Math 375
Week 9

9.1 Normal Subgroups

We have seen the two pieces of the folowing result earlier in the course.

Theorem 1 *Let H be a subgroup of G and let $g \in G$ be a fixed element.*

a) *The set gHg^{-1} is a subgroup of G.*

b) $|H| = |gHg^{-1}|$.

Make sure that you understand this. Try proving this right now before reading the proof below. Begin by using the two-step test to prove the first part and then show that the mapping $\phi : H \to gHg^{-1}$ is an isomorphism which proves the two sets have the same number of elements.

PROOF A few weeks ago we saw that if $\phi : G_1 \to G_2$ is a group homomorphism and $H < G_1$, then $\phi(H) < G_2$. So for a fixed $g \in G$, define $\phi : G \to G$ by $\phi(x) = gxg^{-1}$. We will show that ϕ is an isomorphism and this will prove both parts of the theorem.

ϕ is a homomorphism because

$$\phi(a)\phi(b) = (gag^{-1})(gabg^{-1}) = g(ab)g^{-1} = \phi(ab).$$

ϕ is injective because by cancellation,

$$\phi(a) = \phi(b) \iff gag^{-1} = gbg^{-1} \iff a = b.$$

ϕ is surjective because for any $x \in G$ (codomain), $g^{-1}xg \in G$ (domain) since G is a group (closed). But

$$\phi(g^{-1}xg) = g(g^{-1}xg)g^{-1} = x.$$

Then observe that part (a) of the theorem is true because $\phi(H) = gHg^{-1}$ and part (b) is true because ϕ is an isomorphism: $|H| = |\phi(H)| = |gHg^{-1}|$.

We now single out certain subgroups H for special attention:

Definition 2 *Let H be a subgroup of a group G. Then H is **normal** in G if $gH = Hg$ for all $g \in G$ (left and right cosets are equal). This is denoted by writing $H \triangleleft G$.*

At this point, it is not entirely clear why you would want to look at normal subgroups. But we will see that if $H \triangleleft G$, then the set of all left cosets of H is itself a group in a natural way. By next class, we should be able to prove this.

For the moment, let's review what we done in lab and previous classes where we looked at left and right cosets. We saw that: $A_n \triangleleft S_n$, S_2 is not normal in S_3, and $< J > \triangleleft Q_8$. The next theorem gives us a criterion that makes it possible to show a $H \triangleleft G$ *without* having to actually compute gHg^{-1} for every g in G.

Theorem 3 *Let H be a subgroup of G. Then the following three conditions are equivalent.*

i) H is normal in G, that is, $gH = Hg$ for all $g \in G$;

ii) for all $g \in G$, we have $H = gHg^{-1}$;

iii) $ghg^{-1} \in H$ for every $g \in G$ and every $h \in H$, that is, $gHg^{-1} \subseteq H$.

PROOF Notice that (i) $\iff$ (ii) was done as part of the coset property theorem.

To do (ii) $\Rightarrow$ (iii) is easy: Since $gHg^{-1} = H$, then we must have $ghg^{-1} \in H$ for all $g \in G$ and all $h \in H$.

Finally, for (iii) $\Rightarrow$ (ii):)This is really just the fact that the map ϕ in Theorem 1 is onto.) $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$ implies $gHg^{-1} \subseteq H$. Now we need to show that for any $g \in G$, $H \subseteq gHg^{-1}$. That is, given any $g \in G$, we must show that any $h \in H$ can be written in the form $h = gh'g^{-1}$, for some $h' \in H$. In particular, let $x = g^{-1}$. Now set $h' = xhx^{-1}$. Then by assumption (iii), we know that $h' \in H$. So

$$h = g(g^{-1}hg)g^{-1} = g(xhx^{-1})g = gh'g^{-1} \in gHg^{-1},$$

which is what we wanted to show. ∎

Example **a)** Use condition (iii) to show that $SL(\mathbf{R}, n) \triangleleft GL(\mathbf{R}, n)$.

b) Along the same lines: Let $GL^+(\mathbf{R},)$ be the the set of matrices with positive determinant. (This is a subgroup of $GL(\mathbf{R}, n)$. You might check that.) Show that $GL^+(\mathbf{R}, n) \triangleleft GL(\mathbf{R}), n$. □

Theorem 4 *Let H be a subgroup of G such that $[G : H] = 2$. Then $H \triangleleft G$.*

PROOF The two left cosets are H and $G - H$. But so are the right. ∎

Example 2 Now give another proof that $GL^+(\mathbf{R}, n) \triangleleft GL(\mathbf{R}, n)$.

Theorem 5 *$C(G) \triangleleft G$. Further, if H is a subgroup of $C(G)$, then $H \triangleleft G$.*

PROOF Let H be a subgroup of $C(G)$. Then $\forall g \in G$ and $\forall h \in H$, we have $ghg^{-1} = hgg^{-1} = h \in H$. So by condition (iii) of the Theorem 3, $H \triangleleft G$.

COROLLARY 6 *Let H be a subgroup of an abelian group G. Then $H \triangleleft G$.*

PROOF Note $C(G) = G$. ∎

COROLLARY 7 *Let H be a subgroup of a cyclic group G. Then $H \triangleleft G$.*

PROOF G is cyclic so it is abelian. ∎

There are other simple criteria for normality which depend more on the subgroup H, than the group G. Here's one that is sometimes useful.

COROLLARY 8 *If H is a subgroup of G and no other subgroup of G has the same order as H, then H is normal.*

PROOF We know that for any $g \in G$ that gHg^{-1} is a subgroup of G and that $|H| = |gHg^{-1}|$. But there are no subgroups of the order of H in G except H itself. So we must have $gHg^{-1} = H$ for all g. ∎

EXAMPLE 3 Let $G = S_3 \oplus \mathbf{Z}_5$. Then clearly G is not abelian since S_3 is not. Let

$$H =< ((1),1) >= \{((1),1),\ ((1),2),\ ((1),3),\ ((1),4),\ ((1),0)\}.$$

Show that $H \triangleleft G$.

SOLUTION Notice that $[G : H] = 6 \neq 2$ and G is not abelian so none of the obvious results on normality apply. However if we can show that H is the only subgroup of G of order 5, then $H \triangleleft G$. We can do this by using a proof by contradiction.

Suppose some other subgroup K also had order 5. Then K is cyclic (why?), thus $K =< (\alpha, n) >$ (why?) where $\alpha \in S_3$ and $n \in \mathbf{Z}_5$. We know that

$$5 = |< (\alpha, n) >| = \text{lcm}(|\alpha|, |n|).$$

What are the possible orders of α? What are the possible orders of n? Show that the only way the lcm to be 5 is for $|\alpha| = 1$ and $|n| = 5$. Thus $\alpha = (1)$ and n can be 1, 2, 3, or 4 (why?). In any of these cases $K =< ((1), n) >= H$. Thus $H \triangleleft G$. □

EXAMPLE 4 Can you also do this same problem above by showing H is a subgroup of $C(G)$?

EXAMPLE 5 For another illustration of this last corollary, see p. 187 #39.

Definition 9 *Let H be a subgroup of G. Let*

$$N(H) = \{g \in G \mid\ gHg^{-1} = H\}.$$

$N(H)$ is called the ***normalizer of*** *H in G.*

Example

a) See if you can show that the normalizer of $H = < v >$ in D_4 is $K = \{r_0, r_{180}, v, h\}$. Use the table on page 31.

b) Find the normalizer of another subgroup of a nonabelian group. □

The normalizer of a subgroup is important because:

Lemma 10

a) *$N(H)$ is a subgroup of G;*

b) *$H \subseteq N(H)$;*

c) *$H \lhd N(H)$.*

A Use the one-step test: Let $a, b \in N(H)$. This means that $aHa^{-1} = H$ and $bHb^{-1} = H$. The latter implies that $H = b^{-1}Hb$. We must show $ab^{-1} \in N(H)$, that is, $(ab^{-1})H(ab^{-1})^{-1} = H$. But

$$(ab^{-1})H(ab^{-1})^{-1} = (ab^{-1})H(ba^{-1}) = a(b^{-1}Hb)a^{-1} = aHa^{-1} = H.$$

B Let $h \in H$. We must show that $h \in N(H)$, i.e., $hHh^{-1} = H$. One of the basic coset properties is that $aH = H \iff a \in H$. But here h and $h^{-1} \in H$ so

$$hHh^{-1} = (hH)h^{-1} = Hh^{-1} = H.$$

Thus $h \in N(H)$, that is, $H \subseteq N(H)$.

C Show $H \lhd N(H)$. Let $a \in N(H)$. Then by definition of $N(H)$, $aHa^{-1} = H$. □

Problems for Section 9.1

1. Read Chapter 9.

2.

a) Show that the n rotations in D_n are a normal subgroup of D_n (where $n \geq 3$).

b) p. 185 #8.

c) p. 188 #47.

3. This example shows that normality is not transitive. Let Let $G = D_4$, let $K = \{r_0, r_{180}, v, h\}$, and let $H = \{r_0, v\}$.

a) Show that K is a subgroup of D_4 (just write out the group table for K). Of course H is a subgroup of both D_4 and K since $H =< v >$.

b) What are $[D_4 : K]$ and $[K : H]$?

c) Show that $H \triangleleft K$ and $K \triangleleft G$.

d) Now show that H is not normal in D_4. This shows normality is complicated! $\square$

4. Let $x \in G$. Show that $< x > \triangleleft G$ if and only if for each $g \in G$ there is some integer n so that $gxg^{-1} = x^n$.

5. Here's a more challenging problem. It's not that hard. Do it for extra credit. Let G be an abelian group

a) Show that if $|x| = n$, then $|gxg^{-1}| = n$ for any $g \in G$.

b) Let $T = \{x \in G \mid x^n = e \text{ for some } n \in \mathbf{N}\}$. T is just the set of all elements in G of finite order. Show that $T \triangleleft G$.