**1.** Briefly justify your answers to the following questions.

**a)** Determine $[\mathbf{Z}_{10} \oplus S_4 :< (6, (13)(24)) >]$.

**b)** Is $A_n \triangleleft S_n$? Explain.

**c)** Is $< \pi >$ normal in $\mathbf{R}$? Explain.

**d)** Find the order of $(r_{270}, (1243)(145), 3)$ in $D_4 \oplus S_5 \oplus U(13)$.

**e)** Determine $[\mathbf{C}^* :< i >]$.

**2. a)** Suppose that $G$ is an *abelian* group order 80 with elements of various orders as described below. Use the Fundamental Theorem of Finite Abelian Groups as well as other results we have discussed in class to determine which product of finite cyclic groups is isomorphic to $G$. Be sure to explain how you arrived at your answer.

1 element of order 1;
7 elements of order 2;
8 elements of order 4;
4 elements of order 5;
28 elements of order 10;
32 elements of order 20.

**b)** Which of the following three groups, if any, are isomorphic? Explain.

$$\mathbf{Z}_{50} \oplus \mathbf{Z}_{24} \qquad \mathbf{Z}_{75} \oplus \mathbf{Z}_{16} \qquad \mathbf{Z}_{200} \oplus \mathbf{Z}_{6}$$

**3.** **a)** Find an element of order 8 in $\mathbf{Z}_{10} \oplus \mathbf{Z}_{12}$ or explain why none exists.

**b)** Find a subgroup of order 8 in $\mathbf{Z}_{10} \oplus \mathbf{Z}_{12}$ or explain why none existis.

**c)** Suppose that $|G| = n$ and that $k$ is a divisor of $n$. Must $G$ have an element of order $k$? Explain.

**4.** Each part below is a separate question. Find groups $G$ and $H$ that satisfy the condition or state why it is impossible.

**a)** $|G \oplus H| = 35$ and $G \oplus H$ is not abelian and neither group has order 1.

**b)** $|G \oplus H| = 72$ which is cyclic and neither group has order 1.

**c)** Find two non-isomorphic groups $G$ and $H$ of order 32 that are not abelian. Make sure to justify your answer.

**5. a)** Give a careful definition of what it means for a subgroup $H$ of a group $G$ to be *normal* in $G$.

**b)** Let $\phi : G_1 \to G_2$ be a group homomorphism. On the last exam I asked you to prove that the set

$$H = \{h \in G_1 \mid \phi(h) = e_2\}$$

is a subgroup of $G_1$. For this exam, please prove that $H$ is normal in $G_1$. (Note: $H$ is called the **kernel** of $\phi$.)

**6.** The Cayley table for $D_4$ is given below, though the elements are listed in a different order.

| $*$ | $r_0$ | $r_{180}$ | $h$ | $v$ | $r_{90}$ | $r_{270}$ | $a$ | $b$ |
|---|---|---|---|---|---|---|---|---|
| $r_0$ | $r_0$ | $r_{180}$ | $h$ | $v$ | $r_{90}$ | $r_{270}$ | $a$ | $b$ |
| $r_{180}$ | $r_{180}$ | $r_0$ | $v$ | $h$ | $r_{270}$ | $r_{90}$ | $b$ | $a$ |
| $h$ | $h$ | $v$ | $r_0$ | $r_{180}$ | $b$ | $a$ | $r_{270}$ | $r_{90}$ |
| $v$ | $v$ | $h$ | $r_{180}$ | $r_0$ | $a$ | $b$ | $r_{90}$ | $r_{270}$ |
| $r_{90}$ | $r_{90}$ | $r_{270}$ | $a$ | $b$ | $r_{180}$ | $r_0$ | $v$ | $h$ |
| $r_{270}$ | $r_{270}$ | $r_{90}$ | $b$ | $a$ | $r_0$ | $r_{180}$ | $h$ | $v$ |
| $a$ | $a$ | $b$ | $r_{90}$ | $r_{270}$ | $h$ | $v$ | $r_0$ | $r_{180}$ |
| $b$ | $b$ | $a$ | $r_{270}$ | $r_{90}$ | $v$ | $h$ | $r_{180}$ | $r_0$ |

Let $K = \{r_0, r_{180}, h, v\}$, and let $H =< h >= \{r_0, h\}$.

**a)** Prove in one sentence that $K$ is a subgroup of $D_4$.

**b)** What are $[D_4 : K]$ and $[K : H]$?

**c)** Prove in one sentence that $H \triangleleft K$ and $K \triangleleft D_4$.

**d)** Determine whether $H$ is normal in $D_4$.

**e)** Agree or disagree: Normality is transitive, that is, if $A \triangleleft B$ and $B \triangleleft C$, then $A \triangleleft C$. Explain.

**7. a)** Suppose that $H = \{e, h\}$ is a two element subgroup of a group $G$. Prove: If $H \triangleleft G$, then $H$ is contained in the center of $G$.

**b)** Let $x \in G$. If $< x > \triangleleft G$ , then for each $g \in G$ there is some integer $n$ so that $gxg^{-1} = x^n$.

**0.** **a)** Beware of typos below. For next class, **read** Chapter 9 through page 179. Begin looking at Chapter 10.

**b)** Practice: Let $\phi : GL(\mathbf{R}, n)/SL(\mathbf{R}, n) \to \mathbf{R}^*$ given by $\phi(aH) = \det a$ is an isomorphism. Remember to first show that the map is well-defined, that is, if $aH = bH$, show that $\phi(aH) = \phi(bH)$.

**c)** Practice: Page 185–187 #11, 12, 15, 16, 17, 20–23, 42

**1.** **a)** Determine $[\mathbf{Z}_{10} \oplus S_4 :< (6, (13)(24)) >]$.

$$\frac{|\mathbf{Z}_{10}| \cdot |S_4|}{\operatorname{lcm}(|6|, |(13)(24)|)} = \frac{10 \cdot 24}{\operatorname{lcm}(5, 2)} = 24.$$

**b)** Is $A_n \triangleleft S_n$? Explain. Yes. Since $[S_n : A_n] = 2$, it follows that $A_n \triangleleft S_n$.

**c)** Is $< \pi >$ normal in $\mathbf{R}$? Explain.

Yes, $\mathbf{R}$ is abelian so any subrgroup is normal.

**d)** Find the order of $(r_{270}, (1243)(145), 3)$ in $D_4 \oplus S_5 \oplus U(13)$.

$$|(r_{270}, (1243)(145), 3)| = |(r_{270}, (13)(245), 3)| = \operatorname{lcm}(4, 6, 3) = 12.$$

**e)** Determine $[\mathbf{C}^* :< i >]$.

The index is infinite since $|\mathbf{C}^*|$ is infinite and $|i|$ is 4. So there would need to be an infinite number of cosets to partition $\mathbf{C}^*$.

**2.** **a)** Suppose that $G$ is an *abelian* group order 80 with elements of various orders as described below. Use the Fundamental Theorem of Finite Abelian Groups as well as other results we have discussed in class to determine which product of finite cyclic groups is isomorphic to $G$. Be sure to explain how you arrived at your answer.

1 element of order 1;
7 elements of order 2;
8 elements of order 4;
4 elements of order 5;
28 elements of order 10;
32 elements of order 20.

Note that $80 = 5 \times 2^4$. We can eliminate $\mathbf{Z}_{16} \oplus \mathbf{Z}_5$ and $\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_5$ since each of these has elements of order 80 and 40 respectively. Likewise, we can eliminate $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_5$ since it has no elements of order 10. Next, since $\mathbf{Z}_4$ has only 1 element of order 2 and 1 element of order 1 and $\mathbf{Z}_5$ has no elements of order 2 and 1 element of order 1, $\mathbf{Z}_4 \oplus \mathbf{Z}_4 \oplus \mathbf{Z}_5$ has only 3 elements of order 2. Therefore, the only choice remaining is $\mathbf{Z}_4 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_5 \cong \mathbf{Z}_{20} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

**b)** Which of the following three groups, if any, are isomorphic? Explain.

$$\mathbf{Z}_{50} \oplus \mathbf{Z}_{24} \qquad \mathbf{Z}_{75} \oplus \mathbf{Z}_{16} \qquad \mathbf{Z}_{200} \oplus \mathbf{Z}_6$$

We have $\mathbf{Z}_{50} \oplus \mathbf{Z}_{24} \cong \mathbf{Z}_{25} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_3$, while $\mathbf{Z}_{75} \oplus \mathbf{Z}_{16} \cong \mathbf{Z}_{25} \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_{16}$, and $\mathbf{Z}_{200} \oplus \mathbf{Z}_6 \cong \mathbf{Z}_{25} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_3$. So $\mathbf{Z}_{50} \oplus \mathbf{Z}_{24} \cong \mathbf{Z}_{200} \oplus \mathbf{Z}_6$.

**3.** **a)** Find an element of order 8 in $\mathbf{Z}_{10} \oplus \mathbf{Z}_{12}$ or explain why none exists.

Impossible. The orders of elements in $\mathbf{Z}_{10}$ are 1, 2, 5, an 10 while the elements of $\mathbf{Z}_{12}$ have orders 1, 2, 3, 4, 6, and 12. No combination of these orders has an lcm of 8.

**b)** Find a subgroup of order 8 in $\mathbf{Z}_{10} \oplus \mathbf{Z}_{12}$ or explain why none existis.

Use $< 5 > \oplus < 3 >$.

**c)** Suppose that $|G| = n$ and that $k$ is a divisor of $n$. Must $G$ have an element of order $k$? Explain.

No, as part (a) shows. $8 \mid 120$ but $G = \mathbf{Z}_{10} \oplus \mathbf{Z}_{12}$ has no element of order 8.

**4.** Each part below is a separate question. Find groups $G$ and $H$ that satisfy the condition or state why it is impossible.

**a)** $|G \oplus H| = 35$ and $G \oplus H$ is not abelian and neither group has order 1.

Impossible! Since $|G \oplus H| = |G| \times |H|$, the only possible orders for $G$ and $H$ are 5 and 7 which are prime. All groups of prime order are cyclic and hence abelian. Since the product of abelian groups is abelian, then $G \oplus H$ would have to be abelian. In this case, the product would even be cyclic because $\mathrm{lcm}(5, 7) = 1$.

**b)** $|G \oplus H| = 72$ which is cyclic and neither group has order 1.

Use cylic groups whose order is relatively prime which ensures that the product is cyclic. Use $\mathbf{Z}_8 \oplus \mathbf{Z}_9$.

**c)** Find two non-isomorphic groups $G$ and $H$ of order 32 that are not abelian. Make sure to justify your answer.

Here are thre such groups. Use $D_{16}$ which has order 32 and has 8 elements of order 16. Compare to $D_8 \times \mathbf{Z}_2$. $D_8$ has elements of order 1, 2, 4, and 8 while $\mathbf{Z}_2$ has elements of order 1 and 2. So $D_8 \times \mathbf{Z}_2$ has no elements of order 16, but does have elements of order 8. Similarly, $D_4 \oplus \mathbf{Z}_4$ has no elements of order 16 or 8.

**5.** **a)** Give a careful definition of what it means for a subgroup $H$ of a group $G$ to be *normal* in $G$.

$H$ is normal in $G$ if for all $a \in G$, the left coset $aH$ is the same as the right coset $Ha$.

**b)** Let $\phi : G_1 \to G_2$ be a group homomorphism. On the last exam I asked you to prove that the set

$$H = \{h \in G_1 \mid \phi(h) = e_2\}$$

is a subgroup of $G_1$. For this exam, please prove that $H$ is normal in $G_1$.

We use condition (iii) for normality. To show $H \triangleleft G$, we must show that $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$. That is, we must show that $\phi(ghg^{-1}) = e_2$. But $\phi$ is a group homomorphism. Because $h \in H$, since $\phi(h) = e_2$, so

$$\phi(ghg^{-1}) = \phi(g)\phi(h)\phi(g^{-1}) = \phi(g)e_2(\phi(g))^{-1} = \phi(g)(\phi(g))^{-1} = e_2.$$

So, $ghg^{-1} \in H$.

**6.** The Cayley table for $D_4$ is given below, though the elements are listed in a different order.

| $*$ | $r_0$ | $r_{180}$ | $h$ | $v$ | $r_{90}$ | $r_{270}$ | $a$ | $b$ |
|---|---|---|---|---|---|---|---|---|
| $r_0$ | $r_0$ | $r_{180}$ | $h$ | $v$ | $r_{90}$ | $r_{270}$ | $a$ | $b$ |
| $r_{180}$ | $r_{180}$ | $r_0$ | $v$ | $h$ | $r_{270}$ | $r_{90}$ | $b$ | $a$ |
| $h$ | $h$ | $v$ | $r_0$ | $r_{180}$ | $b$ | $a$ | $r_{270}$ | $r_{90}$ |
| $v$ | $v$ | $h$ | $r_{180}$ | $r_0$ | $a$ | $b$ | $r_{90}$ | $r_{270}$ |
| $r_{90}$ | $r_{90}$ | $r_{270}$ | $a$ | $b$ | $r_{180}$ | $r_0$ | $v$ | $h$ |
| $r_{270}$ | $r_{270}$ | $r_{90}$ | $b$ | $a$ | $r_0$ | $r_{180}$ | $h$ | $v$ |
| $a$ | $a$ | $b$ | $r_{90}$ | $r_{270}$ | $h$ | $v$ | $r_0$ | $r_{180}$ |
| $b$ | $b$ | $a$ | $r_{270}$ | $r_{90}$ | $v$ | $h$ | $r_{180}$ | $r_0$ |

Let $K = \{r_0, r_{180}, h, v\}$, and let $H =< h >= \{r_0, h\}$.

**a)** Prove in one sentence that $K$ is a subgroup of $D_4$.

By the finite subgroup test, $K$ is closed in $D_4$, just look at the upper left $4 \times 4$ corner of the table!

**b)** What are $[D_4 : K]$ and $[K : H]$?

The indices are $[D_4 : K] = \frac{8}{4} = 2$ and $[K : H] = \frac{4}{2} = 2$.

**c)** Prove in one sentence that $H \triangleleft K$ and $K \triangleleft D_4$.

Both subgroups have index 2 so are normal.

**d)** Determine whether $H$ is normal in $D_4$.

No, it is not normal: $aH = \{a, r_{90}\}$ while $Ha = \{a, r_{270}\}$.

**e)** Agree or disagree: Normality is transitive, that is, if $A \triangleleft B$ and $B \triangleleft C$, then $A \triangleleft C$. Explain.

Disagree because we have just seen $H \triangleleft K$ and $K \triangleleft D_4$, but $H$ is not normal in $D_4$.

**7. a)** Suppose that $H = \{e, h\}$ is a two element subgroup of a group $G$. Prove that If $H \triangleleft G$, then $H$ is contained in the center of $G$.

Notice that since $H$ is normal, then for all $g \in G$, $gH = Hg$ or $\{g, gh\} = \{g, hg\}$. But this means $gh = hg$ for all $g \in G$. That is, $h \in C(G)$. Of course, $e \in C(G)$, so $H \subset C(G)$.

**b)** Let $x \in G$. If $< x > \triangleleft G$ , then for each $g \in G$ there is some integer $n$ so that $gxg^{-1} = x^n$.

If $< x > \triangleleft G$ , then by condition (iii) $\forall g \in G$, we have $gxg^{-1} \in < x >$. But then $gxg^{-1} \in < x > \iff gxg^{-1} = x^n$ for some $n \in \mathbf{Z}$.