# Math 375
# Week 4

## 4.1 Mappings

From previous calculus experience you are familiar with the notion of a function. From Math 204 you are familiar with (linear) transformations. These notions can be understood as particular examples of the more general notion of *mapping* or *general function*.

Most often you think if a calculus function as a *rule* or *formula*. For example:

$$y = f(x) = x^2 + 1,$$

or in terms of the points on the graph of this function we have the pairs: $(x, y) = (x, x^2 + 1)$. What makes such a rule a 'function' is that when we input a particular numerical value for $x$ we obtain a unique number $y$ as the output, that is, we have only one $y$ value for each input $x$ value.

Some formulas, of course, fail to yield functions for this very reason. For instance, $x^2 + y^2 = 1$ is the equation of a unit circle. This time certain values of $x$ are paired with 2 values of $y$:

$$(x, \pm\sqrt{1 - x^2}).$$

The second element of the ordered pair is *not uniquely determined* by the first element of the pair. The notion of a function or mapping which we will use uses this notion of uniquely determined ordered pairs.

### The Definition

**Definition 1** *A* ***mapping*** *or* ***function*** *$\phi$ from a set $S$ to a set $T$ is a set of ordered pairs $(s, t)$ where $s \in S$ and $t \in T$ such that each element $s \in S$ occurs as the first element of one and only one ordered pair $(s, t)$. Informally, $\phi$ is a rule that assigns to each element $s \in S$, exactly one element $t \in T$. $S$ is called the* ***domain*** *and $T$ is called the* ***codomain****. If $\phi$ assigns $s$ to $t$, we say that $t$ is the* ***image*** *of $s$ under $\phi$. The subset of $T$ of all the images of elements of $S$ is called the* ***image*** *of $S$ under $\phi$.*

Notation: $\phi : S \to T$, $\phi$ takes (maps) $S$ to $T$.

**Definition 2** *Let $\phi : S \to T$ and $\lambda : T \to U$, The* ***composition*** *$\lambda\phi$ is the mapping defined by $(\lambda\phi)(s) = \lambda(\phi(s))$ for all $s \in S$.*

**Example 1** Draw a picture. Let $\phi : GL(n, \mathbf{R}) \to GL(n, \mathbf{R})$ by $\phi(A) = A^{-1}$ and let $\lambda : GL(n, \mathbf{R}) :\to \mathbf{R}^*$ by $B\lambda = \det B$. Then the composite $\phi\lambda : GL(n, \mathbf{R}) \to \mathbf{R}^*$ by $(\lambda\phi)(A) = \lambda(\phi(A)) = \det(A^{-1}) = 1/\det A$. $\square$

**Example**

**a)** Let $\mathcal{G}$ = set of all finite groups. Let $\phi : \mathcal{G} \to \mathbf{Z}^+$ by $\phi(G) = |G|$.

**b)** Let $\mathcal{C}^1$ = set of continuously differentiable functions on $\mathbf{R}$. Let $\mathcal{C}$ be the continuous functions on $\mathbf{R}$. Define $\delta : \mathcal{C}^1 \to \mathcal{C}$ by $\delta(f) = f'$.

**c)** Let $\lambda : \mathbf{R}^3 \to \mathbf{R}$ by $\lambda(a, b, c) = |(a, b, c)| = \sqrt{a^2 + b^2 + c^2}$.

**d)** Let $f : \mathbf{Z} \to \mathbf{Z}$ by $f(n) = 2n$.

**e)** Let $i_X : X \to X$ by $i_X(x) = x$; $i_X$ is usually called the **identity** function on $X$.

## One-To-One and Onto Mappings

Two of the most important types of mappings are those that are one-to-one and those that are onto. Both of these notions should be familiar to you from your work with transformations in Math 204 and even your work with inverse functions in Math 131 (there a function had an inverse if and only if it was one-to-one).

**Definition 3** *A function $\phi : S \to T$ is* ***one-to-one*** *if whenever $\phi(s_1) = \phi(s_2)$, then $s_1 = s_2$. ($\phi$ is also called* ***injective****.)*

**Definition 4** *A function $\phi : S \to T$ is* ***onto*** *if each $t \in T$ is the image of at least one $s \in S$, that is, there is at least one $s \in S$ so that $\phi(s) = t$. ($\phi$ is also called* ***surjective****.)*

Informally, if everything in $T$ gets "hit" at least once, then $\phi$ is onto; if nothing in $T$ gets hit twice (or more) then $\phi$ is one-to-one. Draw a picture.

**Example**

**a)** Show $\phi : GL(n, \mathbf{R}) \to GL(n, \mathbf{R})$ by $\phi(A) = A^{-1}$ is both injective and surjective.

**b)** Show that $\det : GL(n, \mathbf{R}) \to \mathbf{R}$ is surjective, but not injective.

**Lemma 5** *Suppose that $\phi : S \to T$ and $\lambda : T \to U$ are both one-to-one. Then the composition $\lambda\phi$ is one-to-one.*

PROOF Let $a, b \in S$

$$\begin{aligned}(\lambda\phi)(a) = (\phi\lambda)(b) &\iff \lambda(\phi(a)) = \lambda(\phi(b)) \\ &\iff \phi(a) = \phi(b) && \text{since } \lambda \text{ is injective} \\ &\iff a = b && \text{since } \phi \text{ is injective}\end{aligned}$$

Thus, $\lambda\phi$ is injective. □

**Lemma 6** *Suppose that $\phi : S \to T$ and $\lambda : T \to U$ are both onto. Then $\lambda\phi$ is onto.*

SOLUTION Let $u \in U$. We need to find $s \in S$ so that $(\lambda\phi)(s)) = u$, that is, $\lambda(\phi(s)) = u$. Now $\lambda$ is surjective so there exists $t \in T$ so that $\lambda(t) = u$. But $\phi$ is also surjective so we can hit this value of $t$ with $\phi$. That is, there exists $s \in S$ so that $\phi(s) = t$. Thus, $(\lambda\phi)(s)) = \lambda(t) = u$, as needed. □

**Theorem 7** ***(Properties of Functions)*** *Let $\phi : S \to T$, $\lambda : T \to U$, and $\gamma : U \to V$. Then:*

***a)*** *$\gamma(\lambda\phi) = (\gamma\lambda)\phi$;*

***b)*** *if $\phi$ is one-to-one and onto, then there is a function $\phi^{-1} : T \to S$ so that $(\phi\phi^{-1})(t) = i_T(t) = t$ for all $t \in T$ and $(\phi^{-1}\phi)(s) = i_S(s) = s$ for all $s \in S$. ($\phi^{-1}$ is the* ***inverse*** *of $\phi$.)*

Why would we ever want to prove such a theorem? Can you illustrate this with a picture?

REMARK Since we have a function $\phi : S \to T$ that is both injective and surjective, then each element of $S$ is paired with one element of $T$ and *each element of $T$ is paired with exactly one element of $S$*. That means we are able to define a function backwards from $T$ to $S$. Defining this function is easy: if $(s, t)$ is an ordered pair for the function $\phi$, then there is a corresponding ordered pair of the function $\phi^{-1}$ namely $(t, s)$. We know that $\phi^{-1}$ is defined for all $t \in T$ since $\phi$ is onto, and we know that $\phi^{-1}$ is well-defined for all $t \in T$ because the original function $\phi$ was one-to-one.

PROOF For (a) see the text. For (b): More precisely, take any $t \in T$. Since $\phi$ is onto, there is some $s \in S$ so that $\phi(a) = b$. But $\phi$ is also one-to-one, so there is only one such element $s$ with this property. So define $\phi^{-1}(t)$ to be $s$. Notice that the composites now work as advertised. ∎

Let's see how things could go wrong. Let $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = x^2$. Notice that $f$ is not injective. For example $f(2) = f(-2) = 4$. If we tried to define $f^{-1}(4)$ we would have two choices, 2 or $-2$. But the function $f^{-1}$ should have only one output for each input. Further notice that $f$ is not surjective. For example, there is no value of $x$ such that $f(x) = -1$. So if we tried to define $f^{-1}(-1)$ there is nothing to send it back to.

**Example 4** An important calculus example concerns the log and exponential functions. Let $f : \mathbf{R}^+ \to \mathbf{R}$ by $f(x) = \ln x$. Since the natural log is an increasing function it is injective. By the intermediate value theorem it is also surjective. So it has an inverse $f^{-1} : \mathbf{R}^+ \to \mathbf{R}$ by $f^{-1}(x) = e^x$.

**Example**

**a)** Let $G$ be a group. Let $a$ be a fixed element of $G$. Show that $\phi : G \to G$ by $\phi(g) = ag$ is one-to-one and onto. What is $\phi^{-1}$?

**b)** Let $P$ be the set of polynomials in $x$. Define $\delta : P \to P$ by $\delta(f) = f'$. Why is $\delta$ not one-to-one? Show that $\delta$ is onto. (Let $f \in P$, let $F = \int f(x)\, dx$. Then $\delta(F) = f$.)