1.1 Equivalence Relations

Definition and Examples

One of the themes of this course is to generalize the basic notions you have used in earlier mathematics courses. In this regard, things that are different in one context are often seen as the same in another setting. Today we will generalize the notion of 'equality' that is obviously so important in elementary mathematics. We will come to understand equality in a much broader sense than we have previously. Fortunately you are already familiar with a few such instances of this generalization. Congruence of geometric figures is the classic example. To say that $\Delta ABC \cong \Delta DEF$ does not mean that the two triangles are the same, but rather that the angles and sides of the two different triangles are of equal measures. The most important aspects of such relationships is that they are reflexive, symmetric, and transitive. This allows for indirect comparisons which makes the relationship easier to understand and manipulate.

Definition 0 *Let S be a set. R (or $\sim$) is an* ***equivalence relation*** *on S if R satisfies the following condidtions:*

i) for every $s \in S$, sRs (s is related to itself; reflexive);

ii) for every $s, t \in S$, if sRt then tRs (symmetric);

iii) for every $s, t, u \in S$, if sRt and tRu then sRu (transitive).

A note on notation: often one writes (s, t) instead of sRt. In this way one sees that an equivalence relation is really a subset of the ordered pairs of elements of S.

Another form of notation is used for the more common equivalence relations: usually it involves some type of suggestive symbol such as: $=$, $\cong$, $\equiv$, or $\sim$.

Example 0 Let R be the relationship of equality on the real numbers.

Example 1 Let S be the set of all triangles in the plane. If $s, t \in S$ define sRt to mean that s is *similar* to t, that is, corresponding anlges have the same measure. Usually this is denoted by $s \sim t$. From high school geomentry you know that $\sim$ satisfies the three conditions of an equivalence relation.

Example 2 Let D be the set of all differentiable functions of a single real variable. If $f, g \in D$ define $f \sim g$ if $f' = g'$. Then $\sim$ is an equivalence relation on D. In fact, we know that $f \sim g \iff f = g + c$ for some constant c.

Example 3 Let n be a positive integer. On the set $\mathbf{Z}$, define $j \equiv k \pmod{n}$ to mean $n|(j-k)$. Then we can easily check that $\equiv$ is an equivalence relation. Only transitivity is hard. If $j \equiv k$ and $k \equiv m$ then $n|(j-k)$ and $n|(k-m)$ so $n|((j-k)+(k-m))$. That is $n|(j-m)$ so $j \equiv m$.

Example 4 Let Q be the following set of ordered pairs of integers

$$Q = \{(m,n)|\ m,n \in \mathbf{Z},\ n \neq 0\}.$$

If (m,n) and (j,k) are in Q define $(m,n) \approx (j,k)$ if $mk = jn$. One easily checks that $\approx$ is an equivalence relation on Q. In fact the pairs (m,n) and (j,k) are equivalent if and only if the fractions m/n and j/k are equal.

Example 5 Practice: On $\mathbf{Z}$, let $aRb \iff ab \leq 0$. Determine whether R is an equivalence relation.

Perhaps the most important property of an equivalence relation is that it breaks the set S up into disjoint subsets. This is seen quite easily in the examples above.

Definition 1 *For any $s \in S$, let $[s]$ denote the subset of S consistingt of all $t \in S$ such that tRs. That is,*

$$[s] = \{t \in S|\ tRs\}.$$

We call $[s]$ the ***equivalence class*** *of s under the relation R.*

Example 6 If R is the equivalence relation $\equiv \pmod{5}$ on $\mathbf{Z}$, then

$$[1] = \{\ldots, -9, -4, 1, 6, 11, 16, \ldots\} = \{1 + 5n|\ n \in Z\}.$$

That is, $[1]$ is the set of elements divisible by 5 with a remainder of 1. Notice that two equivalence classes here are either disjoint or exactly the same set. For exapmle $[1] = [21]$, but $[2] \cap [3] = \emptyset$.

Example 7 If R is the equivalence relation $\approx$ on Q defined earlier, then

$$[(1,2)] = \{(k \cdot 1, k \cdot 2)|\ k \in \mathbf{Z},\ k \neq 0\}.$$

Because of the connection we made earlier to fractions, $[(1,2)]$ corresponds to all the ways of writing the fraction $\frac{1}{2}$. It is clear that two equivalence classes are either equal as sets or else disjoint (i.e., are without any intersection). That's why you 'reduce fractions' and put them into proper form so that you can select a single representative of an equivalence class in some standard way.

EXAMPLE 8 If R is the equivalence relation $\sim$ on D, then

$$[f] = \{g \in D | \; g = f + c\}.$$

Again notice that equivalence classes are either disjoint or equal, there is no 'partial' overlap.

DEFINITION 2 *A* ***partition*** *of a set S is a collection of nonempty disjoint subsets of S whose union is all of S.*

EXAMPLE 9 If R is the equivalence relation $\equiv \pmod 5$ on $\mathbf{Z}$, then $[0], [1], [2], [3], [4]$ form a partition of $\mathbf{Z}$. This situation is the norm for any equivalence relation.

THEOREM 3 *Let R be an equivalence relation on S. Then the equivalence classes of R form a partition of S. That is, every element is in exactly one equivalence class. And conversely.*

PROOF Let R be the equivalence relation. Since sRs for any $s \in S$, it follows that $s \in [s]$. That is, no class is empty. Second, the union of all equivalence classes is clearly all of S since every element s of S lies in some equivalence class.

Finally we must show that any two classes are either disjoint or exactly the same. So suppose that two classes $[s]$ and $[t]$ are not disjoint, that is, that there is at least one element a in both $[s]$ and $[t]$. We must show that $[s] = [t]$. (To do this we must show $[s] \subset [t]$ and $[t] \subset [s]$.) To show $[s] \subset [t]$, let $b \in [s]$. Then: bRs. But $a \in [s]$, so sRa and thus bRa. But $a \in [t]$ so aRt and therefore bRt. That is, $b \in [t]$. So $[s] \subset [t]$ and similarly $[t] \subset [s]$.

The proof of the converse is an exercise. We'll never use it. ∎

Another way to say the same thing is :

$$[s] = [t] \iff [s] \cap [t] \neq \emptyset.$$

Notice that it is actually the equivalence classes mod n that we made into a group.

Hand In Homework for Monday

0. Verify that the relation in **Example 4** is, in fact, an equivalence relation.

1. Determine whether that the relation in **Example 5** is, in fact, an equivalence relation.