MATH 375.26

Class 26: Selected Answers

1. **a)** $[Z_{120} \oplus S_5 :< 15, (1235) >] = \frac{120 \cdot 120}{8 \cdot 4} = 450$.

b) $[S_6 \oplus Q_8 :< ((134)(25), K) >] = \frac{6! \cdot 8}{6 \cdot 4} = 240$.

c) $[\mathbf{R}^* \oplus D_4 :< (-1, r_{90}) >] = \infty$ because the subgroup is finite while the group is infinite. Therefore it would take an infinite number of cosets to partition the group.

2. **a)** Let $H =< (1,2) >$ in $\mathbf{Z}_4 \oplus U(5)$.

b) Are $(3,2)$ and $(4,1)$ in the same left coset of H?

Solution: $H =< (1,2) >= \{(0,1), (1,2), (2,4), (3,3)\}$. $(3,2)$ and $(4,1)$ in the same left coset of H if and only if $(0,2)^{-1}(4,1) = (1,3)(4,1) = (0,3) \in H$. But this is not the case. No.

c) Is H normal in $\mathbf{Z}_4 \oplus U(5)$. **Solution:** Yes, $\mathbf{Z}_4 \oplus U(5)$ is abelian.

3. **a)** Suppose that $|G| = 49$. Show that every proper subgroup of G is cyclic. **Solution:** By Lagrange, the only proper subgroups of G have order 1 and 7. Since 7 is prime, any subgroup of order 7 is cyclic. And, of course, if the subgroup has order 1, then it is $\{e\} =< e >$.

4. Consider the following group table with 8 elements

$*$	e	a	b	c	d	x	f	g
e	e	a	b	c	d	x	f	g
a	a	b	c	e	x	f	g	d
b	b	c	e	a	f	g	d	x
c	c	e	a	b	g	d	x	f
d	d	g	f	x	e	c	b	a
x	x	d	g	f	a	e	c	b
f	f	x	d	g	b	a	e	c
g	g	f	x	d	c	b	a	e

a) Use the table to find $C(G)$ (the center). **Solution:** $C(G) = \{e, b\}$.

b) Show that $H =< b >$ is normal in G. **Solution:** $< b >= \mathbf{C}(G)$.

5. Let $x \in G$. Prove: If for each $g \in G$ there is some integer n_g so that $gxg^{-1} = x^{n_g}$, then $< x > \triangleleft G$. **Solution:** Let $x^k \in < x >$. Then using the given condition

$$gx^kg^{-1} = gxg^{-1} \cdot gxg^{-1} \cdots gxg^{-1} = (gxg^{-1})^k = (x^{n_g})^k = x^{kn_g} \in < x > .$$

This means that $gxg^{-1} \in < x >$. By condition (iii) for normality, this means that $< x > \triangleleft G$.

6. Use the FTFAG to determine which product of cylic groups is isomorphic to

a) $U(150) \cong \mathbf{Z}_{20} \oplus \mathbf{Z}_2 \cong \mathbf{Z}_5 \oplus \mathbf{Z}_4 \oplus \mathbf{Z}_2$.

b) $U(320) \cong \mathbf{Z}_{16} \oplus \mathbf{Z}_4 \oplus \mathbf{Z}_2$

c) a group G of order 48 with 1 element of order 1, 7 elements of order 2, 2 elements of order 3, 8 elements of order 4, 14 14 elements of order 6, and 16 elements of order 12. **Solution:** $\mathbf{Z}_{12} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$

d) Here's a useful fact: Suppose that

$$G = \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \cdots \oplus \mathbf{Z}_{n_k} \oplus \mathbf{Z}_{m_1} \oplus \mathbf{Z}_{m_2} \oplus \cdots \oplus \mathbf{Z}_{m_j},$$

where all the n's are even and all the m's are odd. Prove that G has $2^k - 1$ elements of order 2.

Solution: None of the Z_m have elements of order 2, so in manufacturing an element of order 2 for G we will have to use the identity element from each of these. Next, each Z_{n_i} has 1 element of order 2. Then, we have two choices in the first k slots (either the identity or the element of order 2) and 1 choice in the last j slots for a total of 2^k choices. But this counts the identity element which has order 1. Tossing that out, we get $2^k - 1$ elements of order 2.

7. **a)** Find an element of order 4 in $\mathbf{Z}_{10} \oplus \mathbf{Z}_{14}$ or explain why none exists. **Solution:** None exists. $\mathbf{Z}_{10}$ has elements of order 1, 2, 5, and 10 while $\mathbf{Z}_{14}$ has elements of 1, 2, 7, and 14. There is no combination of these orders that has an lcm of 4.

b) Find a subgroup of order 4 in $\mathbf{Z}_{10} \oplus \mathbf{Z}_{14}$ or explain why none existis. **Solution:** Use $< 5 > \oplus < 7 >$.

8. The Cayley Table for the dihedral group D_3 is given below.

	r_0	r_{120}	r_{240}	a	b	c
r_0	r_0	r_{120}	r_{240}	a	b	c
r_{120}	r_{120}	r_{240}	r_0	c	a	b
r_{240}	r_{240}	r_0	r_{120}	b	c	a
a	a	b	c	r_0	r_{120}	r_{240}
b	b	c	a	r_{240}	r_0	r_{120}
c	c	a	b	r_{120}	r_{240}	r_0

a) Write out the left and right cosets of $H =< a >$. **Solution:** The left cosets are: $H = \{r_0, a\} = aH = r_0 H$, $r_{120}H = \{r_{120}, c\} = cH$, and $r_{240}H = \{r_{240}, b\} = bH$. The right cosets are: $H = \{r_0, a\} = Ha = Hr_0$, $Hr_{120} = \{r_{120}, b\} = Hb$, and $Hr_{240} = \{r_{240}, c\} = Hc$.

b) Determine whether H is normal in G. **Solution:** It is not since $bH \neq Hb$, for example.

c) Find elements x and y, z and w in D_3 so that *all* of the following conditions hold simultaneously (or explain why this is impossible):

i. $Hx = Hy$;
ii. $Hz = Hw$;
iii. $Hxz \neq Hyw$.

Solution: $Hr_{120} = Hb$ and $Hr_{240} = Hc$. But $Hr_{120}r_{240} = H$ while $Hbc = Hr_{120}$.

9. Give an example of a group G and a subgroup H of G such that $[G : H] = 3$ and $H \triangleleft G$, or explain why no such example exists. **Solution:** There are lots. Use $[\mathbf{Z} : 3\mathbf{Z}] = 3$ or $[\mathbf{Z}_{3n} :< n >] = 3$ for any positive integer n.

10. On the first exam I asked you to show that: If H and K are subgroups of G, then $H \cap K$ is subgroup of G. Now prove that if H and K are both *normal* subgroups of G, then $H \cap K$ is normal in G. **Solution:** Let $x \in H \cap K$. Then $x \in H$ which is normal in G, so for any $g \in G$, $gxg^{-1} \in H$. Similarly, $gxg^{-1} \in K$. So $gxg^{-1} \in H \cap K$. By condition (iii), $H \cap K$ is normal in G.

11. As ususal, let $SL(\mathbf{R}, n)$ denote the $n \times n$ matrices whose determinants are 1. Determine whether $SL(\mathbf{R}, n)$ is normal in $GL(\mathbf{R}, n)$. **Solution:** See class notes from Wednesday, 3 November 1999.