MATH 375.22

Class 22: Selected Answers

1. Let $G = \mathbf{Z}_4 \oplus \mathbf{Z}_{12}$.

a) What is the order of G? **Solution:**

$$|G| = |\mathbf{Z}_4 \oplus \mathbf{Z}_{12}| = |\mathbf{Z}_4| \times |\mathbf{Z}_{12}| = 4 \times 12 = 48.$$

b) Let H be the cyclic subgroup of G generated by $(2,2)$. What is the order of H? **Solution:**

$$|H| = |(2,2)| = \text{lcm}(|2|, |2|) = \text{lcm}\left(\frac{4}{\gcd(4,2)}, \frac{12}{\gcd(12,2)}\right) = \text{lcm}(2,6) = 6.$$

c) List all the elements of H and their orders. **Solution:** The six elements of $< 2, 2 >$ are:

$$\begin{aligned}
(0,0) &: |(0,0)| = 1 \\
(2,2) &: |(2,2)| = 6 \\
(0,4) &: |(0,4)| = 3 \\
(2,6) &: |(2,6)| = 2 \\
(0,8) &: |(0,8)| = 3 \\
(2,10) &: |(2,10)| = 6
\end{aligned}$$

d) What is $[G : H]$? **Solution:** $[G : H] = \frac{|G|}{|H|} = \frac{48}{6} = 8$.

e) Are $(3,7)$ and $(1,1)$ in the same coset of H? Explain. **Solution:** Yes. Remember from the coset property list that $b \in aH \iff a^{-1}b \in H$. In additive notation, $b \in a + H \iff -a + b \in H$. But $-(3,7) + (1,1) = (1,5) + (1,1) = (2,6) \in H$.

2. **a)** Try the same question for $G = \mathbf{Z}_6 \oplus \mathbf{Z}_8$ where $H =< (2,2) >$. **Solution:** $|G| = |\mathbf{Z}_6 \oplus \mathbf{Z}_8| = |\mathbf{Z}_6| \times |\mathbf{Z}_8| = 6 \times 8 = 48$.

$$|(2,2)| = \text{lcm}(|2|, |2|) = \text{lcm}\left(\frac{6}{\gcd(6,2)}, \frac{8}{\gcd(8,2)}\right) = \text{lcm}(3,4) = 12.$$

The 12 elements of $< 2, 2 >$ and their orders are: $|(0,0)| = 1$, $|(2,2)| = 12$, $|(4,4)| = 6$, $|(0,6)| = 4$, $|(2,0)| = 3$, $|(4,2)| = 12$, $|(0,4)| = 2$, $|(2,6)| = 12$, $|(4,0)| = 3$, $|(0,2)| = 4$, $|(2,4)| = 6$, and $|(4,6)| = 12$. $[G : H] = \frac{|G|}{|H|} = \frac{48}{12} = 4$. $-(3,7) + (1,1) = (3,1) + (1,1) = (4,2) \in H$, so $(3,7)$ and $(1,1)$ are in the same coset.

b) A bit harder: try the same question for $G = U(8) \oplus \mathbf{Z}_8$ where $H =< (3,2) >$. **Solution:** $|G| = |U(8) \oplus \mathbf{Z}_8| = |U(8)| \times |\mathbf{Z}_8| = 4 \times 8 = 32$.

$$|H| = (3,2)| = \text{lcm}(|3|, |2|) = \text{lcm}\left(2, \frac{8}{\gcd(8,2)}\right) = \text{lcm}(2,4) = 4.$$

The four elements of $< 3, 2 >$ and their orders are: $|(1,0)| = 1$, $|(3,2)| = 4$, $|(1,4)| = 2$, and $|(3,6)| = 4$. $[G : H] = \frac{|G|}{|H|} = \frac{32}{4} = 8$. $(3^{-1}, -7)(1,1) = (3,1)(1,1) = (3,2) \in H$, so $(3,7)$ and $(1,1)$ are in the same coset.

3. Determine the orders of each of the following product groups and state whether the group is (a) finite or infinite; (b) abelian or not; (c) cyclic or not.

a) $V_4 \oplus \mathbf{Z}_5$ **b)** $D_4 \oplus S_4$ **c)** $\mathbf{Q} \oplus \mathbf{Q}_8$ **d)** $\mathbf{Z}_6 \oplus \mathbf{Z}_8$
e) $\mathbf{Z}_{12} \oplus \mathbf{Z}_5$ **f)** $\mathbf{C} \oplus \mathbf{R}$

Solution: $|V_4 \oplus \mathbf{Z}_5| = 20$. It is finite, abelian, but not cyclic. $|D_4 \oplus S_4| = 8 \times 24 = 192$. It is finite, not abelian, and not cyclic. $|\mathbf{Q} \oplus \mathbf{Q}_8| = \infty$. It is infinite, not abelian, and not cyclic. $|\mathbf{Z}_6 \oplus \mathbf{Z}_8| = 48$. It is finite, abelian, and not cyclic. $|\mathbf{Z}_{12} \oplus \mathbf{Z}_5| = 60$. It is finite, abelian, and cyclic. $|\mathbf{C} \oplus \mathbf{R}| = \infty$. It is infinite, abelian, and not cyclic.

4. **a)** What is the smallest value of n greater than 1 that makes $\mathbf{Z}_n \oplus \mathbf{Z}_{210}$ cyclic? **Solution:** we need the smallest n such that $\gcd(n, 210) = 1$. $n = 11$.

b) Is $\mathbf{Z}_3 \times \mathbf{Z}_7$ isomorphic to $\mathbf{Z}_{21}$? Explain. **Solution:** Yes, both are cyclic (since $\gcd(3, 7) = 1$) and of order 21.

c) Is $\mathbf{Z}_8 \times \mathbf{Z}_{12}$ isomorphic to $\mathbf{Z}_{96}$? Explain. **Solution:** No. Since $\gcd(8, 12) \neq 1$, then $\mathbf{Z}_8 \times \mathbf{Z}_{12}$ is not cyclic but $\mathbf{Z}_{96}$ is.

5. Find the orders of these elements in their given product groups.

a) $(3, 4) \in \mathbf{Z}_5 \oplus \mathbf{Z}_6$ **b)** $((1243)(13), i) \in S_4 \oplus C^*$

c) $(v, 2) \in D_4 \oplus \mathbf{Z}$ **d)** $(6, r_{180}) \in \mathbf{Z}_8 \oplus D_4$

Solution: $|(3, 4)| = \text{lcm}\left(\frac{5}{\gcd(5,3)}, \frac{6}{\gcd(6,4)}\right) = \text{lcm}(5, 3) = 15$.

$|((1243)(13), i)| = |(243), i| = \text{lcm}(3, 4) = 12$.

$|(v, 2)| = \infty$.

$|(6, r_{180})| = \text{lcm}(4, 2) = 4$.

6. Each part below is a separate question. Find groups G and H or state why it is impossible.

a) $|G \oplus H| = 36$ and $G \oplus H$ is not abelian. **Solution:** One example, $S_3 \oplus \mathbf{Z}_6$.

b) $|G \oplus H| = 36$ and $G \oplus H$ is abelian, but not cyclic. Give an example with $G \oplus H$ cyclic. **Solution:** Try $V_4 \times \mathbf{Z}_9$ and then $\mathbf{Z}_4 \times \mathbf{Z}_9$.

c) $|G \oplus H| = 64$ which is cyclic and neither group has order 1. **Solution:** Impossible. The groups would have to be (isomorphic to) $\mathbf{Z}_k$ and $\mathbf{Z}_m$ with $\gcd(k, m) = 1$ and $km = 64$. But the only factorizations of 64 into two factors are: 2×32, 4×16, and 8×8.

7. **a)** How many non-isomorphic groups of order 24 can you find. **Solution:** Here are the abelian ones: $\mathbf{Z}_{24} \cong \mathbf{Z}_8 \times \mathbf{Z}_3$, $\mathbf{Z}_2 \times \mathbf{Z}_{12} \cong \mathbf{Z}_2 \times \mathbf{Z}_3 \times \mathbf{Z}_4 \cong \mathbf{Z}_6 \times \mathbf{Z}_4$, and $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_3 \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_6$. Some non-abelian ones: $S_3 \times \mathbf{Z}_4$, $S_3 \times \mathbf{Z}_2 \times \mathbf{Z}_2$, $D_4 \times \mathbf{Z}_3$, $A_4 \times \mathbf{Z}_2$.

8. Notice that the three groups $D_{11} \oplus Z_3$, $D_3 \times Z_{11}$, and D_{33} are non-abelian and have order 66. This problem asks you to show that no two of these are isomorphic. (Hint: What else can you check about these groups and their elements that would have to be the same if they were isomorphic?)

a) Prove that $D_{11} \oplus \mathbf{Z}_3 \ncong D_3 \times \mathbf{Z}_{11}$.

b) Prove that $D_{11} \oplus \mathbf{Z}_3 \ncong D_{33}$.

c) Prove that $D_{33} \ncong D_3 \times \mathbf{Z}_{11}$.

Solution: For all three parts, use the table below which lists the elements of various orders in each of the groups. Notice that the elements of order 2 already distinguish them.

Orders	$\mathbf{Z}_3$	$\mathbf{Z}_{11}$	D_3	D_{11}	$\mathbf{Z}_3 \oplus D_{11}$	$Z_{11} \oplus D_3$	D_{33}
1	1	1	1	1	1	1	1
2	0	0	3	11	11	3	33
3	2	0	2	0	2	2	2
6	0	0	0	0	22	0	0
11	0	10	0	10	10	10	10
22	0	0	0	0	0	30	0
33	0	0	0	0	20	20	20

9. We proved in class that: "Let G be a non-abelian group of order $2p$, where $p \neq 2$ is prime. Then G has a cyclic subgroup of order p and it also has p elements of order 2." Let's apply this to the case where $p = 3$. So let G be a non-abelian group of order 6. Then G has an element x of order 3 and G has an element a of order 2. So $< x >= \{e, x, x^2\}$ and G is composed of the two disjoint cosets: $< x >$ and $a < x >$, where $a < x >= \{a, ax, ax^2\}$ and these three elements have order 2. Of course this means that the six elements of the group $G = \{e, x, x^2, a, ax, ax^2\}$. We know that $a^2 = (ax)^2 = (ax^2)^2 = e$ since each has order 2. Fill in the Cayley Table for G. Some slots are easy to fill in: for example, $a \cdot x^2 = ax^2$. The only hard entry to fill in is xa. Note that xa must be one of the six elements listed above in G. What are the choices? Show by filling in the table that if $x \cdot a = ax$, then G turns out to be abelian. So then fill it in again with xa being the only other possible choice. What is that choice?

Solution: If $xa = ax$ we get a contradiction in row 5 of the Cayley table. If $xa = ax^2$, the only other choice, then the Cayley table can be filled in. The format is the same as for $D_3 = S_3$.

$\cdot$	e	x	x^2	a	ax	ax^2
e	e	x	x^2	a	ax	ax^2
x	x	x^2	e	ax	ax^2	a
x^2	x^2	e	x	ax^2	a	ax
a	a	ax	ax^2	e	x	x^2
ax	ax	ax^2	a	x	e	e
ax^2	ax^2					e

$\cdot$	e	x	x^2	a	ax	ax^2
e	e	x	x^2	a	ax	ax^2
x	x	x^2	e	ax^2	a	ax
x^2	x^2	e	x	ax	ax^2	a
a	a	ax	ax^2	e	x	x^2
ax	ax	ax^2	a	x^2	e	x
ax^2	ax^2	a	ax	x	x^2	e

10. Use the software `Marc's U(n)` in the Math/CS Computer lab on either of the Macintoshes. Write each of the following groups as a product of $\mathbf{Z}_n$'s using the FTAG. Describe your reasoning. **Solution:**

a) $U(38) \cong \mathbf{Z}_{18}$ **b)** $U(40) \cong \mathbf{Z}_4 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$

c) $U(66) \cong \mathbf{Z}_{10} \oplus \mathbf{Z}_2$ **d)** $U(318) \cong \mathbf{Z}_{13} \oplus \mathbf{Z}_4 \oplus \mathbf{Z}_2$

e) Do one of your own choosing—make it an interesting one!

f) Extra Credit: **Solution:** $U(760) \cong \mathbf{Z}_9 \oplus \mathbf{Z}_4 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$.