Symbolic Logic: Grammar, Semantics, Syntax

Logic aims to give a precise method or recipe for determining what follows from a given set of sentences. So we need precise definitions of ‘sentence’ and ‘follows from.’

1. Grammar: What’s a sentence?

What are the rules that determine whether a string of symbols is a sentence, and when it is not? These rules are known as the grammar of a particular language. We have actually been operating with two different but very closely related grammars this term: a grammar for propositional logic and a grammar for first-order logic.

The grammar for propositional logic (thus far) is simple:

1. There are an indefinite number of propositional variables, which we have been symbolizing as $A, B, ...$ and $P, Q, ...$; each of these is a sentence. $\bot$ is also a sentence.

2. If $A, B$ are sentences, then:
   - $\neg A$ is a sentence,
   - $A \land B$ is a sentence, and
   - $A \lor B$ is a sentence.

3. Nothing else is a sentence: anything that is neither a member of 1, nor constructable via successive applications of 2 to the members of 1, is not a sentence.

The grammar for first-order logic (thus far) is more complex.
2 and 3 remain exactly the same as above, but 1 must be replaced by something more complex. First, a list of all predicates (e.g. Tet or SameShape) and proper names (e.g. $\text{max}, b$) of a particular language $L$ must be specified. Then we can define:

$1_{FOL}$. A series of symbols $s$ is an atomic sentence $= s$ is an $n$-place predicate followed by an ordered sequence of $n$ proper names.

2. Semantics: Consequence in terms of truth and falsity

Here is the basic semantic notion of ‘follows from,’ which should be familiar from before:
**semantic consequence** (|=) A sentence $C$ (semantically) follows from premises $P_1, P_2, ...$ there is no case in which $P_1, P_2, ...$ are all true and $C$ is false.

The obvious next question is ‘What’s a case?’ This notion is, on its own, imprecise. But we have seen two ways of making it precise: a case is (i) an allowable arrangement of the Tarski’s world checkerboard, or (ii) a row of a truth-table. Substituting (i) in for ‘case’ above gives us one precise notion of following-from, namely *Tarski’s-World consequence*. Substituting (ii) in for ‘case’ gives us another precise notion of following-from, namely *Tautological Consequence*. These are different consequence relations, but they both are defined in terms of truth and falsity; that is, they are both semantic.

3. **Syntax: Consequence in terms of rules**

We have introduction and elimination rules for $\land, \lor, \neg, =,$ and $\bot$. We can define a precise notion of *follows from* using these rules: the basic idea is that $C$ follows from $P_1, P_2, ...$ exactly when there is a formal proof of the conclusion from the premises. This is a *syntactic*, not semantic, notion of consequence, since truth and falsity play no role.

The official definition of syntactic consequence is as follows:

**syntactic consequence** (⊢) A sentence $C$ is a syntactic consequence of sentences $P_1, P_2, ... = C$ can be reached from $P_1, P_2, ...$ using only the introduction- and elimination-rules.

In other words, there is a formal proof of $C$ from $P_1, P_2, ....$ (As this definition makes clear, whether $C$ is a syntactic consequence of $P_1, P_2, ...$ depends on which specific rules you have in your system.)

**Question:** Is it possible for $C$ to be a semantic consequence of $P_1, P_2, ...$, but not a syntactic consequence? What about the converse: can you imagine circumstances in which $C$ is a syntactic consequence of $P_1, P_2, ...$, but not a semantic consequence?

**Logical truth (semantics) vs. Theorem (syntax)**

The textbook often downplays the difference between semantic and syntactic consequence. This is very clear in the ‘Remember Box’ on p.174. ‘Logical truth’ was defined earlier as a sentence that is true in every possible case. But how do we know that the last line of a proof with no premises is true in every possible case? (The usual word for the last line of a proof without any premises is a *theorem*.)