\mathcal{D}_b\tilde{\sigma}^b), \\ C_{\underset{\sim}{N}}(\tilde{\sigma}, A) &= -\sqrt{2}i\int_\Sigma d^3x\, \underset{\sim}{N}\,\mathrm{tr}\,\tilde{\sigma}^a\tilde{\sigma}^b F_{ab}, \end{aligned} \tag{5.74}$$

where $\underline{N}$, N^a, and $\underset{\sim}{N}$ are arbitrary $SU(2)$-, vector-, and scalar-valued test functions respectively.

The Poisson bracket algebra of the constraint functions is

$$
\begin{aligned}
\{C_N, C_M\} &= -C_{[N,M]} \\
\{C_{\vec{N}}, C_M\} &= -C_{\mathcal{L}_{\vec{N}} M} \\
\{C_{\vec{N}}, C_{\vec{M}}\} &= -C_{[\vec{N},\vec{M}]} \\
\{C_N, C_{\underset{\sim}{M}}\} &= 0 \\
\{C_{\vec{N}}, C_{\underset{\sim}{M}}\} &= -C_{\mathcal{L}_{\vec{N}} \underset{\sim}{M}} \\
\{C_{\underset{\sim}{N}}, C_{\underset{\sim}{M}}\} &= C_{\vec{K}} - C_{A_m K^m}.
\end{aligned}
\tag{5.75}
$$

Each Poisson bracket in the algebra equals a linear sum of constraints. The constraints are therefore all first class.

Chapter 6

Classical BRST Structure of Self-dual Gravity

A useful BRST analysis of a physical theory requires the construction of a real BRST charge. The construction of a real BRST charge for the Ashtekar theory is not straightforward, however, because the constraints (5.73) are complex. We clarify the complex structure of these constraints in Sec. 6.1 and show that, in fact, they are a complex mix of real constraints. This is the same structure that we considered in Sec. 4.3. Following the analysis of Sec. 4.3.1, we show in Sec. 6.2 that the BRST charges constructed by Ashtekar, Mazur and Torre [13] are complex and therefore not useful. In Sec. 6.3, we construct a new BRST charge which is real by using the technique developed in Sec. 4.3.2, namely, by adding the complex conjugates of the Ashtekar constraints to the original set of Ashtekar constraints and treating the combined set as a reducible set of constraints. We thus satisfy the criterion of reality, but we show that this is at the cost of polynomiality. The BRST charge constructed in Sec. 6.3 is a viable BRST charge, but the nonpolynomiality makes it difficult to proceed with the BRST analysis. In Sec. 6.4, we construct a BRST charge that is real and nearly polynomial by a judicious remixing of the constraints.

6.1 Complex structure of the constraints

In this section, we examine the complex structure of the Ashtekar constraints. We first separate them into their real and imaginary parts, and then rewrite them as a complex combination of real constraints. The manifestly complex objects in the Ashtekar constraints are the Ashtekar connection $\mathcal{D}_a$ and, what amounts to the same thing, the connection 1-form $A_{aA}{}^B$. Less obvious, however, is the complex behaviour of the $SU(2)$ spinors. Hermitian spinors behave like "real" numbers and anti-Hermitian spinors behave like "imaginary" numbers under Hermitian conjugation. So we must take care also to examine the Hermiticity properties of the spinors in the constraints. The Hermiticity properties of $SU(2)$ spinors, and of the Ashtekar variables in particular, are discussed in Appendix B. For convenience, we drop the $\pm$ signs from the Ashtekar variables, choosing $A_a := {}^+A_a$ in the first two sections of this chapter.

We begin with the Ashtekar connection $\mathcal{D}_a$ and the connection 1-form $A_{aM}{}^N$. To separate the real and imaginary parts of $\mathcal{D}_a$, we take a closer look at Eq. (5.59),

$$D_a\lambda_{bM} := \partial_a\lambda_{bM} + \Gamma_{ab}{}^c\lambda_{cM} + \Gamma_{aM}{}^N\lambda_{bN} + \frac{i}{\sqrt{2}}\Pi_{aM}{}^N\lambda_{bN}. \tag{6.1}$$

Ignoring λ_{bM}, which is included only so that the indices match, we see that the first two terms on the right hand side, which involve only real spacetime operators, are manifestly real. The third term involves the spinorial connection 1-form $\Gamma_{aM}{}^N \equiv \Gamma_{ab}\sigma^b{}_M{}^N$. The tensor Γ_{ab} is real and $\sigma^a{}_M{}^N$ is anti-Hermitian so, under complex conjugation, the third term goes to minus itself and therefore behaves like an imaginary number. In the last term, $\Pi_{aM}{}^N \equiv \Pi_{ab}\sigma^b{}_M{}^N$ is also anti-Hermitian, but the coefficient i makes the overall term Hermitian. It thus behaves like a real number under complex conjugation. So we rearrange the Ashtekar connection into "real" and "imaginary" parts,

$$\mathcal{D}_a\lambda_{bM} := \underbrace{\partial_a\lambda_{bM} + \Gamma_{ab}{}^c\lambda_{cM} + \frac{i}{\sqrt{2}}\Pi_{aM}{}^N\lambda_{bN}}_{\text{"real"}} + \underbrace{\Gamma_{aM}{}^N\lambda_{bN}}_{\text{"imaginary"}}. \tag{6.2}$$

Since the connection 1-form $A_{aM}{}^N$ of Eq. (5.60) consists of just the last two terms of $\mathcal{D}_a$, we can immediately write it in terms of its "real" and "imaginary" parts,

$$A_{aM}{}^N = \underbrace{+\frac{i}{\sqrt{2}}\Pi_{aM}{}^N}_{\text{"real"}} + \underbrace{\Gamma_{aM}{}^N}_{\text{"imaginary"}}. \tag{6.3}$$

We now consider the Gauss constraint. Using the definition of D_a in Eq. (5.56), the fact that $D_a\tilde{\sigma}^a{}_M{}^N \equiv D_a(q^{1/2}\sigma^a{}_M{}^N) = 0$ because both $\tilde{\sigma}^a{}_M{}^N$ and q_{ab} are compatible with D_a, and Eqs. (B5), we can rewrite the Gauss constraint, $\mathcal{D}_a\tilde{\sigma}^a{}_M{}^N = 0$, in a number of equivalent forms,

$$\begin{aligned} \mathcal{D}_a\tilde{\sigma}^a{}_M{}^N &\equiv \frac{i}{\sqrt{2}}[\Pi_a, \tilde{\sigma}^a]_M{}^N \\ &\equiv \sqrt{2}iq^{1/2}\Pi_{[ab]}\sigma^b{}_M{}^P\sigma^a{}_P{}^N \\ &\equiv -iq^{1/2}\Pi_{[ab]}\epsilon^{abc}\sigma_{cM}{}^N. \end{aligned} \tag{6.4}$$

Any of these forms of the Gauss constraint can be used to examine its reality properties, but the last is the simplest to use. $\Pi_{[ab]}$ and ϵ^{abc} are real tensors, $\sigma^c{}_M{}^N$ is anti-Hermitian, and the coefficient i makes the Gauss constraint overall Hermitian. It thus behaves like a "real" number under complex conjugation,

$$\mathcal{D}_a\tilde{\sigma}^a{}_M{}^N = \underbrace{-iq^{1/2}\Pi_{[ab]}\epsilon^{abc}\sigma_{cM}{}^N}_{\text{"real"}}. \tag{6.5}$$

To separate the vector constraint into real and imaginary parts, we begin with Eq. (5.69),

$$\mathrm{tr}(\sigma^a F_{ab}) \equiv \frac{1}{2\sqrt{2}}(\Pi_{am}\Pi_{bn} - \Pi_{bm}\Pi_{an})\epsilon^{mna} - \frac{i}{\sqrt{2}}D^a(\Pi_{ba} - \Pi q_{ba}).$$

The first term contains only the real tensors Π_{ab} and ϵ^{abc} and is manifestly real. In the second term, although D_a is in general "complex" because of the spinorial connection 1-form that it contains, it is, in this case, acting on a tensor so that the spinorial connection does not enter and it therefore behaves as a real operator. The vector constraint is therefore already separated into real and imaginary parts,

$$\mathrm{tr}(\sigma^a F_{ab}) = \underbrace{\frac{1}{2\sqrt{2}}(\Pi_{am}\Pi_{bn} - \Pi_{bm}\Pi_{an})\epsilon^{mna}}_{\text{real}} - \underbrace{\frac{i}{\sqrt{2}}D^a(\Pi_{ba} - \Pi q_{ba})}_{\text{imaginary}}. \tag{6.6}$$

The same arguments apply to the scalar constraint which we immediately recognize as already separated into real and imaginary parts in Eq. (5.71),

$$\mathrm{tr}(\sigma^a\sigma^b F_{ab}) = \underbrace{\tfrac{1}{2}(R + \Pi^2 - \Pi_{ab}\Pi^{ab})}_{\text{real}} - \underbrace{i\epsilon^{abc}D_a\Pi_{bc}}_{\text{imaginary}}. \tag{6.7}$$

Equations (6.5), (6.6), and (6.7) explicitly show the real and imaginary parts of the Ashtekar constraints, but the real and imaginary parts cannot all be independent since we have seven complex constraint equations and only seven true (real) constraints on the phase space. In fact, we saw in Eqs. (5.69) and (5.71) that the vector and scalar constraints implicitly contain the Gauss constraint. We wish to make this dependence on the Gauss constraint explicit. We consider first the vector constraint (6.6). By relabeling dummy indices and using the third of Eqs. (6.4), we can rewrite the first term on the right side of

Eq. (6.6) as

$$\frac{1}{2\sqrt{2}}(\Pi_{am}\Pi_{bn}-\Pi_{bm}\Pi_{an})\epsilon^{mna}=\frac{i}{\sqrt{2}}\Pi_{bN}{}^{M}(\mathcal{D}_a\tilde{\sigma}^a{}_M{}^N). \tag{6.8}$$

In the second term on the right side of Eq. (6.6) we need to separate Π_{ba} into its symmetric and antisymmetric components since the Gauss constraint is related to the antisymmetric part only. Then, solving the third of Eqs. (6.4) for $\Pi_{[ab]}$ allows us to rewrite the second term as

$$\begin{aligned}\frac{i}{\sqrt{2}}D^a(\Pi_{ba}-\Pi q_{ba}) &\equiv \frac{i}{\sqrt{2}}D^a(\Pi_{[ba]}+\Pi_{(ba)}-\Pi q_{ba})\\ &= -\frac{1}{2\sqrt{2}}\epsilon_{bcd}\sigma^c{}_N{}^M D^d(\mathcal{D}_a\sigma^a{}_M{}^N)-\frac{i}{\sqrt{2}}D_a(K_{ab}-Kq_{ab}),\end{aligned} \tag{6.9}$$

where we have used Eq. (5.58). Combining these terms, we can rewrite the vector constraint as

$$\begin{aligned}\mathcal{V}_b \equiv \mathrm{tr}(\tilde{\sigma}^a F_{ab}) = &\underbrace{\frac{i}{\sqrt{2}}\Pi_{bN}{}^{M}(\mathcal{D}_a\tilde{\sigma}^a{}_M{}^N)}_{\text{real}}\\ &-\underbrace{\frac{1}{2\sqrt{2}}\epsilon_{bcd}\tilde{\sigma}^c{}_N{}^M D^d(\mathcal{D}_a\sigma^a{}_M{}^N)-\frac{i}{\sqrt{2}}q^{1/2}D_a(K_{ab}-Kq_{ab})}_{\text{imaginary}}=0.\end{aligned} \tag{6.10}$$

Similarly, we can rewrite the scalar constraint as

$$\mathcal{S}\equiv \mathrm{tr}(\tilde{\sigma}^a\tilde{\sigma}^b F_{ab})=\underbrace{\tfrac{1}{2}q(R+K^2-K_{ab}K^{ab})}_{\text{real}}+\underbrace{\tilde{\sigma}^a{}_N{}^M D_a(\mathcal{D}_b\tilde{\sigma}^b{}_M{}^N)}_{\text{imaginary}}=0, \tag{6.11}$$

where we have again used Eqs. (6.4) and (5.58). Equations (6.10) and (6.11), together with the Gauss constraint,

$$\mathcal{G}_M{}^N \equiv \underbrace{\mathcal{D}_a\tilde{\sigma}^a{}_M{}^N}_{\text{“real”}}=0, \tag{6.12}$$

are the forms of the Ashtekar constraints that we will find useful in the next sections.

Equations (6.10), (6.11), and (6.12) immediately lead to the reality conditions on the constraints,

$$(\mathcal{G}_M{}^N)^\dagger = \mathcal{G}_M{}^N \tag{6.13}$$

$$\mathcal{V}_b^* = -\mathcal{V}_b + \sqrt{2}i \,\mathrm{tr}\, \Pi_b \mathcal{G} \tag{6.14}$$

$$\mathcal{S}^* = \mathcal{S} - 2 \,\mathrm{tr}\, \tilde{\sigma}^a D_a \mathcal{G}, \tag{6.15}$$

where $\dagger$ is Hermitian conjugation and $*$ is ordinary complex conjugation. We observe that the reality conditions on the vector and scalar constraints have nonconstant coefficients. These reality conditions will prove to be useful in determining the reality of the BRST charges in the next sections.

6.2 The Ashtekar, Mazur, and Torre BRST charges

In 1987, Ashtekar, Mazur, and Torre [13] investigated the BRST structure of canonical general relativity in terms of the recently introduced new variables. They used methods developed by Henneaux [6] which assume that the constraints are real, and did not consider the consequences of the complex nature of the Ashtekar constraints. They constructed three different BRST charges, one based on the original set of Ashtekar constraints and two others based on recombinations of the constraints. The recombinations were motivated by physical and computational arguments and were not related to the reality properties of the constraints. In this section, we review the BRST charges constructed by Ashtekar, Mazur, and Torre (AMT) and show that, in fact, all three are intrinsically complex.

6.2.1 The original Ashtekar constraints

We consider first the BRST charge constructed from the standard Ashtekar constraints (5.73),

$$\begin{aligned} \mathcal{D}_a \tilde{\sigma}^a{}_A{}^B &= 0, \\ \mathrm{tr}(\tilde{\sigma}^b F_{ab}) &= 0, \\ \mathrm{tr}(\tilde{\sigma}^a \tilde{\sigma}^b F_{ab}) &= 0. \end{aligned} \tag{6.16}$$

This is not the case preferred by Ashtekar, Mazur, and Torre, but is logically the first case to consider.

The constraints are integrated against test functions to convert them to scalar-valued functions on the phase space,

$$\begin{aligned} U(\underline{N}) &= -i\sqrt{2}\int_\Sigma \mathrm{tr}\, \underline{N}\mathcal{D}_a\tilde{\sigma}^a \\ U(\overrightarrow{N}) &= -i\sqrt{2}\int_\Sigma \mathrm{tr}\, N^a\tilde{\sigma}^b F_{ab} \\ U(\underset{\sim}{N}) &= -i\sqrt{2}\int_\Sigma \mathrm{tr}\, \underset{\sim}{N}\tilde{\sigma}^a\tilde{\sigma}^b F_{ab}. \end{aligned} \tag{6.17}$$

The fields $\underline{N}$, N^a, and $\underset{\sim}{N}$ are,respectively, an Lie-algebra-valued function on Σ, a vector field on Σ, and a scalar density of weight minus one on Σ. These fields play the same role in this field theory as the index a did in the constraints G_a of the finite theories considered in Chap. 2.

Calculation of the Poisson brackets between the constraints yields the structure functions $U(\ ,\ |\ )$, where, comparing to the finite-dimensional case, the two entries in the parentheses on the left of the vertical line are to be thought of as the covariant indices a and b of the structure function $U_{ab}{}^c$, and the entry on the right of the line as the contravariant index c. The only nonvanishing first-order structure functions are

$$
\begin{aligned}
U(\underline{N}, \underline{M}|\widetilde{\underline{L}}) &= -\int_\Sigma \mathrm{tr}\, \underline{N}\underline{M}\widetilde{\underline{L}}, \\
U(\overrightarrow{N}, \overrightarrow{M}|\widetilde{\underline{L}}) &= \tfrac{1}{2}\int_\Sigma \mathrm{tr}\, N^a M^b F_{ab}\widetilde{\underline{L}}, \\
U(\overrightarrow{N}, \overrightarrow{M}|\widetilde{\mathbf{L}}) &= \tfrac{1}{2}\int_\Sigma (\mathcal{L}_{\overrightarrow{N}} M^a)\widetilde{L}_a, \\
U(\overrightarrow{N}, \underset{\sim}{M}|\widetilde{\underline{L}}) &= \int_\Sigma \mathrm{tr}\, \underset{\sim}{M} n^b \tilde{\sigma}^a F_{ab}\widetilde{\underline{L}}, \\
U(\overrightarrow{N}, \underset{\sim}{M}|\tilde{\tilde{L}}\,) &= \tfrac{1}{2}\int_\Sigma (\mathcal{L}_{\overrightarrow{N}} \underset{\sim}{M})\tilde{\tilde{L}}\,, \\
U(\underset{\sim}{N}, \underset{\sim}{M})|\widetilde{\mathbf{L}}) &= \int_\Sigma (\underset{\sim}{N}\partial_a \underset{\sim}{M} - \underset{\sim}{M}\partial_a \underset{\sim}{N})(\mathrm{tr}\, \tilde{\sigma}^a \tilde{\sigma}^b)\widetilde{L}_b.
\end{aligned}
\tag{6.18}
$$

Here $\widetilde{\underline{L}}$ is a density of weight one with values in the $SU(2)$ Lie algebra (representing an index dual to $\underline{N}$), $\widetilde{\mathbf{L}}$ is a covector field of weight one (representing an index dual to $\overrightarrow{N}$), and $\tilde{\tilde{L}}$ is a scalar density of weight two (representing an index dual to $\underset{\sim}{N}$).

The calculation of the second-order structure functions is quite tedious, but AMT show that the only nonvanishing second-order structure functions are

$$
\begin{aligned}
U(\underset{\sim}{L}\,, \underset{\sim}{M}, \overrightarrow{K}|\widetilde{\mathbf{N}}\widetilde{\underline{J}}) &= \frac{\sqrt{2}i}{6}\,\mathrm{tr}\int_\Sigma (\underset{\sim}{M}\partial_a \underset{\sim}{L} - \underset{\sim}{L}\,\partial_a \underset{\sim}{M})\widetilde{N}_b K^{(a}\tilde{\sigma}^{b)}\widetilde{\underline{J}}, \\
U(\underset{\sim}{L}\,, \overrightarrow{M}\overrightarrow{N}|\widetilde{\underline{K}}\widetilde{\underline{J}}) &= \frac{\sqrt{2}i}{6}\,\mathrm{tr}\int_\Sigma \underset{\sim}{L}\, N^a M^b F_{ab}\widetilde{\underline{K}}\widetilde{\underline{J}}.
\end{aligned}
\tag{6.19}
$$

AMT then show that the third-order and fourth-order structure functions all vanish and that the theory is, therefore, rank-two. The BRST charge takes, as they put it, "the rather unwieldy form,"

$$\Omega = \int_\Sigma \mathrm{tr}\left[\frac{\sqrt{2}}{i}\left(\underline{\eta}(\mathcal{D}_a\tilde{\sigma}^a) + \eta^a\tilde{\sigma}^b F_{ab} + \underset{\sim}{\eta}\tilde{\sigma}^a\tilde{\sigma}^b F_{ab}\right) + \underline{\eta\eta}\underline{\widetilde{\mathcal{P}}} - (\eta^b\partial_b\eta^a)\widetilde{\mathcal{P}}_a \right.$$
$$- (\eta^a\partial_a\underset{\sim}{\eta} + \underset{\sim}{\eta}\partial_a\eta^a)\widetilde{\widetilde{\mathcal{P}}} - 2\underset{\sim}{\eta}(\partial_a\underset{\sim}{\eta})(\mathrm{tr}\tilde{\sigma}^a\tilde{\sigma}^b)\widetilde{\mathcal{P}}_b + 2\eta^a\underset{\sim}{\eta}\underline{\widetilde{\mathcal{P}}}\tilde{\sigma}^b F_{ab}$$
$$\left. -\tfrac{1}{2}\eta^a\eta^b\underline{\widetilde{\mathcal{P}}}F_{ab} - 2i\sqrt{2}\underset{\sim}{\eta}(\partial_a\underset{\sim}{\eta})\eta^{(a}\tilde{\sigma}^{b)}\widetilde{\mathcal{P}}_b\underline{\widetilde{\mathcal{P}}} - \frac{i\sqrt{2}}{2}\underset{\sim}{\eta}\eta^a\eta^b F_{ab}\underline{\widetilde{\mathcal{P}}}\underline{\widetilde{\mathcal{P}}}\right]. \tag{6.20}$$

We now wish to investigate the reality of this BRST charge. A BRST charge is an expansion in the antighost number, *i.e.*, in powers of the ghost momenta $\mathcal{P}$, and so must be real at each antighost number for the overall charge to be real. The first three terms on the right side of (6.20) are the antighost number zero part, which we can rewrite as $-\sqrt{2}i(\mathrm{tr}\,\underline{\eta}\mathcal{G} + \eta^a\mathcal{V}_a + \underset{\sim}{\eta}\mathcal{S})$. We require that this expression be real,

$$\begin{aligned} i\,\mathrm{tr}\,\underline{\eta}\mathcal{G} + i\eta^a\mathcal{V}_a + i\underset{\sim}{\eta}\mathcal{S} &= (i\,\mathrm{tr}\,\underline{\eta}\mathcal{G} + i\eta^a\mathcal{V}_a + i\underset{\sim}{\eta}\mathcal{S})^* \\ &= -i\,\mathrm{tr}\,\underline{\eta}^\dagger\mathcal{G}^\dagger - i\eta^{a^*}\mathcal{V}_a^* - i\underset{\sim}{\eta}^*\mathcal{S}^* \\ &= i\,\mathrm{tr}[(-\underline{\eta}^\dagger - \sqrt{2}i\eta^{a^*}\Pi_a + 2\underset{\sim}{\eta}^*\tilde{\sigma}^a D_a)\mathcal{G}] + i\eta^{a^*}\mathcal{V}_a - i\underset{\sim}{\eta}^*\mathcal{S}, \end{aligned} \tag{6.21}$$

where the reality conditions (6.13) were used in the last step. Matching coefficients on the left and right sides and solving for the complex conjugate ghosts, we find the reality conditions on the ghosts,

$$\begin{aligned} (\underline{\eta}_M{}^N)^\dagger &= -\underline{\eta}_M{}^N - \sqrt{2}i\eta^a\Pi_{aM}{}^N - 2\underset{\sim}{\eta}\tilde{\sigma}^a{}_M{}^N D_a, \\ \eta^{a*} &= \eta^a, \\ \underset{\sim}{\eta}^* &= -\underset{\sim}{\eta}. \end{aligned} \tag{6.22}$$

These, in turn, impose reality conditions on the ghost momenta, which are found by complex conjugating the fundamental Poisson brackets between the ghosts and their momenta and imposing the ghost reality conditions,

$$\begin{aligned} \underset{\sim}{\widetilde{\mathcal{P}}}^{\dagger} &= \underset{\sim}{\widetilde{\mathcal{P}}}, \\ \widetilde{\mathcal{P}}_a^{*} &= -\widetilde{\mathcal{P}}_a + \sqrt{2}i\,\mathrm{tr}\,\underset{\sim}{\widetilde{\mathcal{P}}}\Pi_a, \\ \widetilde{\widetilde{\mathcal{P}}}^{*} &= \widetilde{\widetilde{\mathcal{P}}} - 2\,\mathrm{tr}\,\underset{\sim}{\widetilde{\mathcal{P}}}\tilde{\sigma}^a D_a. \end{aligned} \tag{6.23}$$

With the reality conditions (6.22) and (6.23), the antighost number zero part of the BRST charge (6.20) is real. Now let us begin looking at the antighost number one terms. The next term, $\underset{\sim}{\eta}\underset{\sim}{\eta}\underset{\sim}{\widetilde{\mathcal{P}}}$, involves the trace of three $SU(2)$-valued ghosts. $\underset{\sim}{\widetilde{\mathcal{P}}}$ is Hermitian, but, as Eqs. (6.22) show, the Hermiticity properties of $\underset{\sim}{\eta}$ are not well defined. The next term, $-(\eta^b\partial_b\eta^a)\widetilde{P}_a$, is manifestly complex because η^a is real and $\widetilde{\mathcal{P}}_a$ has nonzero real and complex parts. The next term, $-(\eta^a\partial_a\underset{\sim}{\eta} + \underset{\sim}{\eta}\partial_a\eta^a)\widetilde{\widetilde{\mathcal{P}}}$, is also manifestly complex because η^a is real, $\underset{\sim}{\eta}$ is pure imaginary, and $\widetilde{\widetilde{\mathcal{P}}}$ has nonzero real and imaginary parts. Rather than continue, we need simply argue that the imaginary pieces in (6.20) do not cancel each other. We saw in the simple example of Sec. 4.3.1 that the freedom to choose the reality properties of the ghosts and their momenta is "used up" at the antighost zero level, and that the appearance of complex terms at higher antighost numbers makes the BRST charge intrinsically complex. The constraints (6.16) fit the model considered in Sec. 4.3.1. We have explicitly demonstrated here some of the intrinsically complex terms at antighost number one, and we conclude that the BRST charge (6.20) is intrinsically complex.

6.2.2 Modified vector constraint

Having looked at one of the BRST charges constructed by Ashtekar, Mazur, and Torre (AMT) [13] in some detail, we will now look at the other two much more briefly, being satisfied to explicitly show that the constraints upon which they are built are intrinsically complex and concluding from that, that the BRST charge also is intrinsically complex. We look first at the set of constraints and resulting BRST charge that was the primary focus of AMT. The constraints differ from those in (6.16) by the addition of a term to the vector constraint. The additional term is a multiple of the Gauss constraint and therefore preserves the weak equality of the system of constraints, *i.e.*, the modified constraints define the same constraint surface. The modified constraints are

$$\begin{aligned} \mathcal{D}_a \tilde{\sigma}^a{}_A{}^B &= 0, \\ \mathrm{tr}(\tilde{\sigma}^b F_{ab} - A_a \mathcal{D}_b \tilde{\sigma}^b) &= 0, \\ \mathrm{tr}(\tilde{\sigma}^a \tilde{\sigma}^b F_{ab}) &= 0. \end{aligned} \tag{6.24}$$

The extra term was added to the vector constraint for physical and computational reasons. The physical reason is that the modified constraint is the generator of spatial diffeomorphisms and thus has a well-defined geometric meaning. The computational reason is that the Poisson bracket algebra is simplified by the addition of this constraint. Although the motivation of AMT was not to make the constraints real, they do observe in a footnote that the addition of this term "yields a Hermitian function" on the phase space.

We have already shown that the Gauss constraint (6.12) is Hermitian and that the scalar constraint (6.11) has nonzero real and imaginary parts. This already is sufficient to make the BRST charge constructed from the constraints

(6.24) intrinsically complex, but it is enlightening to examine the reality properties of the modified vector constraint and demonstrate that it is, in fact, pure imaginary.

We recall Eq. (6.10), which shows the real and imaginary parts of the original vector constraint,

$$\begin{aligned}\mathcal{V}_b \equiv \mathrm{tr}(\tilde{\sigma}^a F_{ab}) = &\underbrace{\frac{i}{\sqrt{2}}\Pi_{bN}{}^M(\mathcal{D}_a\tilde{\sigma}^a{}_M{}^N)}_{\text{real}} \\ &\underbrace{-\frac{1}{2\sqrt{2}}\epsilon_{bcd}\tilde{\sigma}^c{}_N{}^M D^d(\mathcal{D}_a\sigma^a{}_M{}^N) - \frac{i}{\sqrt{2}}q^{1/2}D_a(K_{ab} - Kq_{ab})}_{\text{imaginary}} = 0.\end{aligned}$$

The last term, involving the extrinsic curvature K_{ab}, is the independent physical constraint and cannot be removed by adding the Gauss or scalar constraints to it. Thus, in order to give the vector constraint well-defined reality properties, it is necessary to cancel the real part, which is, in fact, a multiple of the Gauss constraint. We could simply subtract it off as it is, but it is nonpolynomial and would leave the resulting modified vector constraint nonpolynomial. Instead, we consider the the term $A_a\mathcal{D}_b\tilde{\sigma}^b$ and observe that, using the definition of the Ashtekar connection 1-form in Eq. (5.56), we can separate it into two terms,

$$A_{bN}{}^M\mathcal{D}_a\tilde{\sigma}^a{}_M{}^N = \Gamma_{bN}{}^M\mathcal{D}_a\tilde{\sigma}^a{}_M{}^N + \frac{i}{\sqrt{2}}\Pi_{bN}{}^M\mathcal{D}_a\tilde{\sigma}^a{}_M{}^N. \tag{6.25}$$

The second term is exactly the term we wish to cancel in the vector constraint and, as we have shown, it is real. Furthermore, using Eqs. (6.4) and (B5), we see that the first term,

$$\begin{aligned}\Gamma_{bN}{}^M\mathcal{D}_a\tilde{\sigma}^a{}_M{}^N &= \Gamma_{bc}\sigma^c{}_N{}^M(\sqrt{2}iq^{1/2}\Pi_{[ed]}\sigma^d{}_M{}^P\sigma^e{}_P{}^N) \\ &= -iq^{1/2}\Gamma_{bc}\Pi_{[ed]}\epsilon^{cde}.\end{aligned} \tag{6.26}$$

is purely imaginary. By subtracting the term (6.25) from the vector constraint, we simultaneously cancel the real part and add an imaginary part, leaving the modified vector constraint purely imaginary. The trivial step of multiplying the vector constraint by i thus turns it into a real constraint.

The constraints (6.24) are, from the BRST point of view, an improvement over the constraints (6.16). Nevertheless, we have shown that the BRST charge that Ashtekar, Mazur and Torre construct from them,

$$
\begin{aligned}
Q' \;=\; \int_\Sigma \mathrm{tr}\Bigg[& \frac{\sqrt{2}}{i}\left(\underline{\eta}(\mathcal{D}_a\tilde{\sigma}^a) + \eta^a(\tilde{\sigma}^b F_{ab} - A_a\mathcal{D}_b\tilde{\sigma}^b) + \underset{\sim}{\eta}\tilde{\sigma}^a\tilde{\sigma}^b F_{ab}\right) \\
& + \underline{\eta\eta}\,\underline{\widetilde{\mathcal{P}}} + (\eta^a\partial_a\underline{\eta})\underline{\widetilde{\mathcal{P}}} - (\eta^b\partial_b\eta^a)\widetilde{\mathcal{P}}_a - (\eta^a\partial_a\underset{\sim}{\eta} + \underset{\sim}{\eta}\partial_a\eta^a)\widetilde{\widetilde{\mathcal{P}}} \\
& - 2\underset{\sim}{\eta}(\partial_a\underset{\sim}{\eta})(\mathrm{tr}\tilde{\sigma}^a\tilde{\sigma}^b)(\widetilde{\mathcal{P}}_b - \mathrm{tr}\, A_b\underline{\widetilde{\mathcal{P}}})\Bigg]\,. \qquad (6.27)
\end{aligned}
$$

must necessarily be complex because the scalar constraint remains complex.

6.2.3 Modified scalar constraint

Having achieved some computational simplification by modifying the vector constraint, Ashtekar, Mazur and Torre then do the same, to some extent, by modifying the scalar constraint. The new constraints are

$$
\begin{aligned}
\mathcal{D}_a\tilde{\sigma}^a{}_A{}^B &= 0, \\
\mathrm{tr}(\tilde{\sigma}^b F_{ab} - A_a\mathcal{D}_b\tilde{\sigma}^b) &= 0, \qquad (6.28) \\
\mathrm{tr}[\tilde{\sigma}^a\tilde{\sigma}^b F_{ab} + 2\tilde{\sigma}^a A_a(\mathcal{D}_b\tilde{\sigma}^b)] &= 0.
\end{aligned}
$$

They do not comment on the reality properties of the modified scalar constraint. To determine the reality properties, we investigate the term that they have added. Using Eqs. (5.56), (6.4), and (B5), we find

$$\begin{aligned}\operatorname{tr}\tilde{\sigma}^a A_a(\mathcal{D}_b\tilde{\sigma}^b) &= \operatorname{tr}\tilde{\sigma}^a(\Gamma_a + \frac{i}{\sqrt{2}}\Pi_a)(\sqrt{2}iq^{1/2}\Pi_{[dc]}\sigma^c\sigma^d) \\ &= \underbrace{-q\Pi^{ab}\Pi_{[ab]}}_{\text{real}} + \underbrace{\sqrt{2}iq\Gamma^{ab}\Pi_{[ab]}}_{\text{imaginary}} .\end{aligned} \tag{6.29}$$

We see that the extra term has nonzero real and imaginary parts. Furthermore, comparing with Eq. (6.7), we see that the imaginary parts do not cancel. The scalar constraint thus remains intrinsically complex and we once again conclude that the BRST charge,

$$\begin{aligned}Q'' \;=\; \int_\Sigma \operatorname{tr}\Bigg[\frac{\sqrt{2}}{i}\bigg(&\underline{\eta}(\mathcal{D}_a\tilde{\sigma}^a) + \eta^a(\tilde{\sigma}^b F_{ab} - A_a\mathcal{D}_b\tilde{\sigma}^b) + \underset{\sim}{\eta}(\tilde{\sigma}^a\tilde{\sigma}^b F_{ab} \\ &+ 2\tilde{\sigma}^a A_a\mathcal{D}_b\tilde{\sigma}^b)\bigg) + \underline{\eta\eta}\underline{\widetilde{\mathcal{P}}} + (\eta^a\partial_a\underline{\eta})\underline{\widetilde{\mathcal{P}}} - (\eta^b\partial_b\eta^a)\widetilde{\mathcal{P}}_a \\ &- (\eta^a\partial_a\underset{\sim}{\eta} + \underset{\sim}{\eta}\partial_a\eta^a)\widetilde{\widetilde{\mathcal{P}}} - 2\underset{\sim}{\eta}\tilde{\sigma}^a(\partial_a\underline{\eta})\underline{\widetilde{\mathcal{P}}} - 2\underset{\sim}{\eta}(\partial_a\underset{\sim}{\eta})(\operatorname{tr}\tilde{\sigma}^a\tilde{\sigma}^b)\widetilde{\mathcal{P}}_b\Bigg] ,\end{aligned} \tag{6.30}$$

constructed from the constraints (6.28), must be intrinsically complex.

6.3 The reducible case

In Sec. 4.3.2 we developed a technique for constructing a real BRST charge for a system with complex constraints which satisfy the condition that the constraints together with their complex conjugates are all first-class. We now apply this method to self-dual gravity. In this section, we resume the use of the symbols + and − introduced in Sec. 5.3 to indicate self-dual and antiself-dual variables, respectively.

To the original Ashtekar constraints,

$$\begin{aligned} {}^{+}\mathcal{D}_a \tilde{\sigma}^a{}_A{}^B &= 0, \\ \mathrm{tr}(\tilde{\sigma}^b\, {}^{+}F_{ab}) &= 0, \\ \mathrm{tr}(\tilde{\sigma}^a \tilde{\sigma}^b\, {}^{+}F_{ab}) &= 0, \end{aligned} \tag{6.31}$$

we add the complex conjugate constraints,

$$\begin{aligned} {}^{-}\mathcal{D}_a \tilde{\sigma}^a{}_A{}^B &= 0, \\ \mathrm{tr}(\tilde{\sigma}^b\, {}^{-}F_{ab}) &= 0, \\ \mathrm{tr}(\tilde{\sigma}^a \tilde{\sigma}^b\, {}^{-}F_{ab}) &= 0. \end{aligned} \tag{6.32}$$

We now have a reducible set of constraints, and the reducibility conditions follow from Eqs. (6.10) through (6.12). The relations among the constraints are,

$$\begin{aligned} (\,{}^{+}\mathcal{D}_a \tilde{\sigma}^a)^\dagger = (\,{}^{+}\mathcal{D}_a \tilde{\sigma}_a) &= -\,{}^{-}\mathcal{D}_a \tilde{\sigma}^a = -(\,{}^{-}\mathcal{D}_a \tilde{\sigma}^a)^\dagger, \\ \mathrm{tr}(\tilde{\sigma}^a\, {}^{+}F_{ab}) + \mathrm{tr}({}^{+}A_b\, {}^{+}\mathcal{D}_a \tilde{\sigma}^a) &= -\,\mathrm{tr}(\tilde{\sigma}^a\, {}^{-}F_{ab}) - \mathrm{tr}({}^{-}A_b\, {}^{-}\mathcal{D}_a \tilde{\sigma}^a), \\ \mathrm{tr}(\tilde{\sigma}^a \tilde{\sigma}^b\, {}^{+}F_{ab}) + D_a[\mathrm{tr}(\tilde{\sigma}^a\, {}^{+}\mathcal{D}_b \tilde{\sigma}^b)] &= \mathrm{tr}(\tilde{\sigma}^a \tilde{\sigma}^b\, {}^{-}F_{ab}) + D_a[\mathrm{tr}(\tilde{\sigma}^a\, {}^{-}\mathcal{D}_b \tilde{\sigma}^b)]. \end{aligned} \tag{6.33}$$

In the case of first-class complex constraints linearly dependent with their complex conjugates, we can show that there exists an Hermitian BRST charge Ω satisfying the requirements,

$$\{\Omega, \Omega\} = 0, \qquad \Omega = \eta^a G_a + \eta^{\bar{a}} G_{\bar{a}} + \phi^i (Z_i{}^a \mathcal{P}_a + Z_i{}^{\bar{a}} \mathcal{P}_{\bar{a}}) + \text{“more,”} \tag{6.34}$$

where “more” means terms of higher antighost number. Here G_a and $G_{\bar{a}}$ are the constraints, η^a and $\eta^{\bar{a}}$ the ghosts, and $\mathcal{P}_a$ and $\mathcal{P}_{\bar{a}}$ are the ghost momenta. These quantities satisfy $G_a^\dagger = G_{\bar{a}}$, $\eta^{a\dagger} = \eta^{\bar{a}}$ and $\mathcal{P}_a^\dagger = -\mathcal{P}_{\bar{a}}$. The constraints satisfy the conditions $Z_i{}^a G_a + Z_i{}^{\bar{a}} G_{\bar{a}} \equiv 0$. ϕ^i is a ghost of ghost, which is real and of opposite Grassmann parity to η. We assume that ${Z_i}^{a\dagger} = -{Z_i}^{\bar{a}}$. To see that the BRST charge $\Omega = \sum_{k\geq 0} {}^{(k)}\Omega$ can be chosen real, we look at the relations on the pure anti-ghost number k pieces ${}^{(k)}\Omega$ which follow from nilpotency,

$$2\{{}^{(p+1)}\Omega, {}^{(1)}\Omega\}_{\phi,\pi} + 2\{{}^{(p+1)}\Omega, {}^{(0)}\Omega\}_{\eta,\mathcal{P}} = \\ -\sum_{k=0}^{p}\{{}^{(p-k)}\Omega, {}^{(k)}\Omega\}_{\text{orig}} - \sum_{k=0}^{p-1}\{{}^{(p-k)}\Omega, {}^{(k+1)}\Omega\}_{\eta,\mathcal{P}} - \sum_{k=0}^{p-2}\{{}^{(p-k)}\Omega, {}^{(k+2)}\Omega\}_{\phi,\pi}. \tag{6.35}$$

If the ${}^{(k)}\Omega$, $k \leq p$, are real, then ${}^{(p+1)}\Omega$ and its complex conjugate satisfy the same relations, and thus ${}^{(p+1)}\Omega$ can be chosen real.

The reducibility relations between the Ashtekar constraints and their Hermitian conjugates are almost split-polynomial (*i.e.* having each side be polynomial in either self- or anti-self-dual variables). The relation between the Gauss constraint and the vector constraint is promising, but the relation between the scalar constraint and the Gauss constraint would be split-polynomial if the divergence were of a vector density instead of a double density. This limits the usefulness of the construction and indicates that a useful BRST quantization must entail a more radical reworking of the standard formalism.

6.4 Construction of a real BRST charge

The method of the previous section produces a real BRST charge, but the method is rather cumbersome. Motivated by the form of the reducibility relations (6.33), we now construct an irreducible set of *real* constraints for self-dual gravity. As we have already discussed, the Gauss constraint (6.12) is real and the modified vector constraint (6.24) is purely imaginary. We now describe how to make the scalar constraint real.

We consider the term $\mathrm{tr}(\tilde{\sigma}^a \mathcal{D}_a \mathcal{D}_b \tilde{\sigma}^b)$. We expand the first Ashtekar derivative using $\mathcal{D}_a = D_a + \frac{i}{\sqrt{2}}\Pi_a$ to get

$$\begin{aligned}\mathrm{tr}(\tilde{\sigma}^a \mathcal{D}_a \mathcal{D}_b \tilde{\sigma}^b) = \mathrm{tr}(\tilde{\sigma}^a D_a \mathcal{D}_b \tilde{\sigma}^b) &+ \frac{i}{\sqrt{2}} \tilde{\sigma}^a{}_B{}^A \Pi_{aA}{}^D \mathcal{D}_b \tilde{\sigma}^b{}_D{}^B \\ &- \frac{i}{\sqrt{2}} \tilde{\sigma}^a{}_B{}^A \Pi_{aD}{}^B \mathcal{D}_b \tilde{\sigma}^b{}_A{}^D.\end{aligned} \tag{6.36}$$

By exchanging dummy indices, $A \leftrightarrow B$ in the first Π-term and $B \leftrightarrow D$ in the second Π-term, we can rewrite this as

$$\mathrm{tr}(\tilde{\sigma}^a \mathcal{D}_a \mathcal{D}_b \tilde{\sigma}^b) = \mathrm{tr}(\tilde{\sigma}^a D_a \mathcal{D}_b \tilde{\sigma}^b) + \frac{i}{\sqrt{2}} [\tilde{\sigma}^a, \Pi_a]_B{}^A \mathcal{D}_b \tilde{\sigma}^b{}_A{}^B. \tag{6.37}$$

We recognize the commutator as the Gauss constraint, so that we end up with a term that is quadratic in the Gauss constraint,

$$\begin{aligned}\mathrm{tr}(\tilde{\sigma}^a \mathcal{D}_a \mathcal{D}_b \tilde{\sigma}^b) &= \mathrm{tr}(\tilde{\sigma}^a D_a \mathcal{D}_b \tilde{\sigma}^b) - (\mathcal{D}_a \tilde{\sigma}^a{}_B{}^A)(\mathcal{D}_b \tilde{\sigma}^b{}_A{}^B) \\ &\equiv \mathrm{tr}(\tilde{\sigma}^a D_a \mathcal{D}_b \tilde{\sigma}^b) - \mathrm{tr}[(\mathcal{D}_a \tilde{\sigma}^a)(\mathcal{D}_b \tilde{\sigma}^b)].\end{aligned} \tag{6.38}$$

The first term on the right side of (6.38) is exactly the imaginary part of the standard scalar constraint, while the second term is purely real. Thus by subtracting (6.38) from the standard scalar constraint, we cancel the imaginary part and add a real part, leaving the modified constraint,

$$\mathrm{tr}(\tilde{\sigma}^a \tilde{\sigma}^b F_{ab}) - \mathrm{tr}(\tilde{\sigma}^a \mathcal{D}_a \mathcal{D}_b \tilde{\sigma}^b) = 0, \tag{6.39}$$

purely real. It is also very nearly polynomial. The double derivative in the last term, when expanded, contains a term $\partial_a q^{1/2}$ which unfortunately is not polynomial. The remaining terms are polynomial, as are the Gauss and vector constraints.

We now have a set of real constraints for the Ashtekar formulation of self-dual gravity,

$$\begin{aligned} \mathcal{D}_a \tilde{\sigma}^a{}_A{}^B &= 0, \\ i\,\mathrm{tr}(\tilde{\sigma}^a F_{ab}) - i\,\mathrm{tr}(A_b \mathcal{D}_a \tilde{\sigma}^a) &= 0, \\ \mathrm{tr}(\tilde{\sigma}^a \tilde{\sigma}^b F_{ab}) - \mathrm{tr}(\tilde{\sigma}^a \mathcal{D}_a \mathcal{D}_b \tilde{\sigma}^b) &= 0. \end{aligned} \tag{6.40}$$

From a BRST viewpoint, we have returned to the realm of real constraints on a real phase space in which the standard BRST methods apply. A real BRST charge is therefore known to exist, as proven by Henneaux and Teitelboim [7].

In summary, all three sets of constraints upon which AMT have constructed their BRST charges are intrinsically complex. Their BRST charges are also complex and are therefore not viable precursors to a useful quantization of self-dual gravity. I have given two methods by which a real BRST charge can be constructed for self-dual gravity. (1) Following the procedure I developed for complex extensions of real systems, I have extended the Ashtekar constraints to include their complex conjugates plus the resulting reducibility conditions. The BRST charge constructed by this approach is real but is not polynomial. (2) By a judicious remixing of the original Ashtekar constraints, I have constructed a set of constraints which are real and very nearly polynomial. This again yields a real BRST charge, which is considerably simpler than in the reducible case. My investigations suggest that the goal of constructing a BRST charge for self-dual gravity which is both real and polynomial may yet be achieved by a more clever choice of constraints.

Appendix A

Fermionic variables

In this appendix we collect the rules and notation used in this thesis for calculating with classical variables that satisfy a Grassmann algebra, commonly referred to as *fermionic variables* since they are the classical analogues of the anticommuting quantum operators associated with fermions.

Grassmann numbers ξ^i are classical numbers which have "statistics" opposite to the real numbers, namely they anticommute,

$$\xi^1\xi^2 = -\xi^2\xi^1. \tag{A1}$$

We define the Grassmann parity ϵ by

$$\epsilon(z) = 0, \qquad \epsilon(\xi) = 1, \tag{A2}$$

for any commuting variable z and anticommuting variable ξ. Commuting variables z are also referred to as Grassmann *even* and anticommuting variables ξ as Grassmann *odd*. The Grassmann parity of the product of two numbers is defined by

$$\epsilon(\alpha\beta) = \epsilon(\alpha) + \epsilon(\beta), \quad \text{mod } 2. \tag{A3}$$

In BRST theory, fermionic variables appear as ghost degrees of freedom in an extended phase space. The ghosts η^a and their conjugate momenta $\mathcal{P}_a$, both of which are anticommuting, satisfy the complex conjugation properties:

$$(\eta^a)^* = \eta^a, \qquad \mathcal{P}_a^* = -\mathcal{P}_a, \tag{A4}$$

and the Poisson bracket relation:

$$\{\mathcal{P}_a, \eta^b\} = -\delta_a^b = \{\eta^b, \mathcal{P}_a\}, \tag{A5}$$

The Poisson bracket is symmetric for Grassmann variables. The Poisson brackets of η^b and $\mathcal{P}_a$ with the original phase space variables z^A vanish:

$$\{\eta^a, z^A\} = 0 = \{\mathcal{P}_a, z^A\}, \tag{A6}$$

and the Poisson brackets $\{z^A, z^B\}$ of the original phase space variables are left unchanged. The BRST ghosts η^a are taken to be real by convention. The rule for complex conjugation of the Poisson bracket

$$\{A, B\}^* = -\{B^*, A^*\} \tag{A7}$$

then forces the momenta $\mathcal{P}_a$ to be imaginary. It is also convenient to define an additional structure on the extended phase space, that of *ghost number*, by:

$$\begin{aligned} \mathrm{gh}(z^A) &= 0, \\ \mathrm{gh}(\eta^a) = -\mathrm{gh}(\mathcal{P}_a) &= 1. \end{aligned} \tag{A8}$$

A sum of terms with different ghost numbers is said to have a ghost number which is not well defined or indefinite. The ghost number of a product of variables (with definite ghost number) is equal to the sum of their ghost numbers. We observe that the product $\eta^a\mathcal{P}_a$ is real and has ghost number zero. Similarly, we define the *antighost number* by

$$\begin{aligned} \mathrm{antigh}(z^A) &= 0 \\ \mathrm{antigh}(\mathcal{P}_a) = -\mathrm{antigh}(\eta^a) &= 1. \end{aligned} \tag{A9}$$

Appendix B

Spinors

The relationship between spinors and spacetime is explored in great detail in reference [35]. Many of the rules of spinor algebra and spinor analysis can be found in chapter 5 and Appendix A of reference [18]. For convenience, we collect in this appendix the rules and notation used in this thesis for calculating with spinors in Ashtekar gravity.

The standard representation of $SU(2)$ spinors is in terms of the Pauli matrices ${\tau^i}_A{}^B$, where i identifies the different Pauli matrices and the indices (A, B) identify the matrix elements of τ^i:

$$ {\tau^1}_A{}^B := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad {\tau^2}_A{}^B := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad {\tau^3}_A{}^B := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B1} $$

The algebra of the Pauli matrices is given by:

$$ {\tau^i}_A{}^B {\tau^j}_B{}^C = i\epsilon^{ijk} {\tau_k}_A{}^C + \delta^{ij} \delta_A{}^B. \tag{B2} $$

Given a real vector triad E^a_i, the $SU(2)$ soldering form ${\sigma^a}_A{}^B$ is defined by:

$$ {\sigma^a}_A{}^B \equiv -\frac{i}{\sqrt{2}} E^a_i {\tau^i}_A{}^B. \tag{B3} $$

The fundamental relation between $SU(2)$ spinors and the 3-metric q^{ab} follows from equations (B2) and (B3):

$$ \operatorname{tr} \sigma^a \sigma^b \equiv {\sigma^a}_A{}^B {\sigma^b}_B{}^A = -q^{ab}. \tag{B4} $$

A number of other useful relations also follow from equations (B2) and (B3):

$$[\sigma^a, \sigma^b]_A{}^B = \sqrt{2}\epsilon^{abc}\sigma_{cA}{}^B,$$
$$\mathrm{tr}(\sigma^a\sigma^b\sigma^c) = -\frac{1}{\sqrt{2}}\epsilon_{abc}, \tag{B5}$$
$$\mathrm{tr}(\sigma^a\sigma^b\sigma^c\sigma^d) = \tfrac{1}{2}(q^{ab}q^{cd} - q^{ac}q^{bd} + q^{ad}q^{bc}).$$

$SU(2)$ spinor indices are raised and lowered with the (nowhere vanishing) antisymmetric matrices

$$\epsilon_{AB} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \epsilon^{AB} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{B6}$$

where ϵ^{AB} is the inverse of ϵ_{AB} as defined by the relation $\epsilon^{AB}\epsilon_{AC} = \delta^B{}_C$. The conventions for raising and lowering spinor indices are:

$$\lambda^A = \epsilon^{AB}\lambda_B, \qquad \lambda_B = \lambda^A\epsilon_{AB}, \tag{B7}$$

where care must be taken with the order of the indices because of the antisymmetry of ϵ_{AB}. A mnemonic device for remembering these conventions is to remember that spinor summations are "from upper left to lower right".

In considering the reality properties of expressions in spinor form, we need to consider the Hermiticity properties of spinors. The Pauli spinors (B1) are manifestly Hermitian. By examining the representation of of the $SU(2)$ soldering form $\sigma^a{}_A{}^B$ in terms of Pauli matrices and the real triad E^a_i,

$$\sigma^a{}_A{}^B \equiv -\frac{i}{\sqrt{2}}E^a_i\tau^i{}_A{}^B = \frac{1}{\sqrt{2}}\begin{pmatrix} -iE^a_3 & -E^a_2 - iE^a_1 \\ E^a_2 - iE^a_1 & iE^a_3 \end{pmatrix}, \tag{B8}$$

we see that $\sigma^a{}_A{}^B$ is anti-Hermitian. The $SU(2)$ connection $A_{aA}{}^B$ has a similar representation in terms of Pauli matrices, $A_{aA}{}^B = -\frac{i}{2}A^i_a\tau^i{}_A{}^B$, but the components A^i_a are complex, so that $A_{aA}{}^B$ does not have well-defined Hermiticity properties; it is neither anti-Hermitian nor Hermitian. We recall, however, the definition of $A_{aA}{}^{B}$ in equation (5.60),

$$ {}^{\pm}A_{aA}{}^{B} = \Gamma_{aA}{}^{B} \pm \frac{i}{\sqrt{2}}\Pi_{aA}{}^{B}. $$

The "real" and "imaginary" parts of $A_{aA}{}^{B}$ are $\Gamma_{aA}{}^{B} \equiv \Gamma_{ab}\sigma^{b}{}_{A}{}^{B}$ and $\Pi_{aA}{}^{B} \equiv \Pi_{ab}\sigma^{b}{}_{A}{}^{B}$. The tensorial factors Γ_{ab} and Π_{ab} are real, so $\Gamma_{aA}{}^{B}$ and $\Pi_{aA}{}^{B}$ have the same Hermiticity properties as $\sigma_{aA}{}^{B}$ and are therefore anti-Hermitian.

The product of two $SU(2)$ matrices is, in general, neither Hermitian nor anti-Hermitian, even when the original matrices have well-defined Hermiticity properties. But the symmetrized and antisymmetrized products of Hermitian and anti-Hermitian $SU(2)$ matrices have well defined Hermiticity properties which we state without proof. We let $H_{aM}{}^{N}$ be an arbitrary Hermitian matrix and $A_{aM}{}^{N}$ be an arbitrary anti-Hermitian matrix,

$$ H_a^{\dagger} = H_a, \qquad A_a^{\dagger} = -A_a. \tag{B9} $$

The symmetrized product of two Hermitian matrices is Hermitian,

$$ [H_{(a}H_{b)}]_{M}{}^{N} \equiv H_{aM}{}^{P}H_{bP}{}^{N} + H_{bM}{}^{P}H_{aP}{}^{N} = H_{cM}{}^{N}. \tag{B10} $$

while the antisymmetrized product of two Hermitian matrices is anti-Hermitian,

$$ [H_{[a}H_{b]}]_{M}{}^{N} \equiv H_{aM}{}^{P}H_{bP}{}^{N} - H_{bM}{}^{P}H_{aP}{}^{N} = A_{cM}{}^{N}. \tag{B11} $$

We state these and similar rules more concisely as

$$ \begin{aligned} H_{(a}H_{b)} &= H_c, \qquad H_{[a}H_{b]} = A_c, \\ H_{(a}A_{b)} &= A_c, \qquad H_{[a}A_{b]} = H_c, \\ A_{(a}A_{b)} &= H_c, \qquad A_{[a}A_{b]} = A_c. \end{aligned} \tag{B12} $$

The square of an Hermitian matrix is Hermitian, as is the square of an anti-Hermitian matrix,

$$H_a H_a = H_b, \qquad A_a A_a = H_b. \tag{B13}$$

Finally, we observe that the trace of a Hermitian matrix is always real and that the trace of an anti-Hermitian matrix is always purely imaginary,

$$\operatorname{tr} H_a \equiv H_{aM}{}^M \in \mathbb{R}, \qquad \operatorname{tr} A_a \equiv A_{aM}{}^M \in \mathbb{C}. \tag{B14}$$

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