MATH 375.19

# Class 19: Selected Answers

**1.** The following 8 permuations in $S_4$ are known as the **Octic group**, $O = \{e, a, a^2, a^3, b, g, d, t\}$, where $e = (1)$, $a = (1234)$, $a^2 = (13)(24)$, $a^3 = (1432)$ $b = (14)(23)$, $g = (12)(34)$, $d = (13)$, and $t = (24)$.

**a)** Find all the left cosets of $O$ in $S_4$. **Solution:** The cosets are $O$ itself,

$$(12)O = \{(12), (234), (1324), (143), (1423), (34), (132), (124)\}$$
$$(14)O = \{(14), (123), (1342), (243), (23), (1243), (134), (142)\}$$

**b)** What are the right cosets of 0 is $S_4$? **Solution:** The cosets are $O$ itself,

$$O(12) = \{(12), (134), (1423), (243), (1324), (34), (123), (142)\}$$
$$O(14) = \{(14), (234), (1243), (132), (23), (1342), (143), (124)\}$$

**c)** Is 0 normal in $S_4$? **Solution:** No. $(12)O \neq O(12)$

**2.** Find all the left cosets of $< 4 >$ in $U(15)$. Then find the right cosets. Is $< 4 >$ normal? **Solution:** The left and right cosets must be the same since $U(15) = \{1, 2, 4, 7, 8, 11, 13, 14\}$ is ablelian. So $< 4 >= \{1, 4\}$ is normal. The cosets are: $< 4 >$ itself, $2 < 4 >= \{2, 8\} =< 4 > 2$, $7 < 4 >= \{7, 13\} =< 4 > 7$, and $11 < 4 >= \{11, 14\} =< 4 > 11$.

**3.** Find all the left and right cosets of $A_3$ in $A_4$. Is $A_3$ normal in $A_4$? **Solution:** Using the handout, $A_3 = \{(1), (123), (132)\}$ while

$$A_4 = \{(1), (123), (132), (234), (12)(34), (134), (243), (124), (13)(24), (142), (143), (14)(23)\}.$$

So the left cosets are $A_3$ itself,

$$(234)A_3 = \{(234), (13)(24), (142)\}$$
$$(243)A_3 = \{(243), (143), (12)(34)\}$$
$$(124)A_3 = \{(124), (14)(23), (134)\}$$

The right cosets are $A_3$ itself,

$$A_3(234) = \{(234), (12)(34), (134)\}$$
$$A_3(243) = \{(243), (124), (13)(24)\}$$
$$A_3(143) = \{(143), (14)(23), (142)\}$$

$A$ is not normal since $(234)A_3 \neq A_3(234)$.

**4.** Let $U(\mathbf{R}, n) = \{A \in Gl(\mathbf{R}, n) \mid \det A = \pm 1\}$. Show that $AU(\mathbf{R}, n) = BU(\mathbf{R}, n) \iff |\det a| = |\det b|$. **Solution:**

$$\begin{aligned} AU(\mathbf{R}, n) = BU(\mathbf{R}, n) \iff A^{-1}B \in U(\mathbf{R}, n) &\iff \det(A^{-1}B) = \pm 1 \\ &\iff \det A^{-1} \det B = \pm 1 \\ &\iff \frac{\det B}{\det A} = \pm 1 \\ &\iff \det B = \pm \det A \\ &\iff |\det B| = |\det A| \end{aligned}$$

**5.** Evaluate the following indices (justify your answers)

**a)** $|A_n : A_{n-1}| = \frac{n!/2}{(n-1)!/2} = n$.

**b)** $|\mathbf{Z}_8 :< 2 > | = \frac{8}{|2|} = \frac{8}{4} = 2$.

**c)** $|\mathbf{Z} :< n > | = n$. if $n \neq 0$, there are $n$ cosets: $0+ < n >, 1+ < n >, \ldots, (n-1)+ < n >$. Otherwise if $n = 0$ the index is infinite.

**d)** $|D_4 :< v > | = \frac{8}{|v|} = \frac{8}{2} = 4$.

**e)** $|S_4 : O| = \frac{24}{8} = 3$.

**f)** $|\mathbf{R}^* :< -1 > | = \infty$ because $|R^*| = \infty$ and $|-1| = 2$.

**g)** $[GL(\mathbf{R}, n) : U(\mathbf{R}, n)] = \infty$ because you proved above that there were an infinite number of cosets, one for each nonnegative real number.

**6.** Gallian page 143 #14. **Solution:** Given $K < H < G$ with $|K| = 42$ and $|G| = 420$. So by Lagranges's Theorem, the order of $H$ must be divisible by 42 and a divisor of 420. Possible orders are 84 and 210.

**7.** Gallian page 143 #20. **Solution:** Given $KG$ and Given $H < G$, with $|K| = 12$ and $|H| = 35$. Since $K \cap H$ is a subgroup of both $K$ and $H$, $|K \cap H| \Big| |12|$ and $|K \cap H| \Big| |35|$. Therefore, $|K \cap H| = 1$.

**8.** Gallian page 143 #24. $|G| = 25$. So by Lagrange, if $a \in G$, then $|a| = 1, 5$, or 25. If $|a| = 25$, then $G$ is cyclic. If $G$ is not cyclic, then every element has order 5 (or 1), so $g^5 = e$ for all elements of $G$.

**9.** **a)** Suppose that $G$ is a group such that $g^2 = e$ for all $g \in G$. Prove that $G$ is abelian. **Solution:** We must show that $\forall a, b \in G \; ab = ba$. But

$$\begin{aligned} ab = (ab)e = (ab)(ba)^2 &= ab((ba)ba) \\ &= a(bb)a(ba) \\ &= aea(ba) = (aa)(ba) = e(ba) = ba. \end{aligned}$$

**b)** Suppose that $G$ is a non-abelian group of order 10. Prove that $G$ has an element $a$ of order 5.**Solution:** If all elements $a \in G$ have the property that $a^2 = e$, then we just showed that $G$ would be abelian, a contradiction. So there exists some $a \in G$ with $a \neq e$ and $|x| \neq 2$. By Lagrange $|a| = 5$ or 10. If $|a| = 10$, then $G$ is cyclic, hence abelian. Contradiction. So there is some $a \in G$ so that $|a| = 5$.

**c)** (Continuation of part b): Let $g \in G$ such that $g \notin < a >$. Prove that there are only two left cosets of $< a >$ in $G$, namely $< a >$ and $g < a >$. **Solution:** Since $| < a > | = |a| = 5$, then $[G :< a >] = \frac{10}{5} = 2$. So let $g$ be any element of $G$ not in $< a >$. Then $g < a >$ is the other coset.

**d)** (Continuation of part b and c): Prove that $g^2 < a >=< a >$. **Solution:** Since there are only two cosets, $g^2 < a >$ is either $< a >$ or $g < a >$. But if $g < a >= g^2 < a >$, then by the Coset Property Theorem, $g^{-1}g^2 = g \in < a >$ which contradicts that $< a >$ and $g < a >$ are distinct cosets.

**e)** (Continuation of part b and d): Prove that $g^2 = e$. **Solution:** Use a proof by contradiction. If $g^2 \neq e$, then $|g| = 5$ or 10. But the latter is impossible since $G$ is not cyclid. But if $|g| = 5$, then $g^6 = gg^5 = ge = g$. On the other hand, $g^2 < a >=< a > \Rightarrow g^2 = a^k$. So $g^6 = (g^2)^3 = (a^k)^3$ and so $g \in < a >$. But $< a >$ and $g < a >$ are distinct cosets.

**f)** Ok, look at what you've now shown: If $G$ is a non-abelian group of order 10, then $G$ has an element $a$ of order 5. Further, any element $g \notin < a >$ has order 2. But there are 5 such elements. Since none of the elements of $< a >$ have order 2 (because elements of the cyclic group $< a >$ must have order 5 or 1), then $G$ has exactly 5 elements of order 2. (For example, $D_5$ has 5 flips.) Can you conjecture how many elements a non-abelian group of order $2p$ has? Conjecture $p$.

**10.** **a)** Let $G$ be a group of order $n > 1$. Suppose that the only subgroups of $G$ are $\{e\}$ and $G$ itself. Prove that $G$ is cyclic. **Solution:** Take any element $a \neq e$ in $G$. Since $a \neq e$, then $< a > \neq \{e\}$ is a subgroup of $G$. By assumption, $< a >$ must be $G$, itself. That is, $G$ is cyclic.

**b)** Extra Credit: Show that the number $n$ would have to be prime for the hypothesis in the part above to be true. **Solution:** Let $|G| = n$. If $n$ is not prime, then $n = km$, where $2 \leq k, m \leq n - 1$. Since we have shown that $G$ is cyclic, by the Fundamental Theorem of Cyclic Groups, $G$ has cylic subgroups of orders both $k$ and $m$. This contradicts the fact that the only subgroups of $G$ are $\{e\}$ and $G$ itself.

**11.** Extra Credit: Gallian page 143 #26. This is not hard; there are just several cases to check.