MATH 375.3

Class 3: Selected Answers

1. **Basis Step:** When $n = 1$ we have $1 = 1^2$ so the induction starts. **Inductive Step:** Note the n-th odd integer is $2n - 1$. Given that $1 + 3 + \cdots + (2n - 1) = n^2$. To show that

$$1 + 3 + \cdots + (2n - 1) + (2n + 1) = (n + 1)^2.$$

But

$$\begin{aligned} 1 + 3 + \cdots + (2n - 1) + (2n + 1) = [1 + 3 + \cdots + (2n - 1)] + (2n + 1) &= n^2 + (2n + 1) && \text{given} \\ &= (n + 1)^2. \end{aligned}$$

So the formula holds for all positive integers.

2. **Basis Step:** When $n = 1$ we have $1 \cdot 1! = 2 - 1 = (1 + 1)! - 1$ so the induction starts. **Inductive Step:** Given that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1$. To show that

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! + (n + 1) \cdot (n + 1)! = (n + 2)! - 1.$$

But

$$\begin{aligned} 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! + (n + 1) \cdot (n + 1)! &= [1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!] + (n + 1) \cdot (n + 1)! \\ &= [(n + 1)! - 1] + (n + 1) \cdot (n + 1)! && \text{given} \\ &= -1 + (n + 1 + 1) \cdot (n + 1)! && \text{factor} \\ &= (n + 2)(n + 1)! - 1 \\ &= (n + 2)! - 1. \end{aligned}$$

3. $d = \gcd(a, b)$ is a linear combination of a and b, say $d = ma + nb$. But $t|ma$ and $t|nb$, since $t|a$ and $t|b$, respectively. So $t|(ma + nb)$ and, therefore, $t|d$.

4. **a)** Gallian page 22 #17. Given $a = b \bmod st$, so $st|(a - b)$. Therefore, both $s|(a - b)$ and $t|(a - b)$. Or equivalently, $a = b \bmod s$ and $a = b \bmod t$, respectively.

b) $\gcd(s, t)$ need to be relatively prime for the converse to hold. Can you prove it?

5. **a)** Gallian page 22 #15. Solution:

$$\begin{aligned} 126 &= 34 \cdot 3 + 24 \\ 34 &= 24 \cdot 1 + 10 \\ 24 &= 10 \cdot 2 + 4 \\ 10 &= 4 \cdot 2 + 2 \\ 4 &= 2 \cdot 2 + 0 \end{aligned}$$

So, $\gcd(126, 34) = 2$.

b) Simply work backwards through the chain of equalities in the Euclidean algorithm.

$$\begin{aligned} 2 &= 10 - 2 \cdot 4 && \text{now eliminate 4} \\ &= 10 - 2(24 - 2 \cdot 10) = 5 \cdot 10 - 2 \cdot 24 && \text{now eliminate 10} \\ 5&(34 - 24) - 2 \cdot 24 = 5 \cdot 34 - 7 \cdot 24 && \text{now eliminate 24} \\ &= 5 \cdot 34 - 7(126 - 3 \cdot 34) \\ &= -7 \cdot 126 + 26 \cdot 34. \ \square \end{aligned}$$

6. Gallian page 35 #3. D_3 is not abelian (you should give at least one example of why not). The Cayley table is not symmetric.

$*$	r_0	r_{120}	r_{240}	a	b	c
r_0	r_0	r_{120}	r_{240}	a	b	c
r_{120}	r_{120}	r_{240}	r_0	b	c	a
r_{240}	r_{240}	r_0	r_{120}	c	a	b
a	a	c	b	r_0	r_{240}	r_{120}
b	b	a	c	r_{120}	r_0	r_{240}
c	c	b	a	r_{240}	r_{120}	r_0

7. **(Converse of the Linear Combination Theorem)**. Let $d = \gcd(s, n)$. Then d is the *smallest* positive linear combination of s and n. But by assumption, $1 = as + bn$. That is, 1 is a linear combination of s and n. Therefore d is a positive number such that $d \leq 1$. This forces $d = 1$.

8. Gallian page 51 #1. The odd integers are not closed under addition and there is no identity.

9. Gallian page 51 #25. The table is

$*$	e	a	b	c	d
e	e	a	b	c	d
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c