MATH 375.5

Class 5: Selected Answers

1. Let Q be the following set of ordered pairs of integers

$$Q = \{(m, n) |\ m, n \in \mathbf{Z},\ n \neq 0\}.$$

If (m, n) and (j, k) are in Q define $(m, n) \approx (j, k)$ if $mk = jn$. Check that $\approx$ is an equivalence relation on Q. Solution: As noted in the handout, (m, n) and (j, k) are equivalent if and only if the fractions m/n and j/k are equal. Symmetric: Given $(m, n) \in Q$. Then $mn = mn$ so $(m, n) \sim (m, n)$. Reflexive: Given $(m, n) \sim (j, k)$. Then $mk = jn$, so $jn = mk$. Thus, $(j, k) \sim (m, n)$. Transitive: Given $(m, n) \sim (j, k)$ and $(j, k) \sim (a, b)$. Then $mk = jn$ and $jb = ak$. Thus,

$$mk \cdot jb = jn \cdot ak \Rightarrow mb = na = an \Rightarrow (m, n) \sim (a, b).$$

2. On $\mathbf{Z}$, let $aRb \iff ab \leq 0$. Determine whether R is an equivalence relation. No, it is not reflexive. If $a \neq 0$, then $aa = a^2 > 0$, so a is not related to a. It is symmetric since $ab = ba$. But it is not transitive: $1R-1$, $-1R2$, but $1R \sim 2$.

3. On $\mathbf{Z}$, define $a \sim b$ to mean that $a + b$ is even.

a) Show that $\sim$ is an equivalence relation. Reflexive: $a + a = 2a$ is a always even. Symmetric: If $a \sim b$, then $a + b$ is even. But integer addition is commutative, so $b + a = a + b$ is even, so $b \sim a$. Tranisitive: Given $a \sim b$ and $b \sim c$. Then $a + b$ and $b + c$ are even. Hence $(a + b) + (b + c) = a + 2b + c$ is even. Since $2b$ is even, then

$$(a + 2b + c) - 2b = a + c$$

is even and so $a \sim c$.

b) What are the equivalence classes of this relation? One class is the odd integers, the other the even.

4. Let M be the set of all $n \times n$ matrices. Define $A \sim B$ to mean $\det AB > 0$. Show that this is not an equivalence relation. Which properties fail and which hold? Solution: The relation is not symmetric. If $\det A = 0$ (i.e., A has no inverse), then $\det AA = \det A \det A = 0$. (If $\det A \neq 0$, then $\det AA = (\det A)^2 > 0$.) The relation is symmeteric, since if $A \sim B$, then $\det AB > 0$, so

$$\det BA = \det B \det A = \det A \det B = \det AB > 0 \Rightarrow B \sim A.$$

Finally the relation is transitive. Notice that since $\det AB = \det A \det B$, then $\det AB > 0$ if and only if both determinants have the same sign, i.e., both are positive or both are negative. So $A \sim B$ means $\det A$ and $\det B$ have the same sign and $B \sim C$ means that $\det B$ and $\det C$ have the same sign. It follows that all three determinants must have the same sign and in particular that $\det A$ and $\det C$ have the same sign. Therefore, $\det AC = \det A \det C > 0$ and $A \sim C$.

5. Prove or find a counterexample: Let G be a group. For $a, b \in G$, define $a \sim b$ to mean $ab = ba$. Is $\sim$ an equivalence relation on G? No. Use a non-abelian group like D_3. While $\sim$ is symmetric and reflexive, transitivity can fail. $ar_0 = a = r_0 a$, so $a \sim r_0$. And $r_0 r_{120} = r_{120} = r_{120} r_0$ so $r_0 \sim r_{120}$. But $r_{120} a = b$ while $ar_{120} = c$, so $r_{120} \not\sim a$.