MATH 375.24

Class 24: Selected Answers

1. Gallian page 170 #50. **Solution:** $H \cap K$ is a subgroup of both H and K, so its order must divide both $|H|$ and $|K|$. But $\gcd(|H|, |K|) = 1$, so $|H \cap K| = 1$ and therefore $H \cap K = \{e\}$.

2. You are familiar with the group $\mathbf{R} \oplus \mathbf{R}$ from calculus, linear algebra, and analytic geometry where you thought of it at $\mathbf{R}^2$. Let m be a fixed real number. Define $H_m = \{(x, y) \in \mathbf{R} \oplus \mathbf{R} \mid y = mx\}$. Note that H_m is just the straight line through the origin with slope m.

- **a)** Show that $H_m < \mathbf{R} \oplus \mathbf{R}$. **Solution:** Use the two-step method. Closure: Let (Let $(x_1, y_1), (x_2, y_2) \in H_m$. Show that $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in H_m$. But $y_1 = mx_1$ and $y_2 = mx_2$, so $y_1 + y_2 = m(x_1 + x_2)$. Therefore, $(x_1 + x_2, y_1 + y_2) \in H_m$. Inverses: Let $(x_1, y_1) \in H_m$. Show that $-(x_1, y_1) = (-x_1, -y_1) \in H_m$. But $y_1 = mx_1$ so $-y_1 = -mx_1 = m(-x_1)$, so $(-x_1, -y_1) \in H_m$.
- **b)** Give a quick reason why $H_m \triangleleft \mathbf{R} \oplus \mathbf{R}$. **Solution:** $\mathbf{R} \oplus \mathbf{R}$ is abelian, so all subgroups are normal.
- **c)** What geometric set in $\mathbf{R}^2$ does the coset $(0, b) + H_m$ correspond to? **Solution:** The line parallel to $y = mx$ with intercept b, i.e., $y = mx + b$ because

$$(0, b) + H_m = \{(0, b) + (x, y) = (x, y + b) \in \mathbf{R} \oplus \mathbf{R} \mid y = mx\} = \{(x, y') \in \mathbf{R} \oplus \mathbf{R} \mid y' = mx + b\}.$$

- **d)** What is $[\mathbf{R} \oplus \mathbf{R} : H_m]$? Hint: Use the previous part. **Solution:** $[\mathbf{R} \oplus \mathbf{R} : H_m] = \infty$. There is one coset for each line parallel to $y = mx$.

3.
- **a)** Let $G = \mathbf{Z}_4 \oplus \mathbf{Z}_{12}$. H be the cyclic subgroup of G generated by $(2, 2)$. Give a quick reason why is $H \triangleleft G$. **Solution:** G is abelian.
- **b)** Let $H =< r_{60} >$ in D_6. Give a quick reason why $H \triangleleft D_6$. **Solution:** $[D_6 : H] = \frac{12}{6} = 2$.

4.
- **a)** Let $GU(15) \oplus \mathbf{Z}_{10} \oplus S_5$. Find the order of $(2, 3, (123)(15))$.
  Solution: $|(2, 3, (123)(15))| = \mathrm{lcm}(|2|, |3|, |(1523)|) = \mathrm{lcm}(4, 10, 4) = 20$.
- **b)** Find the inverse of $(2, 3, (123)(15))$. **Solution:** $(2, 3, (123)(15))^{-1} = (8, 7, (51)(321))$.

5.
- **a)** Is $\mathbf{Z}_6 \oplus \mathbf{Z}_{15}$ cyclic? **Solution:** No, $\gcd(6, 15) \neq 1$.
- **b)** Find an element of $\mathbf{Z}_6 \oplus \mathbf{Z}_{15}$ that has order 9 or explain why none exists. **Solution:** None exists. Elements in $\mathbf{Z}_6$ have order $a = 1$, 2, 3, or 6. Elements in $\mathbf{Z}_{15}$ have order $b = 1$, 3, 5, or 15. No combination of a and b produce $\mathrm{lcm}(a, b) = 9$.
- **c)** Let H_1 be a subgroup of G_1 and H_2 be a subgroup of G_2. It is easy to prove (you don't have to) that $H_1 \oplus H_2$ is a subgroup of $G_1 \oplus G_2$. Use this fact to find a subgroup of $\mathbf{Z}_6 \oplus \mathbf{Z}_{15}$ that has order 9. **Solution:** Use $< 2 > \oplus < 5 >$. Both $< 2 >$ and $< 5 >$ have order 3 in their respective groups, so their product has orde 9.

6. Use the Normal Subgroup Test (page 172) to determine whether $U(\mathbf{R}, n)\{A \in GL(\mathbf{R}, n) \mid \det A \pm 1\}$ is normal in $GL(\mathbf{R}, n)$. **Solution:** We must show that $BU(\mathbf{R}, n)B^{-1} \subset U(\mathbf{R}, n)$ for all $B \in GL(\mathbf{R}, n)$. But if $A \in U(\mathbf{R}, n)$, then

$$\det(BAB^{-1}) = \det B \det A \det(B^{-1}) = \frac{\det B \det A}{\det B} = \det A = \pm 1.$$

Therefore, $BU(\mathbf{R}, n)B^{-1} \subset U(\mathbf{R}, n)$.

7.
- **a)** Determine whether $< -I >$ is normal in Q_8. **Solution:** Yes, $< -I > \triangleleft Q_8$. For any element $X \in Q_8$, $X < -I >= \{X, -X\} =< -I > X$.
- **b)** Give a quick reason why $< J > \triangleleft Q_8$. **Solution:** $[Q_8 :< J >] = \frac{8}{4} = 2$.