MATH 375.14

Class 14: Selected Answers

1. Suppose $\alpha, \beta \in S_n$. Prove that if α is even, then so is $\beta^{-1}\alpha\beta$.

2. Let H be any subgroup of S_n. Prove: Either every element of H is even or that exactly half the members of H are even. **Solution:** If all the elements are even, done. Otherwise, let α be any odd permutation in H. You have shown in an earlier problem set that the mapping $f : H \to H$ by $f(\beta) = \alpha\beta$ is one-to-one and onto. [Caution: You need to use an odd permutation α **in** H so that $\alpha\beta \in H$ by closure. An arbtrary odd permutation γ from S_n won't do because then we would not know whether $\gamma\beta \in H$.] But since α is odd, then β is even $\iff$ $\alpha\beta = f(b)$ is odd. That is, f maps the odd permuations to the even permutations of H and vice versa. Therefore, there must be the same number of each.

3.
a) What is the maximum order for an element in S_6? **Solution:** If $\alpha \in S_6$, then it can be written as a product of disjoint cycles, whose lengths sum to 6 (if we include 1-cycles for those elements fixed by α). So the lengths of the disjoint cycles of such splittings are: [6], [5,1], [4,2], [4,1,1], [3,3], [3,2,1], [3,1,1], [2,2,2], [2,2,1], [2,1,1,1,1], [1,1,1,1,1,1]. The maximum order of such a splitting (using Ruffini's Theorem) is 6.

b) What about for A_6? **Solution:** The following are the even splittings: [5,1],[3,3], [3,1,1], [1,1,1,1,1,1]. The maximum order is 5.

c) Find an element of A_8 of order 15. **Solution:** Use a [5,3] splitting: (12345)(678) will do.

d) Find an element of A_{10} of order 21. **Solution:** Use a [7,3] splitting: $(1234567(8, 9, 10)$ will do.

4. Let $\phi : \mathbf{R}^2 \to \mathbf{R}$ by $\phi(a, b) = ab$. Determine whether ϕ is one-to-one and/or onto. **Solutoin:** ϕ is not injective because $\phi(1, 0) = \phi(2, 0) = 0$. ϕ is onto. Let $x \in \mathbf{R}$. Then we must find $(a, b) \in \mathbf{R}^2$ so that $\phi(a, b) = x$. So we need $\phi(ab, b) = ab = x$. There are many choices, but the simplest is to let $(a, b) = (x, 1)$; then $\phi(x, 1) = x \cdot 1 = x$.

5. Let $\phi : V_4 \to V_4$ by $\phi(g) = g^2$ for all $g \in V_4$. Go back to your group table and actually figure out what $\phi(g)$ is for each element in V_4. Is ϕ injective? Surjective? **Solution:** Notice that $\phi(g) = g^2 = e$ for all $g \in V_4$. So ϕ is neither injective nor surjective.

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

6.
a) Let $\alpha = (a_1a_2 \ldots a_k)$ be a k-cycle. Prove that α is odd if and only if k is even. **Solution:** We saw in class that $\alpha = (a_1a_k) \ldots (a_1a_3)(a_2a_1)$ is a product of $k - 1$ transpositions. Therefore, α is odd if and only if $k - 1$ is odd if and only if k is even.

b) Prove that α is odd if and only if $|\alpha|$ is even. **Solution:** As seen in class, the order of a k-cycle is just its length. So $|\alpha|$ is even if and only if k is even and from the previous part k is even if and only if α is odd.

c) OK, here's the hard part on the homework: Now let β be any element of S_n. Prove that if β is odd, then $|\beta|$ is even. Hint: First use Theorem 5.1. Then show at least one of the cycles must be even in length. Then use Ruffini's Theorem. **Solution:** We can write β as a product of n disjoint cycles, say $\beta = \alpha_1\alpha_2 \cdots \alpha_n$. Let k_i be the length of α_i. First use a proof by contradiction to show that some k_i is even in length. Assume not. Then by part (a), all the k_i are odd, so all the α_i are even. So $\beta \in A_n$ and therefore β is even. This contradicts that we are given that β is odd. So some k_i must be even. But then by Ruffini's Theorem,

$$|b| = \text{lcm}(k_1k_2 \cdots k_n)$$

must be even since $k_i \mid \text{lcm}(k_1k_2 \cdots k_n)$ and k_i is even.

Optional Mastery and Review Exercises

7. Let G be a group and let H be a subgroup of G. Let a be some *fixed* element of G. Define the set aHa^{-1} to be $\{aha^{-1} \mid h \in H\}$. Show that aHa^{-1} is a subgroup of G. **Solution:** Closure: Let $ah_1a^{-1}, ah_2a^{-1} \in aHa^{-1}$. Then $h_1, h_2 \in H$. So

$$(ah_1a^{-1})(ah_2a^{-1}) = a(h_1h_2)a^{-1} \in aHa^{-1}.$$

because H is a subgroup so $h_hh - 2 \in H$. Inverses: Let $aha^{-1} \in aHa^{-1}$. Must show $(aha^{-1})^{-1} \in aHa^{-1}$. But $h^{-1} \in H$. So

$$(aha^{-1})^{-1} = ah^{-1}a^{-1} \in aHa^{-1}.$$

8. Suppose G is a group of order 16. If G has 5 elements for which $x^4 = e$, can G be cyclic? Explain. **Solution:** If G were cyclic of order 16, the elements whose order were were 4, 2 and 1 would satisfy this condition. Now if $< y >= G$, then these elements would be y^4, y^{12}, y^8, and e. So it is impossible.

9. Extra Credit: For those who have taken probability: Show that A_5 has 24 elements of order 5 and 20 elements of order 3. **Solution:** Note that the only elements of order 5 in S_5 are 5-cycles. But all 5-cycles are even, so all 5-cycles are in A_5. There are, of course, $5 = 120!$ ways to fill in the 5-cycle $(a_1, a_2, a_3, a_4, a_5)$ with the numbers 1 through 5. But notice that

$$(a_1, a_2, a_3, a_4, a_5) = (a_5, a_1, a_2, a_3, a_4),$$

and, in fact, there are 5 different ways to write $(a_1, a_2, a_3, a_4, a_5)$ as a 5-cycle, depending on which element yo start with. So the number of 5-cycles is $5!/5 = 4! = 24$. Similarly, the only order 3 elements in S_5 are 3-cycles which are even so in A_5. But a there are $5 \cdot 4 \cdot 3 = 60$ ways to fill in a 3-cycle (a_1, a_2, a_3) with the numbers 1 through 5. Each such such cycle can be written 3 different ways so there are $60/3 = 20$ different such.