1. Let β be the product of the following elements of S_6:

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 6 & 4 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 4 & 6 \end{pmatrix}.$$

a) What is $|\beta|$?

b) What is β^{-1}?

c) Is $\beta \in A_6$? Explain, please.

2. **a)** Give an example of non-abelian cyclic group and its generator, or explain why no such example exists.

b) Assume $G =< a >$, where $|a| = 18$. What are all the other generators of G?

c) Find $|200|$ in $\mathbf{Z}_{216}$.

3. a) Let $\phi : G_1 \to G_2$ be a group homomorphism. The **kernel of** ϕ is the set

$$K_1 = \{a_1 \in G_1 \mid \phi(a_1) = e_2\}.$$

(In words, K_1 is the stuff in G_1 that gets mapped to the identity in G_2.) Carefully prove that K_1 a subgroup of G_1.

b) Let $\phi : G_1 \to G_2$ be a group homomorphism. Let H_2 be a subgroup of G_2. Define

$$H_1 = \{a_1 \in G_1 \mid \phi(a_1) = a_2 \in H_2\}.$$

(In words, H_1 is the stuff in G_1 that gets mapped to the subgroup H_2 in G_2.) Carefully prove that H_1 a subgroup of G_1.

c) Can you use (b) to give a one sentence proof of (a)?

4. Briefly *justify* your answers to **any 3** of the following.

a) Let $\alpha \in S_n$. Is $\alpha^2 \in A_n$?

b) Are $\mathbf{Z}$ and $\mathbf{R}^*$ isomorphic groups? Make sure to explain your answer.

c) Give an example of finite groups G and H such that $|G| = |H|$ but G is not isomorphic to H or explain why no example exists.

d) Find an element $\beta \in S_5$ such that $\beta(23)(45) = (34)\beta$, or explain why no such element exists.

5. **a)** Suppose that $\phi : D_4 \to \mathbf{Z}_8$ is a group homomorphism. What is $\phi(r_0)$? Explain how you know this.

b) Suppose, again, that $\phi : D_4 \to \mathbf{Z}_8$ is a group homomorphism. Let v be the vertical flip. If $\phi(v) = k$, what are the possible choices for k in $\mathbf{Z}_8$? Again, explain your answer.

c) Can a homomorphism $\phi : D_4 \to \mathbf{Z}_8$ be an isomorphism? Explain.

d) Can a homomorphism $\phi : D_4 \to \mathbf{Z}_8$ be onto? Explain.

6. Is the mapping $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ by $\phi(a) = \sqrt{a}$ injective? Surjective? A group homomorphism? An isomorphism? Show all details. (Caution: The operation is multiplication.)

7. Let $\phi : \mathbf{Z}_6 \to \mathbf{Z}_3$ be the homomorphism determined by $\phi(5_6) = 2_3$. Make a list of the elements of $\mathbf{Z}_6$ and determine what each is mapped to under ϕ. Is ϕ injective? Surjective? An isomorphism?

8. (Extra Credit) Prove Colin's Theorem: If $H < S_n$ and $|H|$ is odd, then H contains only even elements. Hint: Think about the homework problem with odd and even elements in subgroups of S_n.

1. Let β be the product of the following elements of S_6:

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 6 & 4 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 4 & 6 \end{pmatrix}.$$

a) What is $|\beta|$?

$\beta = (1543)(26)$, so $|\beta| = \text{lcm}(4,2) = 4$.

b) What is β^{-1}?

$\beta^{-1} = (62)(3451)$.

c) Is $\beta \in A_6$? Explain, please.

$\beta = (13)(14)(15)(26)$ is the product of an even number of transpositions, so $\beta \in A_n$.

2. **a)** Give an example of non-abelian cyclic group and its generator, or explain why no such example exists.

Impossible. Cyclic groups are always abelian.

b) Assume $G =< a >$, where $|a| = 18$. What are all the other generators of G?

By Sam's Theorem, we need elements whose powers are relatively prime to 18; so the generators are: $a^5, a^7, a^{11}, a^{13}, a^{17}$.

c) Find $|200|$ in $\mathbf{Z}_{216}$.

Find the gcd: $216 = 1 \cdot 200 + 16 \Rightarrow 200 = 12 \cdot 16 + 8 \Rightarrow 16 = 2 \cdot 8 + 0$. So $\gcd(216, 200) = 8$. Therefore $|200| = \frac{216}{\gcd(216,200)} = \frac{216}{8} = 27$.

3. **a)** Let $\phi : G_1 \to G_2$ be a group homomorphism. The **kernel of** ϕ is the set $K_1 = \{a_1 \in G_1 \mid \phi(a_1) = e_2\}$. Carefully prove that K_1 a subgroup of G_1.

Use the two-step method. Closure. Let $a, b \in K_1$. Show that $ab \in K$. But $a, b \in K_1$ means that $\phi(a) = \phi(b) = e_2$. But then $\phi(ab) = \phi(a)\phi(b) = e_2 e_2 = e_2$. Therefore, $ab \in K_1$. Inverses: Let $a \in K$. Show $a^{-1} \in K_1$. But $a \in K_1$ so $\phi(a) = e_1$. Therefore, $\phi(a^{-1}) = [\phi(a)]^{-1} = e_2^{-1} = e_2$. Yes, it is a subgroup.

b) Let $\phi : G_1 \to G_2$ be a group homomorphism. Let H_2 be a subgroup of G_2. Define $H_1 = \{a_1 \in G_1 \mid \phi(a_1) = a_2 \in H_2\}$. Carefully prove that H_1 a subgroup of G_1.

Use the two-step method. Closure. Let $a_1, b_1 \in H_1$. Show that $a_1 b_1 \in H$. But $a_1, b_1 \in H_1$ means that $\phi(a) = a_2 \in H_2$ and $\phi(b_1) = b_2 \in H_2$. But then $\phi(a_1 b_1) = \phi(a_1)\phi(b_1) = a_2 b_2 \in H_2$, since H_2 is closed. Therefore, $a_1 b_1 \in H_1$. Inverses: Let $a_1 \in H_1$. Show $a_1^{-1} \in H_1$. But $a_1 \in H_1$ so $\phi(a_1) = a_2 \in H_2$. Therefore, $\phi(a_1^{-1}) = [\phi(a_1)]^{-1} = a_2^{-1} \in H_2$ because H_2 is a subgroup. So $a_1^{-1} \in H_1$. Yes, it is a subgroup.

c) Can you use (b) to give a one sentence proof of (a)?

Yes, $\{e_2\} < G_2$, so $K_1 = \{a_1 \in G_1 \mid \phi(a_1) = e_2\}$ is a subgroup of G_1 by (b).

4. Briefly *justify* your answers to **any 3** of the following.

a) Let $\alpha \in S_n$. Is $\alpha^2 \in A_n$?

Yes. If α is the product of k two-cycles, then $\alpha^2 = \alpha\alpha$ is the product of $2k$ two-cycles.

b) Are $\mathbf{Z}$ and $\mathbf{R}^*$ isomorphic groups? Make sure to explain your answer.

No. $\mathbf{R}^*$ has an element of order 2 ($viz.$, -1) but the only element of finite order in $\mathbf{Z}$ is the identity which has order 1.

c) Give an example of finite groups G and H such that $|G| = |H|$ but G is not isomorphic to H or explain why no example exists.

There are many such. See the next problem to which shows that Z_8 and D_4 are not isomorphic. One is abelian and cyclic. The other is neither.

d) Find an element $\beta \in S_5$ such that $\beta(23)(45) = (34)\beta$, or explain why no such element exists.

Impossible. If β is even, then the left side is even and the right is odd. Similarly, if β is odd, then the left side is odd and the right is even.

5. a) Suppose that $\phi : D_4 \to \mathbf{Z}_8$ is a group homomorphism. What is $\phi(r_0)$? Explain how you know this.

$\phi(r_0) = 0$ since the identity is mapped to the identity under any homomorphism.

b) Suppose, again, that $\phi : D_4 \to \mathbf{Z}_8$ is a group homomorphism. Let v be the vertical flip. If $\phi(v) = k$, what are the possible choices for k in $\mathbf{Z}_8$? Again, explain your answer.

From the basic properties of a homomorphism, we must have $|\phi(v)|\Big|\,|v|$. But $|v| = 2$, So the only elements of $\mathbf{Z}_8$ that have orders which divide 2 are 0 and 4 which have orders 1 and 2, respectively.

c) Can a homomorphism $\phi : D_4 \to \mathbf{Z}_8$ be an isomorphism? Explain.

No. D_4 is not abelian or cyclic, but Z_8 is.

d) Can a homomorphism $\phi : D_4 \to \mathbf{Z}_8$ be onto? Explain.

No. Since both groups have 8 elements, if ϕ were onto, it would also be injective. Since we are told that ϕ is a homomorphism, this would make an isomorphism. But that's impossible.

6. Is the mapping $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ by $\phi(a) = \sqrt{a}$ injective? Surjective? A group homomorphism? Show all details. (Caution: The operation is multiplication.)

Injective. Notice that $\phi(a) = \phi(b) \iff \sqrt{a} = \sqrt{b} \iff a = b$. Surjective: Let $c \in \mathbf{R}^+$ (codomain). Find $a \in \mathbf{R}^+$ (domain) so that $\phi(a) = c$. But $\phi(a) = c \iff \sqrt{(a)} = c \iff a = c^2$. Notice that $c^2 \in \mathbf{R}^+$, so ϕ is surjective. Homomorphism: Let $a, b \in R^+$. Show $\phi(ab) = \phi(a)\phi(b)$. But $\phi(ab) = \sqrt{ab} = \sqrt{a}\sqrt{b} = \phi(a)\phi(b)$. Isomorphism: Yes, because we have shown that ϕ is injective, surjective, and a homomorphism.

7. Let $\phi : \mathbf{Z}_6 \to \mathbf{Z}_3$ be the homomorphism determined by $\phi(5_6) = 2_3$. Make a list of the elements of $\mathbf{Z}_6$ and determine what each is mapped to under ϕ. Is ϕ injective? Surjective? An isomorphism?

Use the fact that ϕ is a homomorphism and that 1_6 is the inverse of 5_6 so $\phi(1_6) = \phi(-5_6) = -\phi(5_6) = -2_3 = 1_3$. Then $\phi(2_6) = \phi(2 \cdot 1_6) \to 2 \cdot 1_3 = 2_3$, $\phi(3_6) = \phi(3 \cdot 1_6) \to 3 \cdot 1_3 = 0_3$, $\phi(4_6) = \phi(4 \cdot 1_6) \to 4 \cdot 1_3 = 1_3$, and of course $\phi(0_6) \to 0_3$.

8. (Extra Credit) Prove Colin's Theorem: If $H < S_n$ and $|H|$ is odd, then H contains only even elements. Hint: Think about the homework problem with odd and even elements in subgroups of S_n.

You proved for homework that: If H is any subgroup of S_n, then either every element of H is even or that exactly half the members of H are even. But in this case the second option is not possible since H has an odd number of elements (half can't be odd and half even). So H must contain only even permutations.