Lab 9
Fourier Synthesis and Analysis

In this lab you will use a number of electronic instruments to explore Fourier synthesis and analysis. As you know, any periodic waveform can be represented by a sum of sine and cosine waves (or complex exponentials). If a function \( V(t) \) has a period of \( T \), then the frequencies of the sines and cosines will be multiples of \( \omega_1 = 2\pi/T \).

\[
V(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos(n \omega_1 t) + b_n \sin(n \omega_1 t) \right]
\]

You already know how to calculate the \( a \)'s and \( b \)'s required to produce a particular \( V(t) \). The goal for today is to try to develop a better intuitive understanding of Fourier series.

I. Fourier Synthesis (to be done as a group demonstration)

To demonstrate construction of a non-sinusoidal \( V(t) \) by adding together sine and cosine waves we have a special “Fourier Synthesis Demonstration box.” This box generates a series of sinusoidal waves with frequencies \( f_1, 2f_1, 3f_1, \) etc. with variable phase and amplitude. Any combination of these can be added together to create new waveforms such as square or triangle waves. We will look this over together.

1. Check out the harmonics. We take the ‘trigger’ output from the box to trigger the ‘scope. We plug in the fundamental frequency signal, \( f_1 \), into Channel 2 and can then look at any of the harmonics by plugging into the red "banana plug" socket for that frequency. Notice that there are two channels of fundamental. We will try out the various controls, looking at signals on the ‘scope.

2. Synthesize some waveforms. There is a summing amplifier that adds together various of the harmonics as determined by a row of slide switches. Look over the xeroxed excerpts from the CRC handbook listing various Fourier series representations. (You will notice that in their examples they are considering functions of position (x) and period is defined to be \( 2L \)). We will construct one or two wave shapes from the various choices, setting relative amplitudes and phases of the first few harmonics based on the published series.

II. Fourier Analysis (now you’re on your own)

The inverse process to the one we have just performed is to take a non-sinusoidal waveform and measure the Fourier series coefficients. A device which determines the frequency spectrum of an incoming time-dependent signal is called a “spectrum analyzer.” In principle one could build a spectrum analyzer using a tunable \( R-L-C \) circuit. One could tune the filter through a range of resonant frequencies and record the signal transmitted as a function of the frequency of the filter.
In practical spectrum analyzers the filtering is not done with an R-L-C circuit but many spectrum analyzers do in fact contain a tunable electronic filter.

Today you will be using two oscilloscopes, one to display the voltage as a function of time and the other to display the Fourier spectrum of the signal, i.e to be used as a spectrum analyzer. To produce the Fourier spectrum, the ‘scope records the incoming signal $V(t)$ for some set period of time, calculates the Fourier Series numerically (using an algorithm called “the Fast Fourier Transform”) and then computes and displays the power spectrum. In this case, the stick heights can be thought of as $|c_n|^2$, i.e. the magnitude of the Fourier coefficients, so we lose all information about the phases of the various coefficients. For the scope to properly produce the Fourier spectrum, the signal needs to be on scale i.e., the vertical scale needs to be set appropriately (by going to $V$ vs $t$ mode and making sure that the display looks OK).

Directions for the FFT feature of the scope are xeroxed for your reference. After you have the scope working normally, you can press MATH, and switch to looking at the Fourier spectrum. As you will see, the choice of horizontal scale that makes for a good FFT signal is a bit inconvenient for observing $V(t)$ hence you have a second scope for observing $V(t)$.

The vertical scale for the spectrum measures power i.e. $|c_n|^2$ and is on a logarithmic scale. A dB (pronounced “dee-bee” but meaning “deciBell” in honor of Alexander G. Bell) is a logarithmic way of comparing two signals. Since we are talking about electrical signals, “power” here means average power dissipated in the input resistor of the scope, i.e. $\langle V^2 \rangle / R$. We don’t actually care about the value of $R$, so effectively power means $\langle V^2 \rangle$. If signal A is 3dB more than signal B, that means signal A is a factor of $10^{0.3} = 2$ higher in power, hence $\sqrt{2} = 1.4$ in voltage. If signal A differs from B by -3dB then A has $10^{-0.3} = 0.5$ times the power and hence has $\sqrt{0.5} = .71$ the amplitude. 10 dB means a factor of 10 in power, 20 dB is a factor of 100 in power (hence 10 in amplitude).

In addition to representing the relative size of two signals, one can also use the dB system to express the absolute power of a signal by choosing a standard reference signal. The scope uses dBV’s which means that it measures the power in your signal relative that which would be produced by a 1 V signal. Thus $\text{dBV} = 20 \log(V/V_0)$ or equivalently: $V/V_0 = 10^{\text{dBV}/20}$ where $V_0 = 1$ V. If you measure one signal to be 20 dBV and another to be 0dBV, that means that they differ by 20dB, i.e. a factor of 100 in power (10 in amplitude). The one that measures 0dBV is a 1 Volt signal, the other one is a 10 Volt signal. To complete the picture, one has to distinguish between the various ways of measuring voltage (amplitude, p-p, rms)! 0 dBV on the scope means a signal that is 1 V rms (root-mean-square). Got all that?

Being a logarithmic scale, dB’s are convenient when dealing with signals of greatly differing amplitude. On the scope the vertical scale is 10 dB per division (large block on the screen) and is not intended for making precise amplitude comparisons but rather for showing the relative orders of magnitude of various signals.
While the FFT technique is fast, useful, and ubiquitous one must realize its limitations. It cannot sample the signal infinitely fast, and thus very high harmonics (from rapidly-changing parts of \( V(t) \)) are ‘lost’. Nyquist’s theorem that says no frequency faster than half the rate of sampling can be measured. The sampling rate of the FFT algorithm on the scope changes as you change the horizontal ‘scale’. Thus ‘50 kS/s’ means that we only see frequencies up to \( f_{\text{max}} = 25 \text{ kHz} \). Make sure that your actual signal doesn’t have significant frequency components above this cutoff, or else there will be a confusing phenomenon called ‘aliasing’ where those high frequencies are folded back into the existing spectrum. You can use the horizontal position to center any feature you want in the middle, then use the Math Menu / Zoom feature to zero in on the spectrum (zooming does not change the sampling rate). So you might say “fine, always sample very, very fast!” But the problem is that the FFT only takes a finite number (2048) of samples to compute the transform. Thus, fast sampling means shorter total time meaning fewer periods of oscillation are sampled, reducing the resolution. Bottom line: To get better resolution, reduce the sampling rate. To see out to higher frequencies, increase the sampling rate.

1. Function generator survey and the 'perfect' sine wave. You have a GW Instek function generator and a Wavetek 182A (or a BK Precision 4011) function generator. Plug the Instek function generator output into the scope, set to put out a \( \sim 1\text{kHz} \) sine wave at low amplitude with \( V(t) \) on the scope and the power spectrum on the spectrum analyzer (i.e the scope set to display the FFT). Vary the time axis scale (sample rate) and get the cursors working to measure both the frequency and height of the peaks. Set your sine wave to be 1 V \( \text{p-p} \). Measure the amplitude of the fundamental on the spectrum analyzer. Does the dBV reading come out correctly?

   Note that a sine wave should be a pure frequency. Is it? By what factor is the next harmonic reduced? The vertical scale is 10 dB/box, so that, for example, a ‘four-box’ difference in peak heights = factor of \( 10^4 \) in power = 100 in amplitude. Note that if you increase the amplitude of the input sine wave without readjusting the spectrum analyzer the sine wave appears to get less pure but that is an artifact of overdriving the input to the spectrum analyzer, thus clipping off the peaks of the sine wave, not an actual change in the output of the signal generator.

2. Square wave and triangle wave. Now have the Instek box produce a square wave. Measure the amplitudes of the odd frequency Fourier components by determining the dB change from fundamental frequency to all higher components. (You might have some problems with "aliasing." See pp. 122-124 of the digital scope FFT instructions). Notice that there are also some small even harmonics that (ideally) should not be there. Compare the measured odd harmonic values of the to what you expect for a mathematically perfect square wave.

   Now analyze a triangle wave and compare your results for the Fourier coefficients with the mathematical results for a perfect triangle wave.

3. Pulses w/ GW Instek generator. The Fourier spectrum of a series of pulses of length \( \tau \) spaced by \( T \) is given by:

   \[
   c_n = \left( \tau / T \right) \left[ \sin(\omega_n \tau / 2) / (\omega_n \tau / 2) \right] ; \ \omega_n = n 2\pi / T
   \]
(See also the CRC pages). Start with the Instek generator sending a 1 kHz square wave to both the scope and spectrum analyzer. With the ‘duty’ cycle knob off (pushed in), confirm that we have a 50% pos/neg wave (i.e. $T/\tau = 1$), and confirm which frequency components are ‘missing’ on the spectrum.

Now pull out the duty cycle knob and adjust the duty cycle knob and confirm on the analog scope that only $\tau$ is changing and not $T$. When you reach $T/\tau = 3$ and $T/\tau = 4$ note what is happening in the spectrum (where is the first zero?). Does the frequency of the fundamental Fourier peak change? Leave the pulse at its narrowest (dial fully turned). By the way, does it matter whether the narrow part of the square wave is the ‘positive’ or ‘negative piece’ in terms of the shape of the spectrum?

Now increase the frequency by a factor of ten (by pushing the button on the signal generator), but change nothing else. Look at $V(t)$. Has $T$ changed? Has $\tau$ changed? If so how? What about the ratio of $T/\tau$? Has the location of the first ‘zero’ of frequency spectrum changed? Has the frequency of the fundamental Fourier peak changed? What about the number of frequency peaks in the first ‘lobe’ of the spectrum?

As you make $T$ longer you approach the single pulse “Fourier Transform” limit. Unfortunately, we cannot make the relative width of the pulse any smaller than $\tau/T = \frac{1}{5}$ with this function generator.

4. Amplitude Modulation. The Instek generator will produce both AM (amplitude modulated) and FM (frequency modulated) signals. You will need to send a modulating signal into the back of the Instek. Use the ‘hi’ output from the Wavetek (or BK Precision) function generator and plug it into the VCF/MOD input on the back of the Instek. Take a BNC from the ‘synch’ output of the Wavetek (or the "TTL/CMOS" output of the BK Precision) and put this signal into the EXT trigger of the scope (and set it to trigger on this). You will need to push both the MOD ON and MOD EXT buttons on the front panel of the Instek. Now note the knob which shows AM in orange and FM in black. Pulling this knob enables Amplitude Modulation, while leaving it in produces Frequency Modulation. Adjusting the knob changes the strength of the modulation. The frequency of the modulating signal will be controlled by the Wavetek source.

To begin with, pull the knob for AM modulation. With this setting, the amplitude of the signal from the Instek is set by the (time-dependent) signal from the other signal generator. Set the Instek to put out a $\sim 20$ kHz sine wave. This is called the “carrier” frequency. Start with the Wavetek set to a 1 Hz (very slow!) sine wave output (this is the “modulation” frequency). Look at the time-dependent signal on the scope. Adjust the strength of modulation via either the ‘volume’ knob on the Wavetek or the AM modulation knob on the Instek. Is the modulation of the amplitude clear? What happens when you change the amplitude of the modulation signal? What about when you vary the frequency of the modulation? Does this all make sense?

Now look at the spectrum of the signal. At this slow modulation rate, you should simply see the main peak (the ‘carrier’) bobbing up and down. Vary both amplitude and frequency of the AM slightly and see what happens. Focus in on the main peak using the Zoom function and
the horizontal position knob. In this 'slow' this regime, the time to acquire the FFT is short compared to the modulation period, so we simply see the FFT constantly refreshed with an updated amplitude/frequency. Now switch to 1 kHz modulation frequency. The modulation period is now much shorter than FFT sample time. The spectrum is (should be) completely different looking! The new peaks are referred to as “sidebands.” Now vary both the amplitude and the frequency a bit and describe the effect of these changes. Describe what you see!

Increase the modulation "depth" (i.e. the amplitude of the modulation). Eventually you can achieve 100% modulation depth (and even 'beyond'). Note what happens to the frequency spectrum as we go this extreme. Now back off to a more modest modulation amplitude, and switch the Wavetek to a square wave output. You are now multiplying the carrier wave by a square wave function. Describe what you now see on the FFT spectrum.

5. Frequency Modulation. Now push in the AM/FM knob on the Instek generator to switch to FM modulation. Return to a Wavetek modulation frequency of \( \sim 1 \) Hz. In FM modulation the frequency of the wave is adjusted in response to the modulation input. If we write the wave as \( V(t) = V_0 \cos(\Phi(t)) \) with instantaneous frequency \( \omega = d\Phi/dt \), then the carrier, with \( \omega = \omega_c \) (a constant) has \( \Phi = \int \omega_c dt = \omega_c t + \phi \), giving our usual \( V(t) = V_0 \cos(\omega_c t + \phi) \). If the frequency is modulated by an amount \( \pm \Delta \omega \) at frequency \( \omega_m \), i.e. \( d\Phi / dt = \omega_c + \Delta \omega \sin(\omega_m t) \), then \( V(t) = V_0 \cos(\omega_c t + \Delta \omega \sin(\omega_m t) + \phi) \). This can also be thought of as phase modulation, i.e. \( V(t) = V_0 \cos(\omega_c t + \beta(t)) \) where the phase shift is \( \beta(t) = \Delta \beta \cos(\omega_m t) + \phi \).

Look at the carrier frequency on your spectrum analyzer in this “slow modulation” regime. Adjust the amplitude and/or frequency of the modulation and note what happens. Does the spectrum make sense in this slow modulation regime?

Now adjust the modulation amplitude to vary the frequency by a few hundred Hz (thus setting \( \Delta \omega \)). Now switch again to ‘fast’ FM, by putting the Wavetek frequency (i.e. \( \omega_m \)) to about 1 kHz. Note the dramatic change in the spectrum. The sideband pattern has some similarities to the AM pattern but also some important differences. At high modulation amplitude the sideband pattern grows more complex, than the rather simple AM spectrum. Experiment with this. Notice that you can "suppress the carrier", i.e. remove all of the signal at the original center frequency!

If you try modulating by 100 Hz, it may appear that the carrier is simply broadened out. This is due to insufficient resolution. Increase the resolution on the FFT (by decreasing the sample rate) and see if you can resolve the 100 Hz modulation peaks.

6. Half wave rectifier. On the breadboard set up a half wave rectifier circuit (diode and resistor) as shown. Send in a sine wave. The voltage needs to be \( > 0.6 \) V or so for the diode to conduct, so be sure it is above that. Look at the voltage across the resistor on the scope and make sure you see something that looks like the positive parts of a sine wave. Now look at the Fourier spectrum.
(which you calculated on a problem set). What frequencies are present?

7. **Low-pass filter**: Set up a low pass filter with $R=600 \ \Omega$, $C=0.01 \ \mu F$ and drive it with a square wave with fundamental frequency 300 Hz.

   Compare the spectrum of the source square wave and the filtered signal on a very zoomed out scale at high sample rate (like 500 kS/s) where you can’t see individual spikes, but just the overall envelope of the spectrum. What happens to the spectrum?

   Recall that the ‘corner’ frequency of the filter is defined as $f_c = 1/(2\pi RC)$. Do your results make sense given the components you are using? Try increasing the the fundamental frequency of the square wave and look at the output waveform in time.

### III. Acoustical Spectra *(a group activity)*

Find the fancy spectrum analyzer + scope + microphone + keyboard set up. It should be up and running, but ask for help if you don’t see what you expect. The idea is that you can sing, play, or blow notes and see both the time-dependent waveform, and its spectrum.

1. **Organ pipes.** Find a square cross-sectioned ‘organ pipe’. Without going into a full analysis of the acoustical standing waves, we can think of this as a string with one fixed and one free end. In this case pressure is the variable of choice. At an opening, the pressure of the wave is ‘fixed’ to be atmospheric pressure (node), while in a closed-ended tube the pressure is ‘free’ to oscillate above and below atmospheric pressure (antinode). Taking the speed of sound to be 343 m/s, measure the length of your pipe and blow gently near the microphone. Look at the periodic but complex time dependent signal. Look at its spectrum on the spectrum analyzer. Is the fundamental frequency (lowest mode) what you expect? Why are there multiple frequencies, and what relationship do they have to the fundamental? Note that you ‘hear’ only the pitch of the fundamental. Note also that if you ‘overblow’ the pipe (blow hard), a new pitch results (it is an octave higher (approximately). Look at the spectrum in this case and comment.

    Now firmly place your palm over the end of the tube, closing it off. Holding the hole near the mouthpiece near the microphone, blow gently. Explain the resulting pitch you hear and the spectrum you see in detail. What is the physical explanation for this? Try ‘overblowing’ in this case, and note that you hear a higher pitch (but now it is actually an octave+perfect fifth higher – i.e. a factor of three in frequency). Explain.

2. **Voice.** For fun, time permitting try singing different vowels into the microphone and check out the spectrum. Not only will you see rich harmonic structure, but if you sing a low-ish note, you may be able to see a range of high harmonics which are emphasized relative to nearby ones. This is due to resonance in the vocal tract, referred to as vocal ‘formants’. Control over these formants is what makes the tone quality of a trained voice noticeably different from that of an untrained voice (I’m told).